More Identities

In Section **??**, we saw the utility of identities in finding the values of the circular functions of a given angle as well as simplifying expressions involving the circular functions. In this section, we introduce several collections of identities which have uses in this course and beyond.

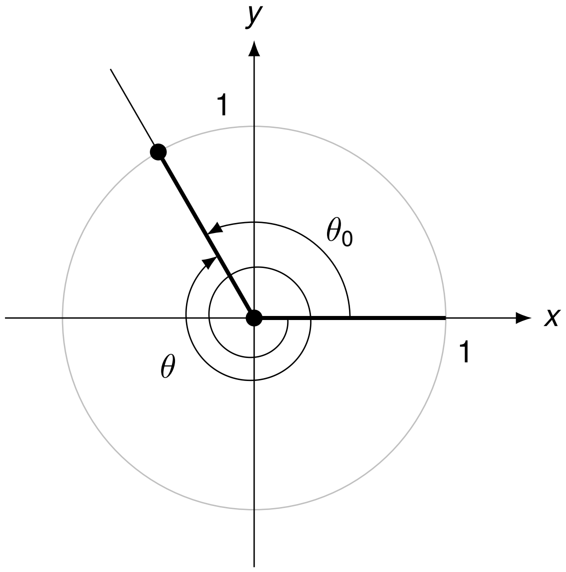
Our first set of identities is the 'Even / Odd' identities. We *observed* the even and odd properties of the circular functions graphically in Sections **??** and **??**. Here, we take the time to *prove* these properties from first principles. We state the theorem below for reference.

Theorem 1. Even / Odd Identities: For all applicable angles θ,

- $\cos(-\theta) = \cos(\theta)$
- $\sin(-\theta) = -\sin(\theta)$
- $\tan(-\theta) = -\tan(\theta)$
- $\sec(-\theta) = \sec(\theta)$
- $\csc(-\theta) = -\csc(\theta)$
- $\cot(-\theta) = -\cot(\theta)$

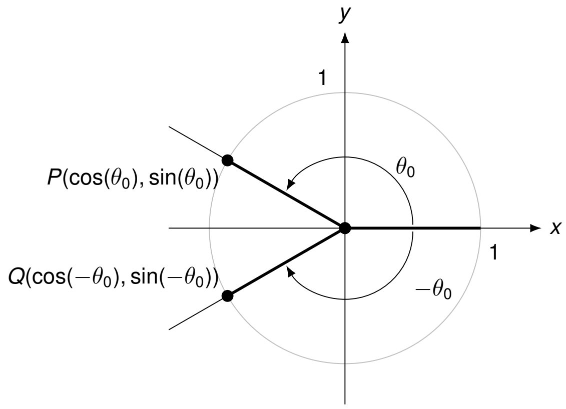
We start by proving $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$.

Consider an angle θ plotted in standard position. Let θ_0 be the angle coterminal with θ with $0 \leq \theta_0 < 2\pi$. (We can construct the angle θ_0 by rotating counter-clockwise from the positive x-axis to the terminal side of θ as pictured below.) Since θ and θ_0 are coterminal, $\cos(\theta) = \cos(\theta_0)$ and $\sin(\theta) = \sin(\theta_0)$.



We now consider the angles $-\theta$ and $-\theta_0$. Since θ is coterminal with θ_0, there is some integer k so that $\theta = \theta_0 + 2\pi \cdot k$. Hence, $-\theta = -\theta_0 - 2\pi \cdot k = -\theta_0 + 2\pi \cdot (-k)$. Since k is an integer, so is $(-k)$, which means $-\theta$ is coterminal with $-\theta_0$. Therefore, $\cos(-\theta) = \cos(-\theta_0)$ and $\sin(-\theta) = \sin(-\theta_0)$.

Let P and Q denote the points on the terminal sides of θ_0 and $-\theta_0$, respectively, which lie on the Unit Circle. By definition, the coordinates of P are $(\cos(\theta_0), \sin(\theta_0))$ and the coordinates of Q are $(\cos(-\theta_0), \sin(-\theta_0))$.



Since θ_0 and $-\theta_0$ sweep out congruent central sectors of the Unit Circle, it follows that the points P and Q are symmetric about the x-axis. Thus, $\cos(-\theta_0) = \cos(\theta_0)$ and $\sin(-\theta_0) = -\sin(\theta_0)$.

Since the cosines and sines of θ_0 and $-\theta_0$ are the same as those for θ and $-\theta$, respectively, we get $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$, as required.

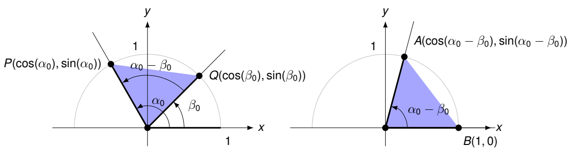
As we saw in Section **??**, the remaining four circular functions 'inherit' their even/odd nature from sine and cosine courtesy of the Reciprocal and Quotient Identities, Theorem **??**.

Our next set of identities establish how the cosine function handles sums and differences of angles.

Theorem 2. Sum and Difference Identities for Cosine: For all angles α and β,

- $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
- $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

We first prove the result for differences. As in the proof of the Even / Odd Identities, we can reduce the proof for general angles α and β to angles α_0 and β_0, coterminal with α and β, respectively, each of which measure between 0 and 2π radians. Since α and α_0 are coterminal, as are β and β_0, it follows that $(\alpha - \beta)$ is coterminal with $(\alpha_0 - \beta_0)$. Consider the case below where $\alpha_0 \geq \beta_0$.



Since the angles POQ and AOB are congruent, the distance between P and Q is equal to the distance between A and B.[1] The distance formula, Equation **??**, yields

$$\sqrt{(\cos(\alpha_0) - \cos(\beta_0))^2 + (\sin(\alpha_0) - \sin(\beta_0))^2} = \sqrt{(\cos(\alpha_0 - \beta_0) - 1)^2 + (\sin(\alpha_0 - \beta_0) - 0)^2}$$

Squaring both sides, we expand the left hand side of this equation as

$$\begin{aligned}(\cos(\alpha_0) - \cos(\beta_0))^2 + (\sin(\alpha_0) - \sin(\beta_0))^2 &= \cos^2(\alpha_0) - 2\cos(\alpha_0)\cos(\beta_0) + \cos^2(\beta_0) \\ &\quad + \sin^2(\alpha_0) - 2\sin(\alpha_0)\sin(\beta_0) + \sin^2(\beta_0) \\ &= \cos^2(\alpha_0) + \sin^2(\alpha_0) + \cos^2(\beta_0) + \sin^2(\beta_0) \\ &\quad -2\cos(\alpha_0)\cos(\beta_0) - 2\sin(\alpha_0)\sin(\beta_0)\end{aligned}$$

[1] In the picture we've drawn, the triangles POQ and AOB are congruent, which is even better. However, $\alpha_0 - \beta_0$ could be 0 or it could be π, neither of which makes a triangle. It could also be larger than π, which makes a triangle, just not the one we've drawn. You should think about those three cases.

From the Pythagorean Identities, $\cos^2(\alpha_0) + \sin^2(\alpha_0) = 1$ and $\cos^2(\beta_0) + \sin^2(\beta_0) = 1$, so

$$(\cos(\alpha_0) - \cos(\beta_0))^2 + (\sin(\alpha_0) - \sin(\beta_0))^2 \quad = \quad 2 - 2\cos(\alpha_0)\cos(\beta_0) - 2\sin(\alpha_0)\sin(\beta_0)$$

Turning our attention to the right hand side of our equation, we find

$$\begin{aligned}(\cos(\alpha_0 - \beta_0) - 1)^2 + (\sin(\alpha_0 - \beta_0) - 0)^2 &= \cos^2(\alpha_0 - \beta_0) - 2\cos(\alpha_0 - \beta_0) + 1 + \sin^2(\alpha_0 - \beta_0) \\ &= 1 + \cos^2(\alpha_0 - \beta_0) + \sin^2(\alpha_0 - \beta_0) - 2\cos(\alpha_0 - \beta_0)\end{aligned}$$

Once again, we simplify $\cos^2(\alpha_0 - \beta_0) + \sin^2(\alpha_0 - \beta_0) = 1$, so that

$$(\cos(\alpha_0 - \beta_0) - 1)^2 + (\sin(\alpha_0 - \beta_0) - 0)^2 \quad = \quad 2 - 2\cos(\alpha_0 - \beta_0)$$

Putting it all together, we get $2 - 2\cos(\alpha_0)\cos(\beta_0) - 2\sin(\alpha_0)\sin(\beta_0) = 2 - 2\cos(\alpha_0 - \beta_0)$, which simplifies to: $\cos(\alpha_0 - \beta_0) = \cos(\alpha_0)\cos(\beta_0) + \sin(\alpha_0)\sin(\beta_0)$.

Since α and α_0, β and β_0, and $(\alpha - \beta)$ and $(\alpha_0 - \beta_0)$ are all coterminal pairs of angles, we have established the identity: $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$.

For the case where $\alpha_0 \leq \beta_0$, we can apply the above argument to the angle $\beta_0 - \alpha_0$ to obtain the identity $\cos(\beta_0 - \alpha_0) = \cos(\beta_0)\cos(\alpha_0) + \sin(\beta_0)\sin(\alpha_0)$. Using this formula in conjunction with the Even Identity of cosine gives us the result in this case, too:

$$\begin{aligned}\cos(\alpha_0 - \beta_0) = \cos(-(\alpha_0 - \beta_0)) = \cos(\beta_0 - \alpha_0) &= \cos(\beta_0)\cos(\alpha_0) + \sin(\beta_0)\sin(\alpha_0) \\ &= \cos(\alpha_0)\cos(\beta_0) + \sin(\alpha_0)\sin(\beta_0).\end{aligned}$$

To get the sum identity for cosine, we use the difference formula along with the Even/Odd Identities

$$\cos(\alpha + \beta) = \cos(\alpha - (-\beta)) = \cos(\alpha)\cos(-\beta) + \sin(\alpha)\sin(-\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta).$$

We put these newfound identities to good use in the following example.

Example 1.

1. Find the exact value of $\cos\left(15^\circ\right)$.
2. Verify the identity: $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$.
3. Suppose α is a Quadrant I angle with $\sin(\alpha) = \frac{3}{5}$ and β is a Quadrant IV angle with $\sec(\beta) = 4$. Find the exact value of $\cos(\alpha + \beta)$.

Solution.

1. In order to use Theorem **??** to find $\cos\left(15^\circ\right)$, we need to write 15° as a sum or difference of angles whose cosines and sines we know. One way to do so is to write $15^\circ = 45^\circ - 30^\circ$. We find:

$$\begin{aligned}\cos\left(15^\circ\right) &= \cos\left(45^\circ - 30^\circ\right) \\ &= \cos\left(45^\circ\right)\cos\left(30^\circ\right) + \sin\left(45^\circ\right)\sin\left(30^\circ\right) \\ &= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) \\ &= \frac{\sqrt{6} + \sqrt{2}}{4}.\end{aligned}$$

2. Using Theorem **??** gives:

$$\begin{aligned}\cos\left(\frac{\pi}{2}-\theta\right) &= \cos\left(\frac{\pi}{2}\right)\cos(\theta)+\sin\left(\frac{\pi}{2}\right)\sin(\theta)\\ &= (0)(\cos(\theta))+(1)(\sin(\theta))\\ &= \sin(\theta).\end{aligned}$$

3. Per Theorem **??**, we know $\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$. Hence, we need to find the sines and cosines of α and β to complete the problem.

   We are given $\sin(\alpha)=\frac{3}{5}$, so our first task is to find $\cos(\alpha)$. We can quickly get $\cos(\alpha)$ using the Pythagorean Identity $\cos^2(\alpha)=1-\sin^2(\alpha)=1-\left(\frac{3}{5}\right)^2=\frac{16}{25}$. We get $\cos(\alpha)=\frac{4}{5}$, choosing the positive root since α is a Quadrant I angle.

   Next, we need the $\sin(\beta)$ and $\cos(\beta)$. Since $\sec(\beta)=4$, we immediately get $\cos(\beta)=\frac{1}{4}$ courtesy of the Reciprocal and Quotient Identities.

   To get $\sin(\beta)$, we employ the Pythagorean Identity: $\sin^2(\beta)=1-\cos^2(\beta)=1-\left(\frac{1}{4}\right)^2=\frac{15}{16}$. Here, since β is a Quadrant IV angle, we get $\sin(\beta)=-\frac{\sqrt{15}}{4}$.

   Finally, we get: $\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)=\left(\frac{4}{5}\right)\left(\frac{1}{4}\right)-\left(\frac{3}{5}\right)\left(-\frac{\sqrt{15}}{4}\right)=\frac{4+3\sqrt{15}}{20}$.

■

The identity verified in Example **??**, namely, $\cos\left(\frac{\pi}{2}-\theta\right)=\sin(\theta)$, is the first of the celebrated 'cofunction' identities. These identities were first hinted at in Exercise **??** in Section **??**.
From $\sin(\theta)=\cos\left(\frac{\pi}{2}-\theta\right)$, we get: $\sin\left(\frac{\pi}{2}-\theta\right)=\cos\left(\frac{\pi}{2}-\left[\frac{\pi}{2}-\theta\right]\right)=\cos(\theta)$, which says, in words, that the 'co'sine of an angle is the sine of its 'co'mplement. Now that these identities have been established for cosine and sine, the remaining circular functions follow suit. The remaining proofs are left as exercises.

Theorem 3. Cofunction Identities: For all applicable angles θ,

- $\cos\left(\frac{\pi}{2}-\theta\right)=\sin(\theta)$
- $\sin\left(\frac{\pi}{2}-\theta\right)=\cos(\theta)$
- $\sec\left(\frac{\pi}{2}-\theta\right)=\csc(\theta)$
- $\csc\left(\frac{\pi}{2}-\theta\right)=\sec(\theta)$
- $\tan\left(\frac{\pi}{2}-\theta\right)=\cot(\theta)$
- $\cot\left(\frac{\pi}{2}-\theta\right)=\tan(\theta)$

The Cofunction Identities enable us to derive the sum and difference formulas for sine. We first convert to sine to cosine and expand:

$$\begin{aligned}\sin(\alpha+\beta) &= \cos\left(\frac{\pi}{2}-(\alpha+\beta)\right)\\ &= \cos\left(\left[\frac{\pi}{2}-\alpha\right]-\beta\right)\\ &= \cos\left(\frac{\pi}{2}-\alpha\right)\cos(\beta)+\sin\left(\frac{\pi}{2}-\alpha\right)\sin(\beta)\\ &= \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)\end{aligned}$$

We can derive the difference formula for sine by rewriting $\sin(\alpha-\beta)$ as $\sin(\alpha+(-\beta))$ and using the sum formula and the Even / Odd Identities. Again, we leave the details to the reader.

Theorem 4. Sum and Difference Identities for Sine: For all angles α and β,

- $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$
- $\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$

We try out these new identities in the next example.

Example 2.

1. Find the exact value of $\sin\left(\dfrac{19\pi}{12}\right)$
2. Suppose α is a Quadrant II angle with $\sin(\alpha) = \dfrac{5}{13}$, and β is a Quadrant III angle with $\tan(\beta) = 2$. Find the exact value of $\sin(\alpha-\beta)$.
3. Derive a formula for $\tan(\alpha+\beta)$ in terms of $\tan(\alpha)$ and $\tan(\beta)$.

Solution.

1. As in Example **??**, we need to write the angle $\dfrac{19\pi}{12}$ as a sum or difference of common angles. The denominator of 12 suggests a combination of angles with denominators 3 and 4. One such combination[2] is $\dfrac{19\pi}{12} = \dfrac{4\pi}{3} + \dfrac{\pi}{4}$. Applying Theorem **??**, we get
$$\begin{aligned} \sin\left(\frac{19\pi}{12}\right) &= \sin\left(\frac{4\pi}{3}+\frac{\pi}{4}\right) \\ &= \sin\left(\frac{4\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{4\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) \\ &= \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \\ &= \frac{-\sqrt{6}-\sqrt{2}}{4} \end{aligned}$$
2. In order to find $\sin(\alpha-\beta)$ using Theorem **??**, we need to find $\cos(\alpha)$ and both $\cos(\beta)$ and $\sin(\beta)$.

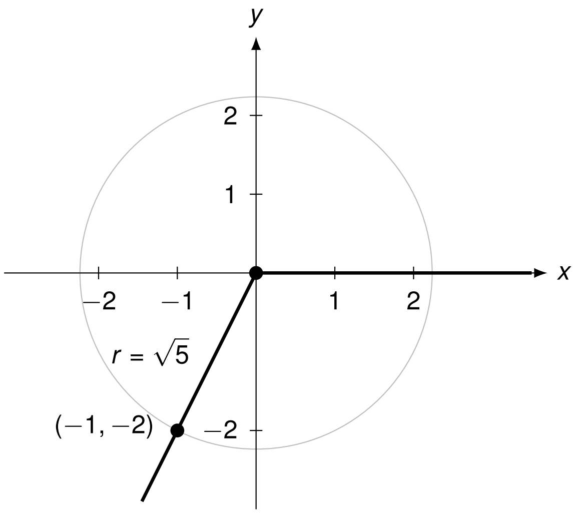
   To find $\cos(\alpha)$, we use the Pythagorean Identity $\cos^2(\alpha) = 1 - \sin^2(\alpha) = 1 - \left(\dfrac{5}{13}\right)^2 = \dfrac{144}{169}$. We get $\cos(\alpha) = -\dfrac{12}{13}$, the negative, here, owing to the fact that α is a Quadrant II angle.

   We now set about finding $\sin(\beta)$ and $\cos(\beta)$. We have several ways to proceed at this point, but since there isn't a direct way to get from $\tan(\beta) = 2$ to either $\sin(\beta)$ or $\cos(\beta)$, we opt for a more geometric approach as presented in Section **??**.

   Since β is a Quadrant III angle with $\tan(\beta) = 2 = \dfrac{-2}{-1}$, we know the point $Q(x,y) = (-1,-2)$ is on the terminal side of β as illustrated below.[3]

[2] It takes some trial and error to find this combination. One alternative is to convert to degrees . . .

[3] Note that even though $\tan(\beta) = \dfrac{\sin(\beta)}{\cos(\beta)}$, we *cannot* take $\sin(\beta) = -2$ and $\cos(\beta) = -1$. Recall that $\sin(\beta)$ and $\cos(\beta)$ are the y and x coordinates on a *specific* circle, the Unit Circle. As we'll see shortly, $(-1,-2)$ lies on a circle of $\sqrt{5}$, so *not* the Unit Circle.



We find $r = \sqrt{x^2+y^2} = \sqrt{(-1)^2+(-2)^2} = \sqrt{5}$, so per Theorem **??**, $\sin(\beta) = \dfrac{-2}{\sqrt{5}} = -\dfrac{2\sqrt{5}}{5}$ and $\cos(\beta) = \dfrac{-1}{\sqrt{5}} = -\dfrac{\sqrt{5}}{5}$.

At last, we have $\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) = \left(\dfrac{5}{13}\right)\left(-\dfrac{\sqrt{5}}{5}\right) - \left(-\dfrac{12}{13}\right)\left(-\dfrac{2\sqrt{5}}{5}\right) = -\dfrac{29\sqrt{5}}{65}$.

3. We can start expanding $\tan(\alpha+\beta)$ using a quotient identity and our sum formulas

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} \\ &= \frac{\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)}\end{aligned}$$

Since $\tan(\alpha) = \dfrac{\sin(\alpha)}{\cos(\alpha)}$ and $\tan(\beta) = \dfrac{\sin(\beta)}{\cos(\beta)}$, it looks as though if we divide both numerator and denominator by $\cos(\alpha)\cos(\beta)$ we will have what we want

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)} \cdot \frac{\frac{1}{\cos(\alpha)\cos(\beta)}}{\frac{1}{\cos(\alpha)\cos(\beta)}} \\ &= \frac{\frac{\sin(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)} + \frac{\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}}{\frac{\cos(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)} - \frac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}} \\ &= \frac{\frac{\sin(\alpha)\cancel{\cos(\beta)}}{\cos(\alpha)\cancel{\cos(\beta)}} + \frac{\cancel{\cos(\alpha)}\sin(\beta)}{\cancel{\cos(\alpha)}\cos(\beta)}}{\frac{\cancel{\cos(\alpha)}\cancel{\cos(\beta)}}{\cancel{\cos(\alpha)}\cancel{\cos(\beta)}} - \frac{\sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}} \\ &= \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}\end{aligned}$$

Naturally, this formula is limited to those cases where all of the tangents are defined. ■

The formula developed in Exercise **??** for $\tan(\alpha+\beta)$ can be used to find a formula for $\tan(\alpha-\beta)$ by rewriting the difference as a sum, $\tan(\alpha+(-\beta))$ and using the odd property of tangent. (The reader is encouraged to fill in the details.) Below we summarize all of the sum and difference formulas.

More Identities

Theorem 5. Sum and Difference Identities: For all applicable angles α and β,

- $\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$
- $\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$
- $\tan(\alpha \pm \beta) = \dfrac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}$

In the statement of Theorem **??**, we have combined the cases for the sum '+' and difference '−' of angles into one formula. The convention here is that if you want the formula for the sum '+' of two angles, you use the top sign in the formula; for the difference, '−', use the bottom sign. For example,

$$\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$$

If we set $\alpha = \beta$ in the sum formulas in Theorem **??**, we obtain the following 'Double Angle' Identities:

Theorem 6. Double Angle Identities: For all applicable angles θ,

- $\cos(2\theta) = \begin{cases} \cos^2(\theta) - \sin^2(\theta) \\ 2\cos^2(\theta) - 1 \\ 1 - 2\sin^2(\theta) \end{cases}$
- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $\tan(2\theta) = \dfrac{2\tan(\theta)}{1 - \tan^2(\theta)}$

The three different forms for $\cos(2\theta)$ can be explained by our ability to 'exchange' squares of cosine and sine via the Pythagorean Identity. For instance, if we substitute $\sin^2(\theta) = 1 - \cos^2(\theta)$ into the first formula for $\cos(2\theta)$, we get $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = \cos^2(\theta) - (1 - \cos^2(\theta)) = 2\cos^2(\theta) - 1$.

It is interesting to note that to determine the value of $\cos(2\theta)$, only *one* piece of information is required: either $\cos(\theta)$ or $\sin(\theta)$. To determine $\sin(2\theta)$, however, it appears that we must know both $\sin(\theta)$ and $\cos(\theta)$. In the next example, we show how we can find $\sin(2\theta)$ knowing just one piece of information, namely $\tan(\theta)$.

Example 3.

1. Suppose $P(-3, 4)$ lies on the terminal side of θ when θ is plotted in standard position.

   Find $\cos(2\theta)$ and $\sin(2\theta)$ and determine the quadrant in which the terminal side of the angle 2θ lies when it is plotted in standard position.
2. If $\sin(\theta) = x$ for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, find an expression for $\sin(2\theta)$ in terms of x.
3. Verify the identity: $\sin(2\theta) = \dfrac{2\tan(\theta)}{1 + \tan^2(\theta)}$.
4. Express $\cos(3\theta)$ as a polynomial in terms of $\cos(\theta)$.

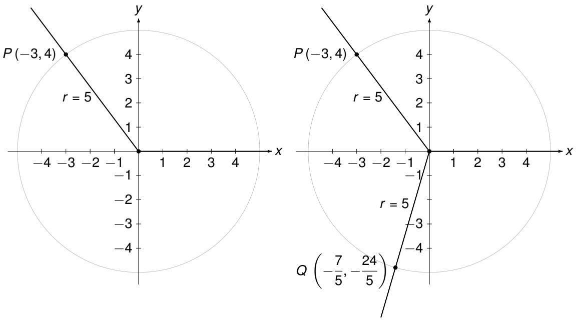
Solution.

1. We sketch the terminal side of θ below on the left. Using Theorem **??** from Section **??** with $x = -3$ and $y = 4$, we find $r = \sqrt{x^2 + y^2} = 5$. Hence, $\cos(\theta) = -\frac{3}{5}$ and $\sin(\theta) = \frac{4}{5}$.

Theorem **??** gives us three different formulas to choose from to find $\cos(2\theta)$. Using the first formula,[4] we get: $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = \left(-\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2 = -\frac{7}{25}$. For $\sin(2\theta)$, we get $\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right) = -\frac{24}{25}$.

Since both cosine and sine of 2θ are negative, the terminal side of 2θ, when plotted in standard position, lies in Quadrant III. To see this more clearly, we plot the terminal side of 2θ, along with the terminal side of θ below on the right.

Note that in order to find the point $Q(x, y)$ on the terminal side of 2θ of a circle of radius 5, we use Theorem **??** again and find $x = r\cos(2\theta) = 5\left(-\frac{7}{25}\right) = -\frac{7}{5}$ and $y = r\sin(2\theta) = 5\left(-\frac{24}{25}\right) = -\frac{24}{5}$.



2. If your first reaction to '$\sin(\theta) = x$' is 'No it's not, $\cos(\theta) = x$!' then you have indeed learned something, and we take comfort in that.

   While we have mostly used 'x' to represent the x-coordinate of the point the terminal side of an angle θ, here, 'x' represents the quantity $\sin(\theta)$ and our task is to express $\sin(2\theta)$ in terms of x.

   Since $\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2x\cos(\theta)$, what remains is to express $\cos(\theta)$ in terms of x.

   Substituting $\sin(\theta) = x$ into the Pythagorean Identity, we get $\cos^2(\theta) = 1 - \sin^2(\theta) = 1 - x^2$, or $\cos(\theta) = \pm\sqrt{1 - x^2}$. Since $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $\cos(\theta) \geq 0$, and thus $\cos(\theta) = \sqrt{1 - x^2}$.

   Our final answer is $\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2x\sqrt{1 - x^2}$.

3. We start with the right hand side of the identity and note that $1 + \tan^2(\theta) = \sec^2(\theta)$. Next, we use the Reciprocal and Quotient Identities to rewrite $\tan(\theta)$ and $\sec(\theta)$ in terms of $\sin(\theta)$ and $\cos(\theta)$:

$$\begin{aligned}
\frac{2\tan(\theta)}{1+\tan^2(\theta)} &= \frac{2\tan(\theta)}{\sec^2(\theta)} = \frac{2\left(\frac{\sin(\theta)}{\cos(\theta)}\right)}{\frac{1}{\cos^2(\theta)}} = 2\left(\frac{\sin(\theta)}{\cos(\theta)}\right)\cos^2(\theta) \\
&= 2\left(\frac{\sin(\theta)}{\cancel{\cos(\theta)}}\right)\cancel{\cos(\theta)}\cos(\theta) = 2\sin(\theta)\cos(\theta) = \sin(2\theta).
\end{aligned}$$

4. In Theorem **??**, one of the formulas for $\cos(2\theta)$, namely $\cos(2\theta) = 2\cos^2(\theta) - 1$, expresses $\cos(2\theta)$ as a polynomial in terms of $\cos(\theta)$. We are now asked to find such an identity for $\cos(3\theta)$.

   Using the sum formula for cosine, we begin with

$$\begin{aligned}
\cos(3\theta) &= \cos(2\theta + \theta) \\
&= \cos(2\theta)\cos(\theta) - \sin(2\theta)\sin(\theta).
\end{aligned}$$

   Our ultimate goal is to express the right hand side in terms of $\cos(\theta)$ only. To that end, we substitute $\cos(2\theta) = 2\cos^2(\theta) - 1$ and $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ which yields:

[4]We invite the reader to check this answer using the other two formulas.

$$\begin{aligned}\cos(3\theta) &= \cos(2\theta)\cos(\theta) - \sin(2\theta)\sin(\theta)\\ &= \left(2\cos^2(\theta) - 1\right)\cos(\theta) - (2\sin(\theta)\cos(\theta))\sin(\theta)\\ &= 2\cos^3(\theta) - \cos(\theta) - 2\sin^2(\theta)\cos(\theta)\end{aligned}$$

Finally, we exchange $\sin^2(\theta) = 1 - \cos^2(\theta)$ courtesy of the Pythagorean Identity, and get

$$\begin{aligned}\cos(3\theta) &= 2\cos^3(\theta) - \cos(\theta) - 2\sin^2(\theta)\cos(\theta)\\ &= 2\cos^3(\theta) - \cos(\theta) - 2\left(1 - \cos^2(\theta)\right)\cos(\theta)\\ &= 2\cos^3(\theta) - \cos(\theta) - 2\cos(\theta) + 2\cos^3(\theta)\\ &= 4\cos^3(\theta) - 3\cos(\theta).\end{aligned}$$

Hence, $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$. ■

In the last problem in Example **??**, we saw how we could rewrite $\cos(3\theta)$ as sums of powers of $\cos(\theta)$. In Calculus, we have occasion to do the reverse; that is, reduce the power of cosine and sine.

Solving the identity $\cos(2\theta) = 2\cos^2(\theta) - 1$ for $\cos^2(\theta)$ and the identity $\cos(2\theta) = 1 - 2\sin^2(\theta)$ for $\sin^2(\theta)$ results in the aptly-named 'Power Reduction' formulas below.

Theorem 7. Power Reduction Formulas: For all angles θ,

- $\cos^2(\theta) = \dfrac{1 + \cos(2\theta)}{2}$
- $\sin^2(\theta) = \dfrac{1 - \cos(2\theta)}{2}$

Our next example is a typical application of Theorem **??** that you'll likely see in Calculus.

Example 4. Rewrite $\sin^2(\theta)\cos^2(\theta)$ as a sum and difference of cosines to the first power.

Solution. We begin with a straightforward application of Theorem **??**

$$\begin{aligned}\sin^2(\theta)\cos^2(\theta) &= \left(\frac{1 - \cos(2\theta)}{2}\right)\left(\frac{1 + \cos(2\theta)}{2}\right)\\ &= \frac{1}{4}\left(1 - \cos^2(2\theta)\right)\\ &= \frac{1}{4} - \frac{1}{4}\cos^2(2\theta)\end{aligned}$$

Next, we apply the power reduction formula to $\cos^2(2\theta)$ to finish the reduction

$$\begin{aligned}\sin^2(\theta)\cos^2(\theta) &= \frac{1}{4} - \frac{1}{4}\cos^2(2\theta)\\ &= \frac{1}{4} - \frac{1}{4}\left(\frac{1 + \cos(2(2\theta))}{2}\right)\\ &= \frac{1}{4} - \frac{1}{8} - \frac{1}{8}\cos(4\theta)\\ &= \frac{1}{8} - \frac{1}{8}\cos(4\theta)\end{aligned}$$

■

Another application of the Power Reduction Formulas is the Half Angle Formulas. To start, we apply the Power Reduction Formula to $\cos^2\left(\dfrac{\theta}{2}\right)$

$$\cos^2\left(\frac{\theta}{2}\right) = \frac{1 + \cos\left(2\left(\frac{\theta}{2}\right)\right)}{2} = \frac{1 + \cos(\theta)}{2}.$$

We can obtain a formula for $\cos\left(\frac{\theta}{2}\right)$ by extracting square roots. In a similar fashion, we may obtain a half angle formula for sine, and by using a quotient formula, obtain a half angle formula for tangent.
We summarize these formulas below.

Theorem 8. Half Angle Formulas: For all applicable angles θ,

- $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}$
- $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$
- $\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}}$

where the choice of $\pm$ depends on the quadrant in which the terminal side of $\frac{\theta}{2}$ lies.

Example 5.

1. Use a half angle formula to find the exact value of $\cos\left(15^\circ\right)$.
2. Suppose $-\pi \leq t \leq 0$ with $\cos(t) = -\frac{3}{5}$. Find $\sin\left(\frac{t}{2}\right)$.
3. Use the identity given in number **??** of Example **??** to derive the identity

$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin(\theta)}{1+\cos(\theta)}$$

Solution.

1. To use the half angle formula, we note that $15^\circ = \dfrac{30^\circ}{2}$ and since 15° is a Quadrant I angle, its cosine is positive. Thus we have

$$\begin{aligned} \cos\left(15^\circ\right) &= +\sqrt{\frac{1+\cos\left(30^\circ\right)}{2}} = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} \\ &= \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}\cdot\frac{2}{2}} = \sqrt{\frac{2+\sqrt{3}}{4}} = \frac{\sqrt{2+\sqrt{3}}}{2} \end{aligned}$$

   Back in Example **??**, we found $\cos\left(15^\circ\right) = \dfrac{\sqrt{6}+\sqrt{2}}{4}$ by using the difference formula for cosine. The reader is encouraged to prove that these two expressions are equal algebraically.

2. If $-\pi \leq t \leq 0$, then $-\dfrac{\pi}{2} \leq \dfrac{t}{2} \leq 0$, which means $\dfrac{t}{2}$ corresponds to a Quadrant IV angle. Hence, $\sin\left(\dfrac{t}{2}\right) < 0$, so we choose the negative root formula from Theorem **??**:

$$\begin{aligned} \sin\left(\frac{t}{2}\right) &= -\sqrt{\frac{1-\cos(t)}{2}} = -\sqrt{\frac{1-\left(-\frac{3}{5}\right)}{2}} \\ &= -\sqrt{\frac{1+\frac{3}{5}}{2}\cdot\frac{5}{5}} = -\sqrt{\frac{8}{10}} = -\frac{2\sqrt{5}}{5} \end{aligned}$$

3. Instead of our usual approach to verifying identities, namely starting with one side of the equation and trying to transform it into the other, we will start with the identity we proved in number **??** of Example **??** and manipulate it into the identity we are asked to prove.

   The identity we are asked to start with is $\sin(2\theta) = \dfrac{2\tan(\theta)}{1+\tan^2(\theta)}$. If we are to use this to derive an identity for $\tan\left(\dfrac{\theta}{2}\right)$, it seems reasonable to proceed by replacing each occurrence of θ with $\dfrac{\theta}{2}$

$$\begin{aligned} \sin\left(2\left(\frac{\theta}{2}\right)\right) &= \frac{2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)} \\ \sin(\theta) &= \frac{2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)} \end{aligned}$$

   We now have the $\sin(\theta)$ we need, but we somehow need to get a factor of $1+\cos(\theta)$ involved. We substitute $1+\tan^2\left(\dfrac{\theta}{2}\right) = \sec^2\left(\dfrac{\theta}{2}\right)$, and continue to manipulate our given identity by converting secants to cosines.

$$\begin{aligned} \sin(\theta) &= \frac{2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)} \\ \sin(\theta) &= \frac{2\tan\left(\frac{\theta}{2}\right)}{\sec^2\left(\frac{\theta}{2}\right)} \\ \sin(\theta) &= 2\tan\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right) \end{aligned}$$

   Finally, we apply a power reduction formula, and then solve for $\tan\left(\dfrac{\theta}{2}\right)$

$$\begin{aligned}
\sin(\theta) &= 2\tan\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right) \\
\sin(\theta) &= 2\tan\left(\frac{\theta}{2}\right)\left(\frac{1+\cos\left(2\left(\frac{\theta}{2}\right)\right)}{2}\right) \\
\sin(\theta) &= \tan\left(\frac{\theta}{2}\right)(1+\cos(\theta)) \\
\tan\left(\frac{\theta}{2}\right) &= \frac{\sin(\theta)}{1+\cos(\theta)}
\end{aligned}$$

■

Our next batch of identities, the Product to Sum Formulas,[5] are easily verified by expanding each of the right hand sides in accordance with Theorem **??** and as you should expect by now we leave the details as exercises. They are of particular use in Calculus, and we list them here for reference.

Theorem 9. Product to Sum Formulas: For all angles α and β,

- $\cos(\alpha)\cos(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$
- $\sin(\alpha)\sin(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$
- $\sin(\alpha)\cos(\beta) = \frac{1}{2}\left[\sin(\alpha-\beta)+\sin(\alpha+\beta)\right]$

Related to the Product to Sum Formulas are the Sum to Product Formulas, which we will have need of in Section **??**. These are essentially restatements of the Product to Sum Formulas (by re-labeling the arguments of the sine and cosine functions) and as such, their proofs are left as exercises.

Theorem 10. Sum to Product Formulas: For all angles α and β,

- $\cos(\alpha)+\cos(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$
- $\cos(\alpha)-\cos(\beta) = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$
- $\sin(\alpha)\pm\sin(\beta) = 2\sin\left(\frac{\alpha\pm\beta}{2}\right)\cos\left(\frac{\alpha\mp\beta}{2}\right)$

Example 6.

1. Write $\cos(2\theta)\cos(6\theta)$ as a sum.
2. Write $\sin(\theta)-\sin(3\theta)$ as a product.

Solution.

1. Identifying $\alpha = 2\theta$ and $\beta = 6\theta$, we find

$$\begin{aligned}
\cos(2\theta)\cos(6\theta) &= \frac{1}{2}\left[\cos(2\theta-6\theta)+\cos(2\theta+6\theta)\right] \\
&= \frac{1}{2}\cos(-4\theta)+\frac{1}{2}\cos(8\theta) \\
&= \frac{1}{2}\cos(4\theta)+\frac{1}{2}\cos(8\theta),
\end{aligned}$$

   where the last equality is courtesy of the even identity for cosine, $\cos(-4\theta) = \cos(4\theta)$.

[5] These are also known as the Prosthaphaeresis Formulas and have a rich history. The authors recommend that you conduct some research on them as your schedule allows.

2. Identifying $\alpha = \theta$ and $\beta = 3\theta$ yields

$$\begin{aligned} \sin(\theta) - \sin(3\theta) &= 2\sin\left(\frac{\theta - 3\theta}{2}\right)\cos\left(\frac{\theta + 3\theta}{2}\right) \\ &= 2\sin(-\theta)\cos(2\theta) \\ &= -2\sin(\theta)\cos(2\theta), \end{aligned}$$

   where the last equality is courtesy of the odd identity for sine, $\sin(-\theta) = -\sin(\theta)$. ■

The reader is reminded that all of the identities presented in this section which regard the circular functions as functions of angles (in radian measure) apply equally well to the circular (trigonometric) functions regarded as functions of real numbers.

1 Sinusoids, Revisited

We first studied sinusoids in Section **??**. Using the sum formulas for sine and cosine, we can expand the forms given to us in Theorem **??**:

$$S(t) = A\sin(\omega t + \phi) + B = A\sin(\omega t)\cos(\phi) + A\cos(\omega t)\sin(\phi) + B,$$

and

$$C(t) = A\cos(\omega t + \phi) + B = A\cos(\omega t)\cos(\phi) - A\sin(\omega t)\sin(\phi) + B.$$

As we'll see in the next example, recognizing these 'expanded' forms of sinusoids allows us to graph functions as sinusoids which, at first glance, don't appear to fit the forms of either $C(t)$ or $S(t)$.

Example 7. Consider the function $f(t) = \cos(2t) - \sqrt{3}\sin(2t)$. Find a formula for $f(t)$:

1. in the form $C(t) = A\cos(\omega t + \phi) + B$ for $\omega > 0$
2. in the form $S(t) = A\sin(\omega t + \phi) + B$ for $\omega > 0$

Check your answers analytically using identities and using a graphing utility.

Solution.

1. The key to this problem is to use the expanded forms of the sinusoid formulas and match up corresponding coefficients. We start by equating $f(t) = \cos(2t) - \sqrt{3}\sin(2t)$ with the expanded form of $C(t) = A\cos(\omega t + \phi) + B$: $\cos(2t) - \sqrt{3}\sin(2t) = A\cos(\omega t)\cos(\phi) - A\sin(\omega t)\sin(\phi) + B$.

   If we take $\omega = 2$ and $B = 0$, we get: $\cos(2t) - \sqrt{3}\sin(2t) = A\cos(2t)\cos(\phi) - A\sin(2t)\sin(\phi)$.

   To determine A and ϕ, a bit more work is involved. We get started by equating the coefficients of the trigonometric functions on either side of the equation.

   On the left hand side, the coefficient of $\cos(2t)$ is 1, while on the right hand side, it is $A\cos(\phi)$. Since this equation is to hold for all real numbers, we must have[6] that $A\cos(\phi) = 1$.

   Similarly, we find by equating the coefficients of $\sin(2t)$ that $A\sin(\phi) = \sqrt{3}$. In conjunction with $A\cos(\phi) = 1$, we have a system of two (nonlinear) equations and two unknowns.

   As usual, our first task is to reduce this system of two equations and two unknowns to one equation and one unknown. We can temporarily eliminate the dependence on ϕ by using a Pythagorean Identity. From $\cos^2(\phi) + \sin^2(\phi) = 1$, we multiply through by A^2 to get $A^2\cos^2(\phi) + A^2\sin^2(\phi) = A^2$.

   In our case, $A\cos(\phi) = 1$ and $A\sin(\phi) = \sqrt{3}$, hence $A^2 = A^2\cos^2(\phi) + A^2\sin^2(\phi) = 1^2 + (\sqrt{3})^2 = 4$ so $A = \pm 2$. In much the same way we fit a sinusoid to a graph in Example **??**, we *choose* $A = 2$, and then find the phase angle ϕ associated with this choice.

   Substituting $A = 2$ into our two equations, $A\cos(\phi) = 1$ and $A\sin(\phi) = \sqrt{3}$, we get $2\cos(\phi) = 1$ and $2\sin(\phi) = \sqrt{3}$. After some rearrangement, $\cos(\phi) = \frac{1}{2}$ and $\sin(\phi) = \frac{\sqrt{3}}{2}$. One such angle ϕ which satisfies this criteria is $\phi = \frac{\pi}{3}$.

   Hence, one way to write $f(t)$ as a sinusoid is $f(t) = 2\cos\left(2t + \frac{\pi}{3}\right)$. We can check our answer using the sum formula for cosine :

$$\begin{aligned} f(t) &= 2\cos\left(2t + \frac{\pi}{3}\right) \\ &= 2\left[\cos(2t)\cos\left(\frac{\pi}{3}\right) - \sin(2t)\sin\left(\frac{\pi}{3}\right)\right] \\ &= 2\left[\cos(2t)\left(\frac{1}{2}\right) - \sin(2t)\left(\frac{\sqrt{3}}{2}\right)\right] \\ &= \cos(2t) - \sqrt{3}\sin(2t). \end{aligned}$$

2. Proceeding as before, we equate $f(t) = \cos(2t) - \sqrt{3}\sin(2t)$ with the expanded form of of the sinusoid $S(t) = A\sin(\omega t + \phi) + B$ to get: $\cos(2t) - \sqrt{3}\sin(2t) = A\sin(\omega t)\cos(\phi) + A\cos(\omega t)\sin(\phi) + B$.

   Taking $\omega = 2$ and $B = 0$, we get $\cos(2t) - \sqrt{3}\sin(2t) = A\sin(2t)\cos(\phi) + A\cos(2t)\sin(\phi)$. We equate[7] the coefficients of $\cos(2t)$ on either side and get $A\sin(\phi) = 1$ and $A\cos(\phi) = -\sqrt{3}$.

   Using $A^2\cos^2(\phi) + A^2\sin^2(\phi) = A^2$ as before, we get $A = \pm 2$, and again we choose $A = 2$.

   This means $2\sin(\phi) = 1$, or $\sin(\phi) = \frac{1}{2}$, and $2\cos(\phi) = -\sqrt{3}$, so $\cos(\phi) = -\frac{\sqrt{3}}{2}$. One such angle which meets these criteria is $\phi = \frac{5\pi}{6}$.

[6] This should remind you of equation coefficients of like powers of x in Section **??**.

[7] Be careful here!

Hence, we have $f(t) = 2\sin\left(2t + \frac{5\pi}{6}\right)$. Checking our work analytically, we have

$$\begin{aligned} f(t) &= 2\sin\left(2t + \frac{5\pi}{6}\right) \\ &= 2\left[\sin(2t)\cos\left(\frac{5\pi}{6}\right) + \cos(2t)\sin\left(\frac{5\pi}{6}\right)\right] \\ &= 2\left[\sin(2t)\left(-\frac{\sqrt{3}}{2}\right) + \cos(2t)\left(\frac{1}{2}\right)\right] \\ &= \cos(2t) - \sqrt{3}\sin(2t) \end{aligned}$$

Graphing the three formulas for $f(t)$ result in the identical curve, verifying the work done analytically.

Desmos link: `https://www.desmos.com/calculator/tjttnvbp11`

A couple of remarks about Example **??** are in order. First, had we chosen $A = -2$ instead of $A = 2$ as we worked through Example **??**, our final answers would have *looked* different. The reader is encouraged to rework Example **??** using $A = -2$ to see what these differences are, and then for a challenging exercise, use identities to show that the formulas are all equivalent.[8]

It is important to note that in order for the technique presented in Example **??** to fit a function into one of the forms in Theorem **??**, the *frequencies* of the sine and cosine terms much match. For example, in the Exercises, you'll be asked to write $f(t) = 3\sqrt{3}\sin(3t) - 3\cos(3t)$ in the form of $S(t)$ and $C(t)$ above, and since both the sine and cosine terms have frequency 3, this is possible.

However, a function such as $f(t) = \sin(t) - \sin(3t)$ cannot be written in the form of $S(t)$ or $C(t)$. The quickest way to see this is to examine its graph below which is decidedly not a sinusoid. That being said, we can still analyze this curve using identities.

Desmos link: `https://www.desmos.com/calculator/vfggmdtn69`

Using our result from number **??** Example **??**, we may rewrite $f(t) = \sin(t) - \sin(3t) = -2\sin(t)\cos(2t)$. Grouping factors, we can view $f(t) = [-2\sin(t)]\cos(2t) = A(t)\cos(2t)$ as the curve $y = \cos(2t)$ with a *variable* amplitude, $A(t) = -2\sin(t)$.

Overlaying the graphs of $f(t)$ with the (dashed) graphs of $A_1(t) = 2\sin(t)$ and $A_2(t) = -2\sin(t)$, we can see the role these two curves play in the graph of $y = f(t)$. They create a kind of 'wave envelope' for the graph of $y = f(t)$. This is an example of the beats phenomenon. Note that when written as a product of sinusoids, it is always the *lower* frequency factor which creates the 'wave-envelope' of the curve.

Note that in order to rewrite a sum or difference of sine and cosine functions with different frequencies into a product using the sum to product identities, Theorem **??**, we need the *amplitudes* of each term to be the same. We explore more examples of these functions and this behavior in the Exercises.

[8]The general equations to fit a function of the form $f(x) = a\cos(\omega x) + b\sin(\omega x) + B$ into one of the forms in Theorem **??** are explored in Exercise **??**.