Exercises for More Trigonometric Identities

Question 1 *In Exercises **??** - **??**, use the Even / Odd Identities to verify the identity. Assume all quantities are defined.*

Problem 1.1 $\sin(3\pi - 2\theta) = -\sin(2\theta - 3\pi)$

Problem 1.2 $\cos\left(-\frac{\pi}{4} - 5t\right) = \cos\left(5t + \frac{\pi}{4}\right)$

Problem 1.3 $\tan(-x^2 + 1) = -\tan(x^2 - 1)$

Problem 1.4 $\csc(-\theta - 5) = -\csc(\theta + 5)$

Problem 1.5 $\sec(-6x) = \sec(6x)$

Problem 1.6 $\cot(9 - 7\theta) = -\cot(7\theta - 9)$

Question 2 *In Exercises **??** - **??**, use the Sum and Difference Identities to find the exact value. You may have need of the Quotient, Reciprocal or Even / Odd Identities as well.*

Problem 2.1 $\cos(75°)$

$\cos(75°) = \frac{\sqrt{6} - \sqrt{2}}{4}$

Problem 2.2 $\sec(165°)$

$\sec(165°) = -\frac{4}{\sqrt{2} + \sqrt{6}} = \sqrt{2} - \sqrt{6}$

Problem 2.3 $\sin(105°)$

$\sin(105°) = \frac{\sqrt{6} + \sqrt{2}}{4}$

Problem 2.4 $\csc(195°)$

$\csc(195°) = \frac{4}{\sqrt{2} - \sqrt{6}} = -(\sqrt{2} + \sqrt{6})$

Problem 2.5 $\cot(255°)$

$\cot(255°) = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = 2 - \sqrt{3}$

Problem 2.6 $\tan(375°)$

$\tan(375°) = \frac{3 - \sqrt{3}}{3 + \sqrt{3}} = 2 - \sqrt{3}$

Problem 2.7 $\cos\left(\frac{13\pi}{12}\right)$

$\cos\left(\frac{13\pi}{12}\right) = -\frac{\sqrt{6} + \sqrt{2}}{4}$

Problem 2.8 $\sin\left(\frac{11\pi}{12}\right)$

$\sin\left(\frac{11\pi}{12}\right) = \frac{\sqrt{6} - \sqrt{2}}{4}$

Problem 2.9 $\tan\left(\frac{13\pi}{12}\right)$

$\tan\left(\frac{13\pi}{12}\right) = \frac{3-\sqrt{3}}{3+\sqrt{3}} = 2-\sqrt{3}$

Problem 2.10 $\cos\left(\frac{7\pi}{12}\right)$

$\cos\left(\frac{7\pi}{12}\right) = \frac{\sqrt{2}-\sqrt{6}}{4}$

Problem 2.11 $\tan\left(\frac{17\pi}{12}\right)$

$\tan\left(\frac{17\pi}{12}\right) = 2+\sqrt{3}$

Problem 2.12 $\sin\left(\frac{\pi}{12}\right)$

$\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$

Problem 2.13 $\cot\left(\frac{11\pi}{12}\right)$

$\cot\left(\frac{11\pi}{12}\right) = -(2+\sqrt{3})$

Problem 2.14 $\csc\left(\frac{5\pi}{12}\right)$

$\csc\left(\frac{5\pi}{12}\right) = \sqrt{6}-\sqrt{2}$

Problem 2.15 $\sec\left(-\frac{\pi}{12}\right)$

$\sec\left(-\frac{\pi}{12}\right) = \sqrt{6}-\sqrt{2}$

Problem 3 *If α is a Quadrant IV angle with $\cos(\alpha) = \frac{\sqrt{5}}{5}$, and $\sin(\beta) = \frac{\sqrt{10}}{10}$, where $\frac{\pi}{2} < \beta < \pi$, find*

1. $\cos(\alpha+\beta)$

   $\cos(\alpha+\beta) = -\frac{\sqrt{2}}{10}$

2. $\sin(\alpha+\beta)$

   $\sin(\alpha+\beta) = \frac{7\sqrt{2}}{10}$

3. $\tan(\alpha+\beta)$

   $\tan(\alpha+\beta) = -7$

4. $\cos(\alpha-\beta)$

   $\cos(\alpha-\beta) = -\frac{\sqrt{2}}{2}$

5. $\sin(\alpha - \beta)$

   $\sin(\alpha - \beta) = \dfrac{\sqrt{2}}{2}$

6. $\tan(\alpha - \beta)$

   $\tan(\alpha - \beta) = -1$

Problem 4 *If* $\csc(\alpha) = 3$*, where* $0 < \alpha < \dfrac{\pi}{2}$*, and* β *is a Quadrant II angle with* $\tan(\beta) = -7$*, find*

1. $\cos(\alpha + \beta)$

   $\cos(\alpha + \beta) = -\dfrac{4 + 7\sqrt{2}}{30}$

2. $\sin(\alpha + \beta)$

   $\sin(\alpha + \beta) = \dfrac{28 - \sqrt{2}}{30}$

3. $\tan(\alpha + \beta)$

   $\tan(\alpha + \beta) = \dfrac{-28 + \sqrt{2}}{4 + 7\sqrt{2}} = \dfrac{63 - 100\sqrt{2}}{41}$

4. $\cos(\alpha - \beta)$

   $\cos(\alpha - \beta) = \dfrac{-4 + 7\sqrt{2}}{30}$

5. $\sin(\alpha - \beta)$

   $\sin(\alpha - \beta) = -\dfrac{28 + \sqrt{2}}{30}$

6. $\tan(\alpha - \beta)$

   $\tan(\alpha - \beta) = \dfrac{28 + \sqrt{2}}{4 - 7\sqrt{2}} = -\dfrac{63 + 100\sqrt{2}}{41}$

Problem 5 *If* $\sin(\alpha) = \dfrac{3}{5}$*, where* $0 < \alpha < \dfrac{\pi}{2}$*, and* $\cos(\beta) = \dfrac{12}{13}$ *where* $\dfrac{3\pi}{2} < \beta < 2\pi$*, find*

1. $\sin(\alpha + \beta)$

   $\sin(\alpha + \beta) = \dfrac{16}{65}$

2. $\cos(\alpha - \beta)$

   $\cos(\alpha - \beta) = \dfrac{33}{65}$

3. $\tan(\alpha - \beta)$

   $\tan(\alpha - \beta) = \dfrac{56}{33}$

Problem 6 *If* $\sec(\alpha) = -\dfrac{5}{3}$*, where* $\dfrac{\pi}{2} < \alpha < \pi$*, and* $\tan(\beta) = \dfrac{24}{7}$*, where* $\pi < \beta < \dfrac{3\pi}{2}$*, find*

1. $\csc(\alpha - \beta)$

   $\csc(\alpha - \beta) = -\dfrac{5}{4}$

2. $\sec(\alpha+\beta)$

   $\sec(\alpha+\beta) = \dfrac{125}{117}$

3. $\cot(\alpha+\beta)$

   $\cot(\alpha+\beta) = \dfrac{117}{44}$

Question 7 *In Exercises* **??** *-* **??***, use Example* **??** *as a guide to show that the function is a sinusoid by rewriting it in the forms* $C(t) = A\cos(\omega t+\phi)+B$ *and* $S(t) = A\sin(\omega t+\phi)+B$ *for* $\omega > 0$ *and* $0 \le \phi < 2\pi$*.*

Problem 7.1 $f(t) = \sqrt{2}\sin(t)+\sqrt{2}\cos(t)+1$

$f(t) = \sqrt{2}\sin(t)+\sqrt{2}\cos(t)+1 = 2\sin\left(t+\frac{\pi}{4}\right)+1 = 2\cos\left(t+\frac{7\pi}{4}\right)+1$

Problem 7.2 $f(t) = 3\sqrt{3}\sin(3t)-3\cos(3t)$

$f(t) = 3\sqrt{3}\sin(3t)-3\cos(3t) = 6\sin\left(3t+\frac{11\pi}{6}\right) = 6\cos\left(3t+\frac{4\pi}{3}\right)$

Problem 7.3 $f(t) = -\sin(t)+\cos(t)-2$

$f(t) = -\sin(t)+\cos(t)-2 = \sqrt{2}\sin\left(t+\frac{3\pi}{4}\right)-2 = \sqrt{2}\cos\left(t+\frac{\pi}{4}\right)-2$

Problem 7.4 $f(t) = -\frac{1}{2}\sin(2t)-\frac{\sqrt{3}}{2}\cos(2t)$

$f(t) = -\frac{1}{2}\sin(2t)-\frac{\sqrt{3}}{2}\cos(2t) = \sin\left(2t+\frac{4\pi}{3}\right) = \cos\left(2t+\frac{5\pi}{6}\right)$

Problem 7.5 $f(t) = 2\sqrt{3}\cos(t)-2\sin(t)$

$f(t) = 2\sqrt{3}\cos(t)-2\sin(t) = 4\sin\left(t+\frac{2\pi}{3}\right) = 4\cos\left(t+\frac{\pi}{6}\right)$

Problem 7.6 $f(t) = \frac{3}{2}\cos(2t)-\frac{3\sqrt{3}}{2}\sin(2t)+6$

$f(t) = \frac{3}{2}\cos(2t)-\frac{3\sqrt{3}}{2}\sin(2t)+6 = 3\sin\left(2t+\frac{5\pi}{6}\right)+6 = 3\cos\left(2t+\frac{\pi}{3}\right)+6$

Problem 7.7 $f(t) = -\frac{1}{2}\cos(5t)-\frac{\sqrt{3}}{2}\sin(5t)$

$f(t) = -\frac{1}{2}\cos(5t)-\frac{\sqrt{3}}{2}\sin(5t) = \sin\left(5t+\frac{7\pi}{6}\right) = \cos\left(5t+\frac{2\pi}{3}\right)$

Problem 7.8 $f(t) = -6\sqrt{3}\cos(3t)-6\sin(3t)-3$

$f(t) = -6\sqrt{3}\cos(3t)-6\sin(3t)-3 = 12\sin\left(3t+\frac{4\pi}{3}\right)-3 = 12\cos\left(3t+\frac{5\pi}{6}\right)-3$

Problem 7.9 $f(t) = \frac{5\sqrt{2}}{2}\sin(t)-\frac{5\sqrt{2}}{2}\cos(t)$

$f(t) = \frac{5\sqrt{2}}{2}\sin(t)-\frac{5\sqrt{2}}{2}\cos(t) = 5\sin\left(t+\frac{7\pi}{4}\right) = 5\cos\left(t+\frac{5\pi}{4}\right)$

Exercises for More Trigonometric Identities

Problem 7.10 $f(t) = 3\sin\left(\frac{t}{6}\right) - 3\sqrt{3}\cos\left(\frac{t}{6}\right)$

$f(t) = 3\sin\left(\frac{t}{6}\right) - 3\sqrt{3}\cos\left(\frac{t}{6}\right) = 6\sin\left(\frac{t}{6} + \frac{5\pi}{3}\right) = 6\cos\left(\frac{t}{6} + \frac{7\pi}{6}\right)$

Problem 8 *In Exercises **??** - **??**, you should have noticed a relationship between the phases ϕ for the $S(t)$ and $C(t)$. Show that if $f(t) = A\sin(\omega t + \alpha) + B$, then $f(t) = A\cos(\omega t + \beta) + B$ where $\beta = \alpha - \frac{\pi}{2}$.*

Problem 9 *Let ϕ be an angle measured in radians and let $P(a, b)$ be a point on the terminal side of ϕ when it is drawn in standard position. Use Theorem **??** and the sum identity for sine in Theorem **??** to show that $f(t) = a\sin(\omega t) + b\cos(\omega t) + B$ (with $\omega > 0$) can be rewritten as $f(t) = \sqrt{a^2 + b^2}\sin(\omega t + \phi) + B$.*

Problem 10 *In Example **??** in Section **??**, we developed two (seemingly) different formulas to model the hours of daylight, $H(t)$: $H_1(t) = 9.25\sin\left(\frac{\pi}{6}t - \frac{\pi}{2}\right) + 12.55$ and $H_2(t) = -8.13\sin\left(\frac{\pi}{6}t - 4.70\right) + 12.5$. Use the difference identities for sine to expand $H_1(t)$ and $H_2(t)$. How different are they?*

Question 11 *In Exercises **??** - **??**, verify the identity.*[1]

Problem 11.1 $\sin\left(\theta + \frac{\pi}{2}\right) = \cos(t)$

Problem 11.2 $\cos\left(\theta - \frac{\pi}{2}\right) = \sin(t)$

Problem 11.3 $\cos(\theta - \pi) = -\cos(\theta)$

Problem 11.4 $\sin(\pi - \theta) = \sin(\theta)$

Problem 11.5 $\tan\left(\theta + \frac{\pi}{2}\right) = -\cot(\theta)$

Problem 11.6 $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin(\alpha)\cos(\beta)$

Problem 11.7 $\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2\cos(\alpha)\sin(\beta)$

Problem 11.8 $\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos(\alpha)\cos(\beta)$

Problem 11.9 $\cos(\alpha + \beta) - \cos(\alpha - \beta) = -2\sin(\alpha)\sin(\beta)$

Problem 11.10 $\dfrac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)} = \dfrac{1 + \cot(\alpha)\tan(\beta)}{1 - \cot(\alpha)\tan(\beta)}$

Problem 11.11 $\dfrac{\cos(\alpha + \beta)}{\cos(\alpha - \beta)} = \dfrac{1 - \tan(\alpha)\tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$

Problem 11.12 $\dfrac{\tan(\alpha + \beta)}{\tan(\alpha - \beta)} = \dfrac{\sin(\alpha)\cos(\alpha) + \sin(\beta)\cos(\beta)}{\sin(\alpha)\cos(\alpha) - \sin(\beta)\cos(\beta)}$

Question 12 *In Exercise **??** - **??**, use the results from Exercise **??** in Section **??**: $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1$ and Exercise **??** in Section **??** : $\lim_{\theta\to 0} \frac{1-\cos(\theta)}{\theta} = 0$ to help you find the derivatives of the sine and cosine functions.*

[1] Note: numbers **??** and **??** are the conversion formulas stated in Theorem **??** in Section **??**.

EXERCISES FOR MORE TRIGONOMETRIC IDENTITIES

Problem 12.1 1. *Verify for* $f(t) = \sin(t)$, $\dfrac{f(t+h) - f(t)}{h} = \dfrac{\sin(t+h) - \sin(t)}{h} = \cos(t)\left(\dfrac{\sin(h)}{h}\right) + \sin(t)\left(\dfrac{\cos(h) - 1}{h}\right)$

2. *Show* $f'(t) = \lim\limits_{h\to 0} \dfrac{f(t+h) - f(t)}{h} = \cos(t)$.

   Hint: $\lim\limits_{\theta\to 0} \dfrac{\cos(\theta) - 1}{\theta} = \lim\limits_{\theta\to 0}\left(-\dfrac{1-\cos(\theta)}{\theta}\right) = -(0) = 0$

3. *Find the equation of the tangent line to the graph of* $y = \sin(t)$ *at* $(0,0)$, $\left(\frac{\pi}{2}, 1\right)$ *and* $(\pi, 0)$.

   Check your answers graphically.

   at $(0,0)$: $y = x$; *at* $\left(\frac{\pi}{2}, 1\right)$: $y = 1$ *at* $(\pi, 0)$: $y = -x + \pi$

Problem 12.2 1. *Verify for* $g(t) = \cos(t)$, $\dfrac{g(t+h) - g(t)}{h} = \dfrac{\cos(t+h) - \cos(t)}{h} = \cos(t)\left(\dfrac{\cos(h) - 1}{h}\right) - \sin(t)\left(\dfrac{\sin(h)}{h}\right)$

2. *Show* $g'(t) = \lim\limits_{h\to 0} \dfrac{g(t+h) - g(t)}{h} = -\sin(t)$.

3. *Find the equation of the tangent line to the graph of* $y = \cos(t)$ *at* $(0,1)$, $\left(\frac{\pi}{2}, 0\right)$ *and* $\left(-\frac{\pi}{2}, 0\right)$.

   Check your answers graphically.

   at $(0,1)$: $y = 1$; *at* $\left(\frac{\pi}{2}, 0\right)$: $y = -x + \frac{\pi}{2}$ *at* $\left(-\frac{\pi}{2}, 0\right)$: $y = x + \frac{\pi}{2}$

Problem 13 1. *Use the fact*[2] *that* $\lim\limits_{\theta\to 0} \dfrac{\sin(\theta)}{\theta} = 1$ *to prove* $\lim\limits_{\theta\to 0} \dfrac{\tan(\theta)}{\theta} = 1$.

   Hint: $\dfrac{\tan(\theta)}{\theta} = \dfrac{\sin(\theta)}{\theta}\,\dfrac{1}{\cos(\theta)} \ldots$

2. *For* $f(t) = \tan(t)$, *show* $\dfrac{f(t+h) - f(t)}{h} = \dfrac{\tan(t+h) - \tan(t)}{h} = \left(\dfrac{\tan(h)}{h}\right)\left(\dfrac{\sec^2(t)}{1 - \tan(t)\tan(h)}\right)$.

3. *Use part* **??** *to show* $f'(t) = \lim\limits_{h\to 0} \dfrac{f(t+h) - f(t)}{h} = \sec^2(t)$

4. *Find the equation of the tangent line to the graph of* $y = \tan(t)$ *at* $(0,0)$, $\left(\frac{\pi}{4}, 1\right)$ *and* $\left(-\frac{\pi}{4}, -1\right)$.

   Check your answers graphically.

   at $(0,0)$: $y = x$; *at* $\left(\frac{\pi}{4}, 1\right)$: $y = 2x - \frac{\pi}{2} + 1$ *at* $\left(-\frac{\pi}{4}, -1\right)$: $y = 2x + \frac{\pi}{2} - 1$

Question 14 *In Exercises* **??** - **??**, *use the Half Angle Formulas to find the exact value. You may have need of the Quotient, Reciprocal or Even / Odd Identities as well.*

Problem 14.1 $\cos(75^\circ)$ *(compare with Exercise* **??***)*

$\cos(75^\circ) = \dfrac{\sqrt{2 - \sqrt{3}}}{2}$

[2] See Exercise **??** in Section **??**.

Problem 14.2 $\sin(105^\circ)$ *(compare with Exercise* **??***)*

$\sin(105^\circ) = \dfrac{\sqrt{2+\sqrt{3}}}{2}$

Problem 14.3 $\cos(67.5^\circ)$

$\cos(67.5^\circ) = \dfrac{\sqrt{2-\sqrt{2}}}{2}$

Problem 14.4 $\sin(157.5^\circ)$

$\sin(157.5^\circ) = \dfrac{\sqrt{2-\sqrt{2}}}{2}$

Problem 14.5 $\tan(112.5^\circ)$

$\tan(112.5^\circ) = -\sqrt{\dfrac{2+\sqrt{2}}{2-\sqrt{2}}} = -1-\sqrt{2}$

Problem 14.6 $\cos\left(\dfrac{7\pi}{12}\right)$ *(compare with Exercise* **??***)*

$\cos\left(\dfrac{7\pi}{12}\right) = -\dfrac{\sqrt{2-\sqrt{3}}}{2}$

Problem 14.7 $\sin\left(\dfrac{\pi}{12}\right)$ *(compare with Exercise* **??***)*

$\sin\left(\dfrac{\pi}{12}\right) = \dfrac{\sqrt{2-\sqrt{3}}}{2}$

Problem 14.8 $\cos\left(\dfrac{\pi}{8}\right)$

$\cos\left(\dfrac{\pi}{8}\right) = \dfrac{\sqrt{2+\sqrt{2}}}{2}$

Problem 14.9 $\sin\left(\dfrac{5\pi}{8}\right)$

$\sin\left(\dfrac{5\pi}{8}\right) = \dfrac{\sqrt{2+\sqrt{2}}}{2}$

Problem 14.10 $\tan\left(\dfrac{7\pi}{8}\right)$

$\tan\left(\dfrac{7\pi}{8}\right) = -\sqrt{\dfrac{2-\sqrt{2}}{2+\sqrt{2}}} = 1-\sqrt{2}$

Question 15 *In Exercises* **??** *-* **??***, use the given information about* θ *to find the exact values of*

- $\sin(2\theta)$
- $\sin\left(\dfrac{\theta}{2}\right)$

- $\cos(2\theta)$
- $\cos\left(\frac{\theta}{2}\right)$
- $\tan(2\theta)$
- $\tan\left(\frac{\theta}{2}\right)$

Problem 15.1 $\sin(\theta) = -\frac{7}{25}$ *where* $\frac{3\pi}{2} < \theta < 2\pi$

- $\sin(2\theta) = -\frac{336}{625}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{\sqrt{2}}{10}$
- $\cos(2\theta) = \frac{527}{625}$
- $\cos\left(\frac{\theta}{2}\right) = -\frac{7\sqrt{2}}{10}$
- $\tan(2\theta) = -\frac{336}{527}$
- $\tan\left(\frac{\theta}{2}\right) = -\frac{1}{7}$

Problem 15.2 $\cos(\theta) = \frac{28}{53}$ *where* $0 < \theta < \frac{\pi}{2}$

- $\sin(2\theta) = \frac{2520}{2809}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{5\sqrt{106}}{106}$
- $\cos(2\theta) = -\frac{1241}{2809}$
- $\cos\left(\frac{\theta}{2}\right) = \frac{9\sqrt{106}}{106}$
- $\tan(2\theta) = -\frac{2520}{1241}$
- $\tan\left(\frac{\theta}{2}\right) = \frac{5}{9}$

Problem 15.3 $\tan(\theta) = \frac{12}{5}$ *where* $\pi < \theta < \frac{3\pi}{2}$

- $\sin(2\theta) = \frac{120}{169}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{3\sqrt{13}}{13}$

- $\cos(2\theta) = -\frac{119}{169}$
- $\cos\left(\frac{\theta}{2}\right) = -\frac{2\sqrt{13}}{13}$
- $\tan(2\theta) = -\frac{120}{119}$
- $\tan\left(\frac{\theta}{2}\right) = -\frac{3}{2}$

Problem 15.4 $\csc(\theta) = 4$ *where* $\frac{\pi}{2} < \theta < \pi$

- $\sin(2\theta) = -\frac{\sqrt{15}}{8}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{\sqrt{8+2\sqrt{15}}}{4}$
- $\cos(2\theta) = \frac{7}{8}$
- $\cos\left(\frac{\theta}{2}\right) = \frac{\sqrt{8-2\sqrt{15}}}{4}$
- $\tan(2\theta) = -\frac{\sqrt{15}}{7}$
- $\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{8+2\sqrt{15}}{8-2\sqrt{15}}}$
  $\tan\left(\frac{\theta}{2}\right) = 4+\sqrt{15}$

Problem 15.5 $\cos(\theta) = \frac{3}{5}$ *where* $0 < \theta < \frac{\pi}{2}$

- $\sin(2\theta) = \frac{24}{25}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{\sqrt{5}}{5}$
- $\cos(2\theta) = -\frac{7}{25}$
- $\cos\left(\frac{\theta}{2}\right) = \frac{2\sqrt{5}}{5}$
- $\tan(2\theta) = -\frac{24}{7}$
- $\tan\left(\frac{\theta}{2}\right) = \frac{1}{2}$

Problem 15.6 $\sin(\theta) = -\frac{4}{5}$ *where* $\pi < \theta < \frac{3\pi}{2}$

- $\sin(2\theta) = \frac{24}{25}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{2\sqrt{5}}{5}$
- $\cos(2\theta) = -\frac{7}{25}$
- $\cos\left(\frac{\theta}{2}\right) = -\frac{\sqrt{5}}{5}$
- $\tan(2\theta) = -\frac{24}{7}$
- $\tan\left(\frac{\theta}{2}\right) = -2$

Problem 15.7 $\cos(\theta) = \frac{12}{13}$ *where* $\frac{3\pi}{2} < \theta < 2\pi$

- $\sin(2\theta) = -\frac{120}{169}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{\sqrt{26}}{26}$
- $\cos(2\theta) = \frac{119}{169}$
- $\cos\left(\frac{\theta}{2}\right) = -\frac{5\sqrt{26}}{26}$
- $\tan(2\theta) = -\frac{120}{119}$
- $\tan\left(\frac{\theta}{2}\right) = -\frac{1}{5}$

Problem 15.8 $\sin(\theta) = \frac{5}{13}$ *where* $\frac{\pi}{2} < \theta < \pi$

- $\sin(2\theta) = -\frac{120}{169}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{5\sqrt{26}}{26}$
- $\cos(2\theta) = \frac{119}{169}$
- $\cos\left(\frac{\theta}{2}\right) = \frac{\sqrt{26}}{26}$
- $\tan(2\theta) = -\frac{120}{119}$

- $\tan\left(\frac{\theta}{2}\right) = 5$

Problem 15.9 $\sec(\theta) = \sqrt{5}$ *where* $\frac{3\pi}{2} < \theta < 2\pi$

- $\sin(2\theta) = -\frac{4}{5}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{\sqrt{50 - 10\sqrt{5}}}{10}$
- $\cos(2\theta) = -\frac{3}{5}$
- $\cos\left(\frac{\theta}{2}\right) = -\frac{\sqrt{50 + 10\sqrt{5}}}{10}$
- $\tan(2\theta) = \frac{4}{3}$
- $\tan\left(\frac{\theta}{2}\right) = -\sqrt{\frac{5 - \sqrt{5}}{5 + \sqrt{5}}}$
  $\tan\left(\frac{\theta}{2}\right) = \frac{5 - 5\sqrt{5}}{10}$

Problem 15.10 $\tan(\theta) = -2$ *where* $\frac{\pi}{2} < \theta < \pi$

- $\sin(2\theta) = -\frac{4}{5}$
- $\sin\left(\frac{\theta}{2}\right) = \frac{\sqrt{50 + 10\sqrt{5}}}{10}$
- $\cos(2\theta) = -\frac{3}{5}$
- $\cos\left(\frac{\theta}{2}\right) = \frac{\sqrt{50 - 10\sqrt{5}}}{10}$
- $\tan(2\theta) = \frac{4}{3}$
- $\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{5 + \sqrt{5}}{5 - \sqrt{5}}}$
  $\tan\left(\frac{\theta}{2}\right) = \frac{5 + 5\sqrt{5}}{10}$

Exercises for More Trigonometric Identities

Question 16 *In Exercises **??** - **??**, verify the identity. Assume all quantities are defined.*

Problem 16.1 $(\cos(\theta)+\sin(\theta))^2 = 1+\sin(2\theta)$

Problem 16.2 $(\cos(\theta)-\sin(\theta))^2 = 1-\sin(2\theta)$

Problem 16.3 $\tan(2t) = \dfrac{1}{1-\tan(t)} - \dfrac{1}{1+\tan(t)}$

Problem 16.4 $\csc(2\theta) = \dfrac{\cot(\theta)+\tan(\theta)}{2}$

Problem 16.5 $8\sin^4(x) = \cos(4x) - 4\cos(2x) + 3$

Problem 16.6 $8\cos^4(x) = \cos(4x) + 4\cos(2x) + 3$

Problem 16.7 $\sin(3\theta) = 3\sin(\theta) - 4\sin^3(\theta)$

Problem 16.8 $\sin(4\theta) = 4\sin(\theta)\cos^3(\theta) - 4\sin^3(\theta)\cos(\theta)$

Problem 16.9 $32\sin^2(t)\cos^4(t) = 2+\cos(2t) - 2\cos(4t) - \cos(6t)$

Problem 16.10 $32\sin^4(t)\cos^2(t) = 2-\cos(2t) - 2\cos(4t) + \cos(6t)$

Problem 16.11 $\cos(4\theta) = 8\cos^4(\theta) - 8\cos^2(\theta) + 1$

Problem 16.12 $\cos(8\theta) = 128\cos^8(\theta) - 256\cos^6(\theta) + 160\cos^4(\theta) - 32\cos^2(\theta) + 1$ *(HINT: Use the result to **??**.)*

Problem 16.13 $\sec(2x) = \dfrac{\cos(x)}{\cos(x)+\sin(x)} + \dfrac{\sin(x)}{\cos(x)-\sin(x)}$

Problem 16.14 $\dfrac{1}{\cos(\theta)-\sin(\theta)} + \dfrac{1}{\cos(\theta)+\sin(\theta)} = \dfrac{2\cos(\theta)}{\cos(2\theta)}$

Problem 16.15 $\dfrac{1}{\cos(\theta)-\sin(\theta)} - \dfrac{1}{\cos(\theta)+\sin(\theta)} = \dfrac{2\sin(\theta)}{\cos(2\theta)}$

Problem 17 *Suppose θ is a Quadrant I angle with $\sin(\theta) = x$. Verify the following formulas*

1. $\cos(\theta) = \sqrt{1-x^2}$
2. $\sin(2\theta) = 2x\sqrt{1-x^2}$
3. $\cos(2\theta) = 1-2x^2$

Problem 18 *Discuss with your classmates how each of the formulas, if any, in Exercise **??** change if we change assume θ is a Quadrant II, III, or IV angle.*

Problem 19 *Suppose θ is a Quadrant I angle with $\tan(\theta) = x$. Verify the following formulas*

1. $\cos(\theta) = \dfrac{1}{\sqrt{x^2+1}}$

2. $\sin(\theta) = \dfrac{x}{\sqrt{x^2+1}}$
3. $\sin(2\theta) = \dfrac{2x}{x^2+1}$
4. $\cos(2\theta) = \dfrac{1-x^2}{x^2+1}$

Problem 20 *Discuss with your classmates how each of the formulas, if any, in Exercise* **??** *change if we change assume θ is a Quadrant II, III, or IV angle.*

Problem 21 *If $\sin(t) = x$ for $-\frac{\pi}{2} < t < \frac{\pi}{2}$, find an expression for $\tan(t)$ in terms of x.*

$\tan(t) = \dfrac{x}{\sqrt{1-x^2}}$

Problem 22 *If $\tan(\theta) = x$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, find an expression for $\sec(\theta)$ in terms of x.*

$\sec(\theta) = \sqrt{1+x^2}$

Problem 23 *If $\sec(\theta) = x$ where θ is a Quadrant II angle, find an expression for $\tan(\theta)$ in terms of x.*

$\tan(\theta) = -\sqrt{x^2-1}$

Problem 24 *If $\sin(t) = \frac{x}{2}$ for $-\frac{\pi}{2} < t < \frac{\pi}{2}$, find an expression for $\cos(2t)$ in terms of x.*

$\cos(2t) = 1 - \dfrac{x^2}{2}$

Problem 25 *If $\tan(\theta) = \frac{x}{7}$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, find an expression for $\sin(2\theta)$ in terms of x.*

$\sin(2\theta) = \dfrac{14x}{x^2+49}$

Problem 26 *If $\sec(t) = \frac{x}{4}$ for $0 < t < \frac{\pi}{2}$, find an expression for $\ln|\sec(t)+\tan(t)|$ in terms of x.*

$\ln|\sec(t)+\tan(t)| = \ln|x+\sqrt{x^2+16}| - \ln(4)$

Problem 27 *Show that $\cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$ for all θ.*

Problem 28 *Let θ be a Quadrant III angle with $\cos(\theta) = -\frac{1}{5}$. Show that this is not enough information to determine the sign of $\sin\left(\frac{\theta}{2}\right)$ by first assuming $3\pi < \theta < \frac{7\pi}{2}$ and then assuming $\pi < \theta < \frac{3\pi}{2}$ and computing $\sin\left(\frac{\theta}{2}\right)$ in both cases.*

Problem 29 *Without using your calculator, show that $\dfrac{\sqrt{2+\sqrt{3}}}{2} = \dfrac{\sqrt{6}+\sqrt{2}}{4}$*

Problem 30 *In part* **??** *of Example* **??***, we wrote $\cos(3\theta)$ as a polynomial in terms of $\cos(\theta)$. In Exercise* **??***, we had you verify an identity which expresses $\cos(4\theta)$ as a polynomial in terms of $\cos(\theta)$. Can you find a polynomial in terms of $\cos(\theta)$ for $\cos(5\theta)$? $\cos(6\theta)$? Can you find a pattern so that $\cos(n\theta)$ could be written as a polynomial in cosine for any natural number n?*

Problem 31 *In Exercise* **??**, *we has you verify an identity which expresses* $\sin(3\theta)$ *as a polynomial in terms of* $\sin(\theta)$. *Can you do the same for* $\sin(5\theta)$? *What about for* $\sin(4\theta)$? *If not, what goes wrong?*

Question 32 *In Exercises* **??** - **??**, *verify the identity by graphing the right and left hand using a graphing utility.*

Problem 32.1 $\sin^2(t) + \cos^2(t) = 1$

Problem 32.2 $\sec^2(x) - \tan^2(x) = 1$

Problem 32.3 $\cos(t) = \sin\left(\frac{\pi}{2} - t\right)$

Problem 32.4 $\tan(x + \pi) = \tan(x)$

Problem 32.5 $\sin(2t) = 2\sin(t)\cos(t)$

Problem 32.6 $\tan\left(\frac{x}{2}\right) = \frac{\sin(x)}{1 + \cos(x)}$

Question 33 *In Exercises* **??** - **??**, *write the given product as a sum. Note: you may need to use an Even/Odd Identity to match the answer provided.*

Problem 33.1 $\cos(3\theta)\cos(5\theta)$

$\frac{\cos(2\theta) + \cos(8\theta)}{2}$

Problem 33.2 $\sin(2t)\sin(7t)$

$\frac{\cos(5t) - \cos(9t)}{2}$

Problem 33.3 $\sin(9x)\cos(x)$

$\frac{\sin(8x) + \sin(10x)}{2}$

Problem 33.4 $\cos(2\theta)\cos(6\theta)$

$\frac{\cos(4\theta) + \cos(8\theta)}{2}$

Problem 33.5 $\sin(3t)\sin(2t)$

$\frac{\cos(t) - \cos(5t)}{2}$

Problem 33.6 $\cos(x)\sin(3x)$

$\frac{\sin(2x) + \sin(4x)}{2}$

Question 34 *In Exercises* **??** - **??**, *write the given sum as a product. Note: you may need to use an Even/Odd or Cofunction Identity to match the answer provided.*

Problem 34.1 $\cos(3\theta) + \cos(5\theta)$

$2\cos(4\theta)\cos(\theta)$

Problem 34.2 $\sin(2t) - \sin(7t)$

$-2\cos\left(\frac{9}{2}t\right)\sin\left(\frac{5}{2}t\right)$

Problem 34.3 $\cos(5x) - \cos(6x)$

$2\sin\left(\frac{11}{2}x\right)\sin\left(\frac{1}{2}x\right)$

Problem 34.4 $\sin(9x) - \sin(-x)$

$2\cos(4x)\sin(5x)$

Problem 34.5 $\sin(t) + \cos(t)$

$\sqrt{2}\cos\left(t - \frac{\pi}{4}\right)$

Problem 34.6 $\cos(x) - \sin(x)$

$-\sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$

Question 35 *In Exercises* **??** - **??**, *using the remarks following Example* **??** *on page* **??** *as a guide, rewrite the given function $f(t)$ as a product of sinusoids. Identify the functions which create the 'wave envelope.' Check your answer by graphing the function along with the 'wave-envelope' using a graphing utility.*

Problem 35.1 $f(t) = \cos(3t) + \cos(5t)$

$f(t) = [2\cos(t)]\cos(4t)$, $A(t) = 2\cos(t)$, *wave-envelope:* $y = \pm 2\cos(t)$

Problem 35.2 $f(t) = 3\cos(5t) - 3\cos(6t)$

$f(t) = \left[6\sin\left(\frac{1}{2}t\right)\right]\sin\left(\frac{11}{2}t\right)$, $A(t) = 6\sin\left(\frac{1}{2}t\right)$, *wave-envelope:* $y = \pm 6\sin\left(\frac{1}{2}t\right)$

Problem 35.3 $f(t) = \frac{1}{2}\sin(9t) + \frac{1}{2}\sin(t)$

$f(t) = [\cos(4t)]\sin(5t)$, $A(t) = \cos(4t)$, *wave-envelope:* $y = \pm\cos(4t)$

Problem 35.4 $f(t) = \frac{2}{3}\sin(2t) - \frac{2}{3}\sin(7t)$

$f(t) = \left[-\frac{4}{3}\sin\left(\frac{5}{2}t\right)\right]\cos\left(\frac{9}{2}t\right)$, $A(t) = -\frac{4}{3}\sin\left(\frac{5}{2}t\right)$, *wave-envelope:* $y = \pm\frac{4}{3}\sin\left(\frac{5}{2}t\right)$

Problem 36 *Verify the Even / Odd Identities for tangent, secant, cosecant and cotangent.*

Problem 37 *Verify the Cofunction Identities for tangent, secant, cosecant and cotangent.*

Problem 38 *Verify the Difference Identities for sine and tangent.*

Problem 39 *Verify the Product to Sum Identities.*

Problem 40 *Verify the Sum to Product Identities.*