Exercises for Logarithmic Equations and Inequalities

Question 1 *In Exercises **??** - **??**, solve the equation analytically.*

Problem 1.1 $\log(3x-1) = \log(4-x)$

$x = \frac{5}{4}$

Problem 1.2 $\log_2\left(x^3\right) = \log_2(x)$

$x = 1$

Problem 1.3 $\ln\left(8-t^2\right) = \ln(2-t)$

$t = -2$

Problem 1.4 $\log_5\left(18-t^2\right) = \log_5(6-t)$

Select All Correct Answers:

1. $t = -3$ ✓
2. $t = -2$
3. $t = 1$
4. $t = 2$
5. $t = 4$ ✓
6. $t = 6$

Problem 1.5 $\log_3(7-2x) = 2$

$x = -1$

Problem 1.6 $\log_{\frac{1}{2}}(2x-1) = -3$

$x = \frac{9}{2}$

Problem 1.7 $\ln\left(t^2-99\right) = 0$

Select All Correct Answers:

1. $t = -33$
2. $t = -10$ ✓
3. $t = -9$
4. $t = 9$
5. $t = 10$ ✓
6. $t = 33$

Problem 1.8 $\log(t^2-3t) = 1$

Select All Correct Answers:

1. $t = -3$
2. $t = -2$ ✓
3. $t = 1$
4. $t = 5$ ✓
5. $t = 6$
6. $t = 8$

Problem 1.9 $\log_{125}\left(\dfrac{3x-2}{2x+3}\right) = \dfrac{1}{3}$

$x = -\dfrac{17}{7}$

Problem 1.10 $\log\left(\dfrac{x}{10^{-3}}\right) = 4.7$

$x = 10^{1.7}$

Problem 1.11 $-\log(x) = 5.4$

$x = 10^{-5.4}$

Problem 1.12 $10\log\left(\dfrac{x}{10^{-12}}\right) = 150$

$x = 10^{3}$

Problem 1.13 $6 - 3\log_5(2t) = 0$

$t = \dfrac{25}{2}$

Problem 1.14 $3\ln(t) - 2 = 1 - \ln(t)$

$t = e^{3/4}$

Problem 1.15 $\log_3(t-4) + \log_3(t+4) = 2$

Select All Correct Answers:

1. $t = -\sqrt{7}$
2. $t = -5$
3. $t = 5$ ✓
4. $t = \sqrt{7}$

Problem 1.16 $\log_5(2t+1) + \log_5(t+2) = 1$

$t = \dfrac{1}{2}$

Problem 1.17 $\log_{169}(3x+7) - \log_{169}(5x-9) = \dfrac{1}{2}$

$x = 2$

Problem 1.18 $\ln(x+1) - \ln(x) = 3$

$x = \dfrac{1}{e^3 - 1}$

Problem 1.19 $2\log_7(t) = \log_7(2) + \log_7(t+12)$

$t = 6$

Problem 1.20 $\log(t) - \log(2) = \log(t+8) - \log(t+2)$

$t = 4$

Problem 1.21 $\log_3(x) = \log_{\frac{1}{3}}(x) + 8$

$x = 81$

Problem 1.22 $\ln(\ln(x)) = 3$

$x = e^{e^3}$

Problem 1.23 $(\log(t))^2 = 2\log(t) + 15$

Select All Correct Answers:

1. $t = 10^{-5}$
2. $t = 10^{-3}$ ✓
3. $t = 10^{3}$
4. $t = 10^{5}$ ✓

Problem 1.24 $\ln(t^2) = (\ln(t))^2$

$t = 1,\ x = e^2$

Question 2 *In Exercises* **??** - **??**, *solve the inequality analytically.*

Problem 2.1 $\dfrac{1 - \ln(t)}{t^2} < 0$

(e, ∞)

Problem 2.2 $t\ln(t) - t > 0$

(e, ∞)

Problem 2.3 $10\log\left(\dfrac{x}{10^{-12}}\right) \geq 90$

$(10^{-3}, \infty)$

Problem 2.4 $5.6 \leq \log\left(\dfrac{x}{10^{-3}}\right) \leq 7.1$

$\left[10^{2.6}, 10^{4.1}\right]$

Problem 2.5 $2.3 < -\log(x) < 5.4$

$\left(10^{-5.4}, 10^{-2.3}\right)$

Problem 2.6 $\ln(t^2) \leq (\ln(t))^2$

$(0, 1] \cup [e^2, \infty)$

Question 3 *In Exercises* **??** - **??**, *use a graphing utility to help you solve the equation or inequality.*

Problem 3.1 $\ln(t) = e^{-t}$

$t \approx 1.3098$

Problem 3.2 $\ln(x) = \sqrt[4]{x}$

$x \approx 4.177,\ x \approx 5503.665$

Problem 3.3 $\ln(t^2 + 1) \geq 5$

$\approx (-\infty, -12.1414) \cup (12.1414, \infty)$

Problem 3.4 $\ln(-2x^3 - x^2 + 13x - 6) < 0$

$\approx (-3.0281, -3) \cup (0.5, 0.5991) \cup (1.9299, 2)$

Question 4 *In Exercises* **??** - **??**, *find the domain of the function.*

Problem 4.1 $r(x) = \dfrac{x}{1 - \ln(x)}$

$(-\infty, e) \cup (e, \infty)$

Problem 4.2 $R(x) = \dfrac{x\ln(x)}{1 - \ln(x)}$

$(0, e) \cup (e, \infty)$

Problem 4.3 $s(t) = \sqrt{2 - \log(t)}$

$(0, 100]$

Problem 4.4 $c(t) = (2\ln(t) - 1)^{\frac{2}{3}}$

$(0, \infty)$

Problem 4.5 $\ell(t) = \ln(\ln(t))$

$(1, \infty)$

Problem 4.6 $L(x) = \log\left(\dfrac{x\ln(x)}{1 - \ln(x)}\right)$

$(1, e)$

Problem 5 *Since* $f(x) = e^x$ *is a strictly increasing function, if* $a < b$ *then* $e^a < e^b$. *Use this fact to solve the inequality* $\ln(2x + 1) < 3$ *without a sign diagram. Use this technique to solve the inequalities in Exercises* **??** *-* **??**. *(Compare this to Exercise* **??** *in Section* **??***.)*

$-\dfrac{1}{2} < x < \dfrac{e^3 - 1}{2}$, *so* $\left(-\dfrac{1}{2}, \dfrac{e^3 - 1}{2}\right)$

Problem 6 *Solve* $\ln(3 - y) - \ln(y) = 2x + \ln(5)$ *for* y.

$y = \dfrac{3}{5e^{2x} + 1}$

Problem 7 *In Example **??** we found the inverse of* $f(x) = \dfrac{\log(x)}{1-\log(x)}$ *to be* $f^{-1}(x) = 10^{\frac{x}{x+1}}$.

1. *Algebraically check our answer by verifying* $\left(f^{-1} \circ f\right)(x) = x$ *for all* x *in the domain of* f *and that* $\left(f \circ f^{-1}\right)(x) = x$ *for all* x *in the domain of* f^{-1}.
2. *Find the range of* f *by finding the domain of* f^{-1}.
3. *Let* $g(x) = \dfrac{x}{1-x}$ *and* $h(x) = \log(x)$. *Show that* $f = g \circ h$ *and* $(g \circ h)^{-1} = h^{-1} \circ g^{-1}$.

   *NOTE: We know this is true in general by Exercise **??** in Section **??**, but it's nice to see a specific example of the property.*

Problem 8 *Let* $f(x) = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$. *Compute* $f^{-1}(x)$ *and find its domain and range.*

$f^{-1}(x) = \dfrac{e^{2x}-1}{e^{2x}+1} = \dfrac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$.

*To see why we rewrite this in this form, see Exercise **??** in Section **??**.*

The domain of f^{-1} *is* $(-\infty, \infty)$ *and its range is the same as the domain of* f, *namely* $(-1, 1)$.

Problem 9 *Explain the equation in Exercise **??** and the inequality in Exercise **??** above in terms of the Richter scale for earthquake magnitude. (See Exercise **??** in Section **??**.)*

Problem 10 *Explain the equation in Exercise **??** and the inequality in Exercise **??** above in terms of sound intensity level as measured in decibels. (See Exercise **??** in Section **??**.)*

Problem 11 *Explain the equation in Exercise **??** and the inequality in Exercise **??** above in terms of the pH of a solution. (See Exercise **??** in Section **??**.)*

Problem 12

1. *With the help of your classmates, numerically and graphically investigate* $\lim\limits_{x\to\infty} \dfrac{\ln(x)}{x^p}$ *for various real number powers,* $p > 0$.
2. *With the help of your classmates, numerically and graphically investigate* $\lim\limits_{x\to 0^+} x^p \ln(x)$ *for various real number powers,* $p > 0$.
3. *What do **??** and **??** suggest about the relative growth rates of powers of* x *and* $\ln(x)$*?*

Question 13 *In Exercises **??** - **??** a function* f *along with its derivatives* f' *and* f'' *are given.*

- *Find the domain of* f.
- *Find the* x*- and* y*-intercepts of the graph of each function, if any.*
- *Use limits to determine the end behavior and behavior at the endpoints of the domain.*
- *Use* f' *to determine the open intervals over which* f *is increasing or decreasing.*
- *Determine the local extrema, if any.*
- *Use* f'' *to determine the open intervals over which the graph of* f *is concave up or concave down.*
- *Determine the inflection points of the graph, if any.*

Problem 13.1 $f(x) = \ln(x) - \ln(5-x)$, $f'(x) = \dfrac{1}{x} + \dfrac{1}{5-x}$, $f''(x) = \dfrac{1}{(5-x)^2} - \dfrac{1}{x^2}$

- *Domain:* $(0, 5)$.
- *x-intercept:* $\left(\frac{5}{2}, 0\right)$*; there is no y-intercept.*
- $\lim_{x\to 0^+} f(x) = -\infty$, $\lim_{x\to 5^-} f(x) = \infty$*; we have two vertical asymptotes:* $x = 0$ *and* $x = 5$.
- *f is always increasing:* $(0, 5)$.
- *There are no local extrema.*
- *The graph of f is concave up on* $\left(0, \frac{5}{2}\right)$ *and concave down on* $\left(\frac{5}{2}, 5\right)$.
- *The inflection point is* $\left(\frac{5}{2}, 0\right)$.

Problem 13.2 $f(x) = \dfrac{\ln(x)}{x}$, $f'(x) = \dfrac{1 - \ln(x)}{x^2}$, $f''(x) = \dfrac{2\ln(x) - 3}{x^3}$.

- *Domain:* $(0, \infty)$.
- *x-intercept:* $(1, 0)$*; there is no y-intercept.*
- $\lim_{x\to 0^+} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = 0$*; we have a vertical and horizontal asymptote:* $x = 0$ *and* $y = 0$.
- *f is increasing on* $(0, e)$ *and decreasing on* (e, ∞).
- *There is a local (absolute) max at* $\left(e, \frac{1}{e}\right)$.
- *The graph of f is concave up on* $\left(e^{\frac{3}{2}}, \infty\right)$ *and concave down on* $\left(0, e^{\frac{3}{2}}\right)$.
- *The inflection point is* $\left(e^{\frac{3}{2}}, \frac{3}{2e^{\frac{3}{2}}}\right)$.