

EXERCISES FOR APPLICATIONS OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Question 1 For each of the scenarios given in Exercises ?? - ??,

- Find the amount A in the account as a function of the term of the investment t in years.
- To the nearest cent, determine how much is in the account after 5, 10, 30 and 35 years.
- To the nearest year, determine how long will it take for the initial investment to double.
- Find and interpret the average rate of change of the amount in the account from the end of the fourth year to the end of the fifth year, and from the end of the thirty-fourth year to the end of the thirty-fifth year. Round your answer to two decimal places.

Problem 1.1 \$500 is invested in an account which offers 0.75%, compounded monthly.

$$A(t) = 500 \left(1 + \frac{0.0075}{12} \right)^{12t}$$

$$A(5) \approx \$519.10, A(10) \approx \$538.93, A(30) \approx \$626.12, A(35) \approx \$650.03$$

It will take approximately 92 years for the investment to double.

The average rate of change from the end of the fourth year to the end of the fifth year is approximately 3.88. This means that the investment is growing at an average rate of \$3.88 per year at this point. The average rate of change from the end of the thirty-fourth year to the end of the thirty-fifth year is approximately 4.85. This means that the investment is growing at an average rate of \$4.85 per year at this point.

Problem 1.2 \$500 is invested in an account which offers 0.75%, compounded continuously.

$$A(t) = 500e^{0.0075t}$$

$$A(5) \approx \$519.11, A(10) \approx \$538.94, A(30) \approx \$626.16, A(35) \approx \$650.09$$

It will take approximately 92 years for the investment to double.

The average rate of change from the end of the fourth year to the end of the fifth year is approximately 3.88. This means that the investment is growing at an average rate of \$3.88 per year at this point. The average rate of change from the end of the thirty-fourth year to the end of the thirty-fifth year is approximately 4.86. This means that the investment is growing at an average rate of \$4.86 per year at this point.

Problem 1.3 \$1000 is invested in an account which offers 1.25%, compounded monthly.

$$A(t) = 1000 \left(1 + \frac{0.0125}{12} \right)^{12t}$$

$$A(5) \approx \$1064.46, A(10) \approx \$1133.07, A(30) \approx \$1454.71, A(35) \approx \$1548.48$$

It will take approximately 55 years for the investment to double.

The average rate of change from the end of the fourth year to the end of the fifth year is approximately 13.22. This means that the investment is growing at an average rate of \$13.22 per year at this point. The average rate of change from the end of the thirty-fourth year to the end of the thirty-fifth year is approximately 19.23. This means that the investment is growing at an average rate of \$19.23 per year at this point.

Problem 1.4 \$1000 is invested in an account which offers 1.25%, compounded continuously.

$$A(t) = 1000e^{0.0125t}$$

$$A(5) \approx \$1064.49, A(10) \approx \$1133.15, A(30) \approx \$1454.99, A(35) \approx \$1548.83$$

It will take approximately 55 years for the investment to double.

The average rate of change from the end of the fourth year to the end of the fifth year is approximately 13.22. This means that the investment is growing at an average rate of \$13.22 per year at this point. The average rate of change from the end of the thirty-fourth year to the end of the thirty-fifth year is approximately 19.24. This means that the investment is growing at an average rate of \$19.24 per year at this point.

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Problem 1.5 \$5000 is invested in an account which offers 2.125%, compounded monthly.

$$A(t) = 5000 \left(1 + \frac{0.02125}{12} \right)^{12t}$$

$$A(5) \approx \$5559.98, A(10) \approx \$6182.67, A(30) \approx \$9453.40, A(35) \approx \$10512.13$$

It will take approximately 33 years for the investment to double.

The average rate of change from the end of the fourth year to the end of the fifth year is approximately 116.80. This means that the investment is growing at an average rate of \$116.80 per year at this point. The average rate of change from the end of the thirty-fourth year to the end of the thirty-fifth year is approximately 220.83. This means that the investment is growing at an average rate of \$220.83 per year at this point.

Problem 1.6 \$5000 is invested in an account which offers 2.125%, compounded continuously.

$$A(t) = 5000e^{0.02125t}$$

$$A(5) \approx \$5560.50, A(10) \approx \$6183.83, A(30) \approx \$9458.73, A(35) \approx \$10519.05$$

It will take approximately 33 years for the investment to double.

The average rate of change from the end of the fourth year to the end of the fifth year is approximately 116.91. This means that the investment is growing at an average rate of \$116.91 per year at this point. The average rate of change from the end of the thirty-fourth year to the end of the thirty-fifth year is approximately 221.17. This means that the investment is growing at an average rate of \$221.17 per year at this point.

Problem 2 Look back at your answers to Exercises ?? - ?? . What can be said about the difference between monthly compounding and continuously compounding the interest in those situations? With the help of your classmates, discuss scenarios where the difference between monthly and continuously compounded interest would be more dramatic. Try varying the interest rate, the term of the investment and the principal. Use computations to support your answer.

Problem 3 How much money needs to be invested now to obtain \$2000 in 3 years if the interest rate in a savings account is 0.25%, compounded continuously? Round your answer to the nearest cent.

$$P = \frac{2000}{e^{0.0025 \cdot 3}} \approx \$1985.06$$

Problem 4 How much money needs to be invested now to obtain \$5000 in 10 years if the interest rate in a CD is 2.25%, compounded monthly? Round your answer to the nearest cent.

$$P = \frac{5000}{\left(1 + \frac{0.0225}{12} \right)^{12 \cdot 10}} \approx \$3993.42$$

Problem 5 On May, 31, 2009, the Annual Percentage Rate listed at Jeff's bank for regular savings accounts was 0.25% compounded monthly. Use Equation ?? to answer the following.

- If $P = 2000$ what is $A(8)$?

$$A(8) = 2000 \left(1 + \frac{0.0025}{12} \right)^{12 \cdot 8} \approx \$2040.40$$

- Solve the equation $A(t) = 4000$ for t .

$$t = \frac{\ln(2)}{12 \ln \left(1 + \frac{0.0025}{12} \right)} \approx 277.29 \text{ years}$$

- What principal P should be invested so that the account balance is \$2000 in three years?

$$P = \frac{2000}{\left(1 + \frac{0.0025}{12} \right)^{36}} \approx \$1985.06$$

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Problem 6 Jeff's bank also offers a 36-month Certificate of Deposit (CD) with an APR of 2.25%.

- If $P = 2000$ what is $A(8)$?

$$A(8) = 2000 \left(1 + \frac{0.0225}{12} \right)^{12 \cdot 8} \approx \$2394.03$$

- Solve the equation $A(t) = 4000$ for t .

$$t = \frac{\ln(2)}{12 \ln \left(1 + \frac{0.0225}{12} \right)} \approx 30.83 \text{ years}$$

- What principal P should be invested so that the account balance is \$2000 in three years?

$$P = \frac{2000}{\left(1 + \frac{0.0225}{12} \right)^{36}} \approx \$1869.57$$

- The Annual Percentage Yield is the simple interest rate that returns the same amount of interest after one year as the compound interest does. With the help of your classmates, compute the APY for this investment.

$$\left(1 + \frac{0.0225}{12} \right)^{12} \approx 1.0227 \text{ so the APY is } 2.27\%$$

Problem 7 A finance company offers a promotion on \$5000 loans. The borrower does not have to make any payments for the first three years, however interest will continue to be charged to the loan at 29.9% compounded continuously. What amount will be due at the end of the three year period, assuming no payments are made? If the promotion is extended an additional three years, and no payments are made, what amount would be due?

$$A(3) = 5000e^{0.299 \cdot 3} \approx \$12,226.18, A(6) = 5000e^{0.299 \cdot 6} \approx \$30,067.29$$

Problem 8 Use Equation ?? to show that the time it takes for an investment to double in value does not depend on the principal P , but rather, depends only on the APR and the number of compoundings per year. Let $n = 12$ and with the help of your classmates compute the doubling time for a variety of rates r . Then look up the Rule of 72 and compare your answers to what that rule says. If you're really interested¹ in Financial Mathematics, you could also compare and contrast the Rule of 72 with the Rule of 70 and the Rule of 69.

Question 9 In Exercises ?? - ??, we list some radioactive isotopes and their associated half-lives. Assume that each decays according to the formula $A(t) = A_0 e^{kt}$ where A_0 is the initial amount of the material and k is the decay constant. For each isotope:

- Find the decay constant k . Round your answer to four decimal places.
- Find a function which gives the amount of isotope A which remains after time t . (Keep the units of A and t the same as the given data.)
- Determine how long it takes for 90% of the material to decay. Round your answer to two decimal places. (HINT: If 90% of the material decays, how much is left?)

Problem 9.1 Cobalt 60, used in food irradiation, initial amount 50 grams, half-life of 5.27 years.

$$k = \frac{\ln(1/2)}{5.27} \approx -0.1315$$

$$A(t) = 50e^{-0.1315t}$$

$$t = \frac{\ln(0.1)}{-0.1315} \approx 17.51 \text{ years.}$$

¹ Awesome pun!

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Problem 9.2 Phosphorus 32, used in agriculture, initial amount 2 milligrams, half-life 14 days.

$$k = \frac{\ln(1/2)}{14} \approx -0.0495$$

$$A(t) = 2e^{-0.0495t}$$

$$t = \frac{\ln(0.1)}{-0.0495} \approx 46.52 \text{ days.}$$

Problem 9.3 Chromium 51, used to track red blood cells, initial amount 75 milligrams, half-life 27.7 days.

$$k = \frac{\ln(1/2)}{27.7} \approx -0.0250$$

$$A(t) = 75e^{-0.0250t}$$

$$t = \frac{\ln(0.1)}{-0.025} \approx 92.10 \text{ days.}$$

Problem 9.4 Americium 241, used in smoke detectors, initial amount 0.29 micrograms, half-life 432.7 years.

$$k = \frac{\ln(1/2)}{432.7} \approx -0.0016$$

$$A(t) = 0.29e^{-0.0016t}$$

$$t = \frac{\ln(0.1)}{-0.0016} \approx 1439.11 \text{ years.}$$

Problem 9.5 Uranium 235, used for nuclear power, initial amount 1 kg, half-life 704 million years.

$$k = \frac{\ln(1/2)}{704} \approx -0.0009846$$

$$A(t) = e^{-0.0009846t}$$

$$t = \frac{\ln(0.1)}{-0.0009846} \approx 2338.60 \text{ million years, or 2.339 billion years.}$$

Problem 10 With the help of your classmates, show that the time it takes for 90% of each isotope listed in Exercises ?? - ?? to decay does not depend on the initial amount of the substance, but rather, on only the decay constant k . Find a formula, in terms of k only, to determine how long it takes for 90% of a radioactive isotope to decay.

Problem 11 In Example ?? in Section ??, the exponential function $V(x) = 25 \left(\frac{4}{5}\right)^x$ was used to model the value of a car over time. Use a change of base formula to rewrite the model in the form $V(t) = 25e^{kt}$.

$$V(t) = 25e^{\ln(\frac{4}{5})t} \approx 25e^{-0.22314355t}$$

Problem 12 The Gross Domestic Product (GDP) of the US (in billions of dollars) t years after the year 2000 can be modeled by:

$$G(t) = 9743.77e^{0.0514t}$$

- Find and interpret $G(0)$.

$G(0) = 9743.77$ This means that the GDP of the US in 2000 was \$9743.77 billion dollars.

- According to the model, what should have been the GDP in 2007? In 2010? (According to the [US Department of Commerce](#), the 2007 GDP was \$14,369.1 billion and the 2010 GDP was \$14,657.8 billion.)

$G(7) = 13963.24$ and $G(10) = 16291.25$, so the model predicted a GDP of \$13,963.24 billion in 2007 and \$16,291.25 billion in 2010.

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Problem 13 The diameter D of a tumor, in millimeters, t days after it is detected is given by:

$$D(t) = 15e^{0.0277t}$$

- What was the diameter of the tumor when it was originally detected?
 $D(0) = 15$, so the tumor was 15 millimeters in diameter when it was first detected.
- How long until the diameter of the tumor doubles?
 $t = \frac{\ln(2)}{0.0277} \approx 25$ days.

Problem 14 Under optimal conditions, the growth of a certain strain of *E. Coli* is modeled by the Law of Uninhibited Growth $N(t) = N_0 e^{kt}$ where N_0 is the initial number of bacteria and t is the elapsed time, measured in minutes. From numerous experiments, it has been determined that the doubling time of this organism is 20 minutes. Suppose 1000 bacteria are present initially.

- Find the growth constant k . Round your answer to four decimal places.
 $k = \frac{\ln(2)}{20} \approx 0.0346$
- Find a function which gives the number of bacteria $N(t)$ after t minutes.
 $N(t) = 1000e^{0.0346t}$
- How long until there are 9000 bacteria? Round your answer to the nearest minute.
 $t = \frac{\ln(9)}{0.0346} \approx 63$ minutes

Problem 15 Yeast is often used in biological experiments. A research technician estimates that a sample of yeast suspension contains 2.5 million organisms per cubic centimeter (cc). Two hours later, she estimates the population density to be 6 million organisms per cc. Let t be the time elapsed since the first observation, measured in hours. Assume that the yeast growth follows the Law of Uninhibited Growth $N(t) = N_0 e^{kt}$.

- Find the growth constant k . Round your answer to four decimal places.
 $k = \frac{1}{2} \frac{\ln(6)}{2.5} \approx 0.4377$
- Find a function which gives the number of yeast (in millions) per cc $N(t)$ after t hours.
 $N(t) = 2.5e^{0.4377t}$
- What is the doubling time for this strain of yeast?
 $t = \frac{\ln(2)}{0.4377} \approx 1.58$ hours

Problem 16 The Law of Uninhibited Growth also applies to situations where an animal is re-introduced into a suitable environment. Such a case is the reintroduction of wolves to Yellowstone National Park. According to the [National Park Service](#), the wolf population in Yellowstone National Park was 52 in 1996 and 118 in 1999. Using these data, find a function of the form $N(t) = N_0 e^{kt}$ which models the number of wolves t years after 1996. (Use $t = 0$ to represent the year 1996. Also, round your value of k to four decimal places.) According to the model, how many wolves were in Yellowstone in 2002? (The recorded number is 272.)

$$N_0 = 52, k = \frac{1}{3} \ln\left(\frac{118}{52}\right) \approx 0.2731, N(t) = 52e^{0.2731t}. N(6) \approx 268.$$

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Problem 17 During the early years of a community, it is not uncommon for the population to grow according to the Law of Uninhibited Growth. According to the Painesville Wikipedia entry, in 1860, the Village of Painesville had a population of 2649. In 1920, the population was 7272. Use these two data points to fit a model of the form $N(t) = N_0 e^{kt}$ where $N(t)$ is the number of Painesville Residents t years after 1860. (Use $t = 0$ to represent the year 1860. Also, round the value of k to four decimal places.) According to this model, what was the population of Painesville in 2010? (The 2010 census gave the population as 19,563) What could be some causes for such a vast discrepancy? For more on this, see Exercise ??.

$N_0 = 2649$, $k = \frac{1}{60} \ln \left(\frac{7272}{2649} \right) \approx 0.0168$, $N(t) = 2649e^{0.0168t}$. $N(150) \approx 32923$, so the population of Painesville in 2010 based on this model would have been 32,923.

Problem 18 The population of Sasquatch in Bigfoot county is modeled by

$$P(t) = \frac{120}{1 + 3.167e^{-0.05t}}$$

where $P(t)$ is the population of Sasquatch t years after 2010.

- Find and interpret $P(0)$.

$$P(0) = \frac{120}{4.167} \approx 29. \text{ There are 29 Sasquatch in Bigfoot County in 2010.}$$

- Find the population of Sasquatch in Bigfoot county in 2013 rounded to the nearest Sasquatch.

$$P(3) = \frac{120}{1 + 3.167e^{-0.05(3)}} \approx 32 \text{ Sasquatch.}$$

- To the nearest year, when will the population of Sasquatch in Bigfoot county reach 60?

$$t = 20 \ln(3.167) \approx 23 \text{ years.}$$

- Find and interpret $\lim_{t \rightarrow \infty} P(t)$ analytically. Check your answer using a graphing utility.

We find $\lim_{t \rightarrow \infty} P(t) = 120$. As time goes by, the Sasquatch Population in Bigfoot County will approach 120. Graphically, $y = P(x)$ has a horizontal asymptote $y = 120$.

Problem 19 Let $f(x) = \frac{10}{1 + e^{-x+1}}$.

- From Calculus, we know the inflection point of the graph of $y = f(x)$ is $(1, 5)$. This means the function is increasing the fastest at $x = 1$, or, equivalently, the slope at $(1, 5)$ is the largest anywhere on the graph. Graph $y = f(x)$ using a graphing utility and convince yourself of the reasonableness of this claim.
- Find average rate of change of f over each of the intervals below. What do you guess the slope of the curve is at $(1, 5)$? Zoom in on the graph near $(1, 5)$ to check your guess.

$$[0.75, 1] \approx 2.487$$

$$[0.9, 1] \approx 2.498$$

$$[0.99, 1] \approx 2.500$$

$$[1, 1.01] \approx 2.500$$

$$[1, 1.1] \approx 2.498$$

$$[1, 1.25] \approx 2.487$$

The average rates of change are listed in order above. They suggest slope at $(1, 5)$ is 2.5.

Problem 20 The half-life of the radioactive isotope Carbon-14 is about 5730 years.

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- Use Equation ?? to express the amount of Carbon-14 left from an initial N milligrams as a function of time t in years.

$$A(t) = Ne^{-\left(\frac{\ln(2)}{5730}\right)t} \approx Ne^{-0.00012097t}$$

- What percentage of the original amount of Carbon-14 is left after 20,000 years?
 $A(20000) \approx 0.088978 \cdot N$ so about 8.9% remains
- If an old wooden tool is found in a cave and the amount of Carbon-14 present in it is estimated to be only 42% of the original amount, approximately how old is the tool?

$$t \approx \frac{\ln(.42)}{-0.00012097} \approx 7171 \text{ years old}$$

- Radiocarbon dating is not as easy as these exercises might lead you to believe. With the help of your classmates, research radiocarbon dating and discuss why our model is somewhat over-simplified.

Problem 21 Carbon-14 cannot be used to date inorganic material such as rocks, but there are many other methods of radiometric dating which estimate the age of rocks. One of them, Rubidium-Strontium dating, uses Rubidium-87 which decays to Strontium-87 with a half-life of 50 billion years. Use Equation ?? to express the amount of Rubidium-87 left from an initial 2.3 micrograms as a function of time t in billions of years. Research this and other radiometric techniques and discuss the margins of error for various methods with your classmates.

$$A(t) = 2.3e^{-0.0138629t}$$

Problem 22 Find and interpret the relative rate of change of $A(t)$ in Equation ?? over the interval $\left[t, t + \frac{1}{n}\right]$.

The relative rate of change of $A(t)$ over $\left[t, t + \frac{1}{n}\right]$ is $\frac{r}{n}$ which is the annual percentage rate divided by the number of compoundings per year – that is, the percentage growth rate over one compounding.

Problem 23 Use Equation ?? to show that $k = -\frac{\ln(2)}{h}$ where h is the half-life of the radioactive isotope.

Problem 24 A pork roast² was taken out of a hardwood smoker when its internal temperature had reached 180°F and it was allowed to rest in a 75°F house for 20 minutes after which its internal temperature had dropped to 170°F . Assuming that the temperature of the roast follows Newton's Law of Cooling (Equation ??),

- Express the temperature T (in $^\circ\text{F}$) as a function of time t (in minutes).
 $T(t) = 75 + 105e^{-0.005005t}$
- Find the time at which the roast would have dropped to 140°F had it not been eaten.
The roast would have cooled to 140°F in about 95 minutes.

Problem 25 In reference to Exercise ?? in Section ??, if Fritzy the Fox's speed is the same as Chewbacca the Bunny's speed, Fritzy's pursuit curve is given by

$$y(x) = \frac{1}{4}x^2 - \frac{1}{4}\ln(x) - \frac{1}{4}$$

Graph this path for $x > 0$ using a graphing utility. Investigate $\lim_{x \rightarrow 0^+} y(x)$ and interpret.

From the graph, it appears that $\lim_{x \rightarrow 0^+} y(x) = \infty$. This is due to the presence of the $\ln(x)$ term in the function. This means that Fritzy will never catch Chewbacca, which makes sense since Chewbacca has a head start and Fritzy only runs as fast as he does.

²This roast was enjoyed by Jeff and his family on June 10, 2009. This is real data, folks!

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Problem 26 The current i measured in amps in a certain electronic circuit with a constant impressed voltage of 120 volts is given by $i(t) = 2 - 2e^{-10t}$ where $t \geq 0$ is the number of seconds after the circuit is switched on. Determine $\lim_{t \rightarrow \infty} i(t)$. (This is called the **steady state** current.)

The steady state current is 2 amps.

Problem 27 If the voltage in the circuit in Exercise ?? above is switched off after 30 seconds, the current is given by the piecewise-defined function

$$i(t) = \begin{cases} 2 - 2e^{-10t} & \text{if } 0 \leq t < 30 \\ (2 - 2e^{-300})e^{-10(t+300)} & \text{if } t \geq 30 \end{cases}$$

With the help of a graphing utility, graph $y = i(t)$ and discuss with your classmates the physical significance of the two parts of the graph $0 \leq t < 30$ and $t \geq 30$.

Problem 28 In Exercise ?? in Section ??, we stated that the cable of a suspension bridge formed a parabola but that a free hanging cable did not. A free hanging cable forms a catenary and its basic shape is given by $y = \frac{1}{2}(e^x + e^{-x})$.

- Use a graphing utility to graph this function. What are its domain and range? What is its end behavior? Is it invertible? How do you think it is related to the function given in Exercise ?? in Section ?? and the one given in the answer to Exercise ?? in Section ???
- When flipped upside down, the catenary makes an arch. The Gateway Arch in St. Louis, Missouri has the shape

$$y = 757.7 - \frac{127.7}{2} \left(e^{\frac{x}{127.7}} + e^{-\frac{x}{127.7}} \right)$$

where x and y are measured in feet and $-315 \leq x \leq 315$. Find the highest point on the arch.
630 feet

Problem 29 In Exercise ?? in Section ??, we examined the data set given below which showed how two cats and their surviving offspring can produce over 80 million cats in just ten years.

- Plot x versus $\ln(x)$ as was done on page ?? using a graphing utility.
- Find a linear model for this new data and comment on its goodness of fit and find an exponential model for the original data and comment on its goodness of fit.

Year x	1	2	3	4	5	6	7	8	9	10
Number of Cats $N(x)$	12	66	382	2201	12680	73041	420715	2423316	13968290	80399780

The linear regression on the data below is $y = 1.74899x + 0.70739$ with $r^2 \approx 0.999995$.

This is an excellent fit.

x	1	2	3	4	5	6	7	8	9	10
$\ln(N(x))$	2.4849	4.1897	5.9454	7.6967	9.4478	11.1988	12.9497	14.7006	16.4523	18.2025

$N(x) = 2.02869(5.74879)^x = 2.02869e^{1.74899x}$ with $r^2 \approx 0.999995$. This is also an excellent fit and corresponds to our linearized model because $\ln(2.02869) \approx 0.70739$.

Problem 30 In Example ?? in Section ??, we fit a power function of the form $L(x) = ax^p$ to a set of data, $(x, L(x))$. In this exercise, we use logs to linearize this data using the same methods presented on page ??, but with a slight difference in interpretation.

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- Starting with $L(x) = ax^p$, take natural logs of both sides of the equation and use log properties to rewrite the resulting equation as: $\ln(L(x)) = p \ln(x) + \ln(a)$.
- Use a graphing utility to find a least squares regression line using the data $(\ln(x), \ln(L(x)))$.
NOTE: In this situation, we are plotting $\ln(x)$ versus $\ln(L(x))$ instead of x versus $\ln(L(x))$.
The linearized model is: $\ln(L(x)) \approx 2.106 \ln(x) - 5.268$ with an $r^2 \approx 0.9914$.
- Find the slope p of the regression line and the intercept $\ln(a)$. Use these to construct a model of the form $L(x) = ax^p$. Find and interpret $L(90)$.
 $L(x) = 0.005154x^{2.106}$. $L(90) \approx 67.3$ meaning the bottom 90% of wage earners take home 67.3% of the total national income. Said differently, according to this model, the top 10% of wage earners take home 32.7% of the total national income.
- Graph both the model obtained in Example ?? and the model obtained in part ?? along with the original data. What do you notice?

Desmos link: <https://www.desmos.com/calculator/9vupqszxtq>

Problem 31 This exercise is a follow-up to Exercise ?? which more thoroughly explores the population growth of Painesville, Ohio. According to [Wikipedia](#), the population of Painesville, Ohio is given by

Year t	1860	1870	1880	1890	1900	1910	1920	1930	1940	1950
Population	2649	3728	3841	4755	5024	5501	7272	10944	12235	14432

Year t	1960	1970	1980	1990	2000
Population	16116	16536	16351	15699	17503

- Use a graphing utility to perform an exponential regression on the data from 1860 through 1920 only, letting $t = 0$ represent the year 1860 as before. How does this model compare with the model you found in Exercise ??? Use the graphing utility's exponential model to predict the population in 2010. (The 2010 census gave the population as 19,563)
We get: $y = 2895.06(1.0147)^x$. Graphing this along with our answer from Exercise ?? over the interval $[0, 60]$ shows that they are pretty close. From this model, $y(150) \approx 25840$ which once again overshoots the actual data value.
- The logistic model fit to all of the given data points for the population of Painesville t years after 1860 (again, using $t = 0$ as 1860) is

$$P(t) = \frac{18691}{1 + 9.8505e^{-0.03617t}}$$

According to this model, what should the population of Painesville have been in 2010? (The 2010 census gave the population as 19,563.) What is the population limit of Painesville?

$P(150) \approx 17914$, so this model predicts 17,914 people in Painesville in 2010, a more conservative number than was recorded in the 2010 census. We have $\lim_{t \rightarrow \infty} P(t) = 18691$, so the limiting population of Painesville based on this model is 18,691 people.

Problem 32 According to [OhioBiz](#), the census data for Lake County, Ohio is as follows:

Year t	1860	1870	1880	1890	1900	1910	1920	1930	1940	1950
Population	15576	15935	16326	18235	21680	22927	28667	41674	50020	75979

Year t	1960	1970	1980	1990	2000
Population	148700	197200	212801	215499	227511

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- Use a graphing utility to fit a logistic model to these data with $x = 0$ representing the year 1860.

$$y = \frac{242526}{1 + 874.63e^{-0.07113x}}, \text{ where } x \text{ is the number of years since 1860.}$$

- Graph the data and your model using a graphing utility to judge the reasonableness of the fit.

Desmos link: <https://www.desmos.com/calculator/jae0y1pdxw>

- Use this model to estimate the population of Lake County in 2010. (The 2010 census gave the population to be 230,041.)

$y(140) \approx 232884$, so this model predicts 232,884 people in Lake County in 2010.

- According to your model, what is the population limit of Lake County, Ohio?

We get $\lim_{x \rightarrow \infty} y = 242526$, so the limiting population of Lake County based on this model is 242,526 people.

Problem 33 According to [facebook](#), the number of active users of facebook has grown significantly since its initial launch from a Harvard dorm room in February 2004. The chart below has the approximate number $U(x)$ of active users, in millions, x months after February 2004. For example, the first entry (10, 1) means that there were 1 million active users in December 2004 and the last entry (77, 500) means that there were 500 million active users in July 2010.

Month x	10	22	34	38	44	54	59	60	62	65	67	70	72	77
Active Users in Millions $U(x)$	1	5.5	12	20	50	100	150	175	200	250	300	350	400	500

With the help of your classmates, find a model for this data.

Problem 34 Each Monday during the registration period before the Fall Semester at LCCC, the Enrollment Planning Council gets a report prepared by the data analysts in Institutional Effectiveness and Planning.³ While the ongoing enrollment data is analyzed in many different ways, we shall focus only on the overall headcount. Below is a chart of the enrollment data for Fall Semester 2008. It starts 21 weeks before "Opening Day" and ends on "Day 15" of the semester, but we have relabeled the top row to be $x = 1$ through $x = 24$ so that the math is easier. (Thus, $x = 22$ is Opening Day.)

Week x	1	2	3	4	5	6	7	8
Total Headcount	1194	1564	2001	2475	2802	3141	3527	3790

Week x	9	10	11	12	13	14	15	16
Total Headcount	4065	4371	4611	4945	5300	5657	6056	6478

Week x	17	18	19	20	21	22	23	24
Total Headcount	7161	7772	8505	9256	10201	10743	11102	11181

With the help of your classmates, find a model for this data. Unlike most of the phenomena we have studied in this section, there is no single differential equation which governs the enrollment growth. Thus there is no scientific reason to rely on a logistic function even though the data plot may lead us to that model. What are some factors which influence enrollment at a community college and how can you take those into account mathematically?

³Thanks to Dr. Wendy Marley and her staff for this data and Dr. Marcia Ballinger for the permission to use it in this problem.

EXERCISES FOR APPLICATIONS OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Problem 35 When we wrote this exercise, the Enrollment Planning Report for Fall Semester 2009 had only 10 data points for the first 10 weeks of the registration period. Those numbers are given below.

Week x	1	2	3	4	5	6	7	8	9	10
Total Headcount	1380	2000	2639	3153	3499	3831	4283	4742	5123	5398

With the help of your classmates, find a model for this data and make a prediction for the Opening Day enrollment as well as the Day 15 enrollment. (WARNING: The registration period for 2009 was one week shorter than it was in 2008 so Opening Day would be $x = 21$ and Day 15 is $x = 23$.)