EXERCISES FOR EXPONENTIAL FUNCTIONS

Question 1 *In Exercises* **??** *-* **??***, sketch the graph of g by starting with the graph of f and using transformations. Track at least three points of your choice and the horizontal asymptote through the transformations. State the domain and range of g.*

Problem 1.1 $f(x) = 2^x$, $g(x) = 2^x - 1$

Use the Desmos graph below with the settings $f(x) = 2^x,\ a = 1,\ b = 1,\ h = 0,\ k = -1$

Desmos link: https://www.desmos.com/calculator/zr6jvgonqk

domain: $(-\infty, \infty)$

range: $(-1, \infty)$

Problem 1.2 $f(x) = \left(\frac{1}{3}\right)^x$, $g(x) = \left(\frac{1}{3}\right)^{x-1}$

Use the Desmos graph below with the settings $f(x) = \left(\frac{1}{3}\right)^x,\ a = 1,\ b = 1,\ h = 1,\ k = 0$

Desmos link: https://www.desmos.com/calculator/qyxiza2f7s

domain: $(-\infty, \infty)$

range: $(0, \infty)$

Problem 1.3 $f(x) = 3^x$, $g(x) = 3^{-x} + 2$

Use the Desmos graph below with the settings $f(x) = 3^x,\ h = 0,\ k = 2,\ a = 1,\ b = -1$

Desmos link: https://www.desmos.com/calculator/jq10j4p3ng

domain: $(-\infty, \infty)$

range: $(2, \infty)$

Problem 1.4 $f(x) = 10^x$, $g(x) = 10^{\frac{x+1}{2}} - 20$

Use the Desmos graph below with the settings $f(x) = 10^x,\ a = 1,\ b = \frac{1}{2},\ h = -\frac{1}{2},\ k = -20$

Desmos link: https://www.desmos.com/calculator/wrlj0hxwll

domain: $(-\infty, \infty)$

range: $(-20, \infty)$

Problem 1.5 $f(t) = (0.5)^t$, $g(t) = 100(0.5)^{0.1t}$

Use the Desmos graph below with the settings $f(t) = 0.5^t,\ a = 100,\ b = 0.1,\ h = 0,\ k = 0$

Desmos link: https://www.desmos.com/calculator/sklxycmgqa

domain: $(-\infty, \infty)$

range: $(0, \infty)$

CC BY-NC-SA 4.0

Problem 1.6 $f(t) = (1.25)^t$, $g(t) = 1 - (1.25)^{t-2}$
Use the Desmos graph below with the settings $f(t) = 1.25^t,\ a = -1,\ b = 1,\ h = 2,\ k = 1$

Desmos link: https://www.desmos.com/calculator/u38r5x258e

domain: $(-\infty, \infty)$

range: $(-\infty, 1)$

Problem 1.7 $f(t) = e^t$, $g(t) = 8 - e^{-t}$
Use the Desmos graph below with the settings $f(t) = e^t,\ h = 0,\ k = 8,\ a = -1,\ b = -1$

Desmos link: https://www.desmos.com/calculator/cawqhkpowm

domain: $(-\infty, \infty)$

range: $(-\infty, 8)$

Problem 1.8 $f(t) = e^t$, $g(t) = 10e^{-0.1t}$
Use the Desmos graph below with the settings $f(t) = e^t,\ h = 0,\ k = 0,\ a = 10,\ b = -0.1$

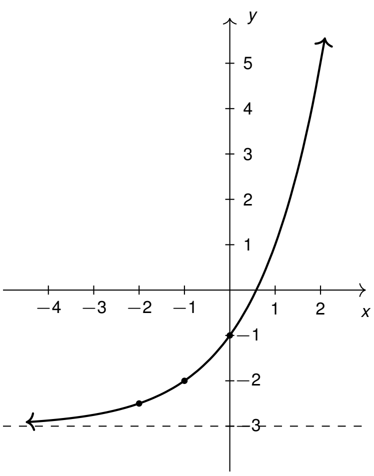
Desmos link: https://www.desmos.com/calculator/ccnzvlqqye

domain: $(-\infty, \infty)$

range: $(0, \infty)$

Question 2 *In Exercises,* **??** - **??***, the graph of an exponential function is given. Find a formula for the function in the form* $F(x) = a \cdot 2^{bx-h} + k$.

Problem 2.1 *Points:* $\left(-2, -\dfrac{5}{2}\right)$, $(-1, -2)$, $(0, -1)$,
Asymptote: $y = -3$.



Use the Desmos graph below with the settings $f(x) = 2^x,\ a = 1,\ b = 1,\ h = -1,\ k = -3$

Desmos link: https://www.desmos.com/calculator/r24nw6gmxp

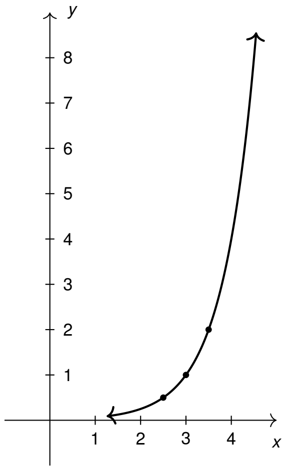
CC BY-NC-SA 4.0

Problem 2.2 *Points:* $(-1,1)$, $(0,2)$, $\left(1,\dfrac{5}{2}\right)$, *Asymptote:* $y=3$.

Use the Desmos graph below with the settings $f(x)=2^x,\ a=-1,\ b=-1,\ h=0,\ k=3$

Desmos link: https://www.desmos.com/calculator/d7emrkstf9

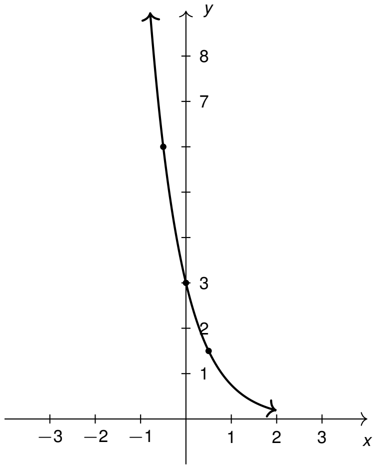
Problem 2.3 *Points:* $\left(\dfrac{5}{2},\dfrac{1}{2}\right)$, $(3,1)$, $\left(\dfrac{7}{2},2\right)$,
Asymptote: $y=0$.



Use the Desmos graph below with the settings $f(x)=2^x,\ a=1,\ b=2,\ h=6,\ k=0$

Desmos link: https://www.desmos.com/calculator/oqffuc1nxc

Problem 2.4 *Points:* $\left(-\dfrac{1}{2},6\right)$, $(0,3)$, $\left(\dfrac{1}{2},\dfrac{3}{2}\right)$,
Asymptote: $y=0$.



Use the Desmos graph below with the settings $f(x)=2^x,\ a=3,\ b=-2,\ h=0,\ k=0$

Desmos link: https://www.desmos.com/calculator/qu4tanz7o2

Problem 3 *Find a formula for each graph in Exercises* **??** - **??** *of the form* $G(x)=a\cdot 4^{bx-h}+k$*. Did you change your solution methodology? What is the relationship between your answers for* $F(x)$ *and* $G(x)$ *for each graph?*

CC BY-NC-SA 4.0

You can use the same Desmos links in the solutions for Exercises **??** - **??** *to check your formulas for G(x).*

Since $2 = 4^{\frac{1}{2}}$*, one way to obtain the formulas for* $G(x)$ *is to use properties of exponents. For example,* $F(x) = 2^{x+1} - 3 = \left(4^{\frac{1}{2}}\right)^{x+1} - 3 = 4^{\frac{1}{2}(x+1)} - 3 = 4^{\frac{1}{2}x+\frac{1}{2}} - 3$*. In order, the formulas for* $G(x)$ *are:*

- $G(x) = 4^{\frac{1}{2}x+\frac{1}{2}} - 3$
- $G(x) = -4^{-\frac{1}{2}x} + 3$
- $G(x) = 4^{x-3}$
- $G(x) = 3 \cdot 4^{-x}$

Problem 4 *In Example* **??** *number* **??***, we obtained the solution* $F(x) = -2^{x+3} + 4$ *as one formula for the given graph by making a simplifying assumption that* $a = -1$*. This exercises explores if there are any other solutions for different choices of a.*

1. *Show* $G(x) = -4 \cdot 2^{x+1} + 4$ *also fits the data for the given graph, and use properties of exponents to show* $G(x) = F(x)$*. (Use the fact that* $4 = 2^2 \ldots$*)*
2. *With help from your classmates, find solutions to Example* **??** *number* **??** *using* $a = -8$*,* $a = -16$ *and* $a = -\dfrac{1}{2}$*. Show all your solutions can be rewritten as:* $F(x) = -2^{x+3} + 4$*.*
3. *Using properties of exponents and the fact that the range of* 2^x *is* $(0, \infty)$*, show that any function of the form* $f(x) = -a \cdot 2^{bx-h} + k$ *for* $a > 0$ *can be rewritten as* $f(x) = -2^c\, 2^{bx-h} + k = -2^{bx-h+c} + k$*. Relabeling, this means every function of the form* $f(x) = -a \cdot 2^{bx-h} + k$ *with four parameters (a, b, h, and k) can be rewritten as* $f(x) = -2^{bx-H} + k$*, a formula with just three parameters: b, H, and k. Conclude that every solution to Example* **??** *number* **??** *reduces to* $F(x) = -2^{x+3} + 4$ *.*

Question 5 *In Exercises* **??** - **??***, write the given function as a nontrivial decomposition of functions as directed.*

Problem 5.1 *For* $f(x) = e^{-x} + 1$*, find functions g and h so that* $f = g + h$*.*

One possible solution is $g(x) = e^{-x}$ *and* $h(x) = 1$*.*

Problem 5.2 *For* $f(x) = e^{2x} - x$*, find functions g and h so that* $f = g - h$*.*

One possible solution is $g(x) = e^{2x}$ *and* $h(x) = x$*.*

Problem 5.3 *For* $f(t) = t^2 e^{-t}$*, find functions g and h so that* $f = gh$*.*

One possible solution is $g(t) = t^2$ *and* $h(t) = e^{-t}$*.*

Problem 5.4 *For* $r(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$*, find functions f and g so* $r = \dfrac{f}{g}$*.*

One possible solution is $f(x) = e^x - e^{-x}$ *and* $g(x) = e^x + e^{-x}$*.*

Problem 5.5 *For* $k(x) = e^{-x^2}$*, find functions f and g so that* $k = g \circ f$*.*

One possible solution is $f(x) = -x^2$ *and* $g(x) = e^x$*.*

Problem 5.6 *For* $s(x) = \sqrt{e^{2x} - 1}$*, find functions f and g so* $s = g \circ f$*.*

One possible solution is $f(x) = e^{2x} - 1$ *and* $g(x) = \sqrt{x}$*.*

CC BY-NC-SA 4.0

Problem 6 *The amount of money in a savings account, $A(t)$, in dollars, t years after an initial investment is made is given by: $A(t) = 500(1.05)^t$, for $t \geq 0$.*

1. *Find and interpret $A(0)$, $A(1)$, and $A(2)$.*

   $A(0) = 500$, so the principal is \$500. $A(1) = 525$, so after 1 year, there is \$525 in the savings account. $A(2) = 551.25$, so after 2 years, there is \$551.25 in the savings account.

2. *Find and interpret the relative rate of change of A over the intervals $[0,1]$, $[1,2]$, $[0,2]$.*

   *The relative rate of change of A over the intervals $[0,1]$ and $[1,2]$ is 0.05 which means the savings account is growing by 5% each year for those two years. Over the interval $[0,2]$, the relative rate of change is 0.1025 meaning the account has grown by 10.25% over the course of the first two years. Note this is greater than the sum of the two rates $5\% + 5\% = 10\%$. This is due to the 'compounding effect' and will be discussed in greater detail in Section **??**.*

3. *Find, simplify, and interpret the relative rate of change of A over the $[t, t+1]$. Assume $t \geq 0$.*

   The relative rate of change of A over the $[t, t+1]$ is 0.05. This means over the course of one year, the savings account grows by 5%.

4. *Use a graphing utility to estimate how long until the savings account is worth \$1500. Round your answer to the nearest year.*

   Graphing $y = A(t)$ and $y = 1500$, we find they intersect when $t \approx 22.5$ so it takes approximately $22-23$ years for the savings account to grow to \$1500 in value.

Problem 7 *Based on census data,[1] the population of Lake County, Ohio, in 2010 was 230,041 and in 2015, the population was 229,437.*

1. *Show the percentage change in the population from 2010 to 2015 is approximately -0.263%.*

   $\dfrac{229437 - 230041}{230041} \approx 0.263\%$.

2. *If this percentage change remains constant, predict the population of Lake County in 2020.*

   Since 2020 is five years after 2015, we expect the population to decrease by 0.263% of 229437, or approximately 603 people. Hence, we approximate the population in 2020 as 228834.

3. *Assuming this percentage change per five years remains constant, find an expression for the population $P(t)$ of Lake County where t is the number of five year intervals after 2010. (So $t = 0$ corresponds to 2010, $t = 1$ corresponds to 2015, $t = 2$ corresponds to 2020, etc.)*

   Hint: *Definitions **??** and **??** and ensuing discussion on that page is useful here.*

   $P(t) = 230041(1 - 0.00263)^t = 230041(0.99737)^t$, $t \geq 0$.

4. *Use your answer to **??** to predict the population of Lake County in the year 2017.*

   Since 2017 is 7 years after 2010, we set $t = \dfrac{7}{5} = 1.4$ and find $P(1.4) \approx 229194$. So the population is approximately 229, 194 in 2017.

5. *Let $A(t)$ represent the population of Lake County t years after 2010 where the we approximate the percentage change in population per year as $-\dfrac{0.263\%}{5} = -0.0526\%$. Find a formula for $A(t)$ and compare your predictions with $A(t)$ to those given by $P(t)$. In particular, what population does each model give for the year 2050? Discuss any discrepancies with your classmates.*

[1] See here.

CC BY-NC-SA 4.0

$A(t) = 230041(1 - 0.0005626)^t = 230041(0.999474)^t$, $t \geq 0$. Since 2050 is 40 years after 2010, using the model $P(t)$, we divide $\frac{40}{5} = 8$ and find $P(8) \approx 225,245$. On the other hand, $A(40) \approx 225,250$. This is more than roundoff error. There is a compounding effect which makes the functions $A(t)$ and $P(t)$ different. [2]

Problem 8 *Show that the average rate of change of a function over the interval $[x, x+2]$ is average of the average rates of change of the function over the intervals $[x, x+1]$ and $[x+1, x+2]$. Can the same be said for the average rate of change of the function over $[x, x+3]$ and the average of the average rates of change over $[x, x+1]$, $[x+1, x+2]$, and $[x+2, x+3]$? Generalize.*

Problem 9 *If $f(x) = b^x$ where $b > 0$, $b \neq 1$, show $f(x_0 + \Delta x) = f(x_0)b^{\Delta x}$.*

Problem 10 *Which is larger: e^π or π^e? How do you know? Can you find a proof that doesn't use technology?*

Problem 11

1. *Use properties of exponential functions to show that if $f(x) = e^x$, then*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} e^x \left(\frac{e^h - 1}{h} \right)$$

2. *Numerically and graphically investigate the limit: $\lim_{h \to 0} \frac{e^h - 1}{h}$.*
3. *Use parts **??** and **??** to show to your surprise and delight that the derivative of e^x is … e^x.*
4. *Write the equations of the tangent lines to $y = e^x$ at the following points:*
   - (i) $(0, 1)$
   - (ii) $(1, e)$
   - (iii) $\left(-1, e^{-1}\right)$

[2] See number **??** above or, for more, see Section **??**.

CC BY-NC-SA 4.0