Exercises for Properties of Logarithms

Question 1 *In Exercises **??** - **??**, expand the given logarithm and simplify. Assume when necessary that all quantities represent positive real numbers.*

Problem 1.1 $\ln(x^3y^2)$

$3\ln(x) + 2\ln(y)$

Problem 1.2 $\log_2\left(\frac{128}{x^2+4}\right)$

$7 - \log_2(x^2+4)$

Problem 1.3 $\log_5\left(\frac{z}{25}\right)^3$

$3\log_5(z) - 6$

Problem 1.4 $\log(1.23 \times 10^{37})$

$\log(1.23) + 37$

Problem 1.5 $\ln\left(\frac{\sqrt{z}}{xy}\right)$

$\frac{1}{2}\ln(z) - \ln(x) - \ln(y)$

Problem 1.6 $\log_5\left(x^2 - 25\right)$

$\log_5(x-5) + \log_5(x+5)$

Problem 1.7 $\log_{\sqrt{2}}\left(4x^3\right)$

$3\log_{\sqrt{2}}(x) + 4$

Problem 1.8 $\log_{\frac{1}{3}}(9x(y^3-8))$

$-2 + \log_{\frac{1}{3}}(x) + \log_{\frac{1}{3}}(y-2) + \log_{\frac{1}{3}}(y^2+2y+4)$

Problem 1.9 $\log\left(1000x^3y^5\right)$

$3 + 3\log(x) + 5\log(y)$

Problem 1.10 $\log_3\left(\frac{x^2}{81y^4}\right)$

$2\log_3(x) - 4 - 4\log_3(y)$

Problem 1.11 $\ln\left(\sqrt[4]{\frac{xy}{ez}}\right)$

$\frac{1}{4}\ln(x) + \frac{1}{4}\ln(y) - \frac{1}{4} - \frac{1}{4}\ln(z)$

Problem 1.12 $\log_6\left(\frac{216}{x^3y}\right)^4$

$12 - 12\log_6(x) - 4\log_6(y)$

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Problem 1.13 $\log\left(\frac{100x\sqrt{y}}{\sqrt[3]{10}}\right)$

$\frac{5}{3}+\log(x)+\frac{1}{2}\log(y)$

Problem 1.14 $\log_{\frac{1}{2}}\left(\frac{4\sqrt[3]{x^2}}{y\sqrt{z}}\right)$

$-2+\frac{2}{3}\log_{\frac{1}{2}}(x)-\log_{\frac{1}{2}}(y)-\frac{1}{2}\log_{\frac{1}{2}}(z)$

Problem 1.15 $\ln\left(\frac{\sqrt[3]{x}}{10\sqrt{yz}}\right)$

$\frac{1}{3}\ln(x)-\ln(10)-\frac{1}{2}\ln(y)-\frac{1}{2}\ln(z)$

Question 2 *In Exercises* **??** - **??***, use the properties of logarithms to write the expression as a single logarithm.*

Problem 2.1 $4\ln(x)+2\ln(y)$

$\ln(x^4y^2)$

Problem 2.2 $\log_2(x)+\log_2(y)-\log_2(z)$

$\log_2\left(\frac{xy}{z}\right)$

Problem 2.3 $\log_3(x)-2\log_3(y)$

$\log_3\left(\frac{x}{y^2}\right)$

Problem 2.4 $\frac{1}{2}\log_3(x)-2\log_3(y)-\log_3(z)$

$\log_3\left(\frac{\sqrt{x}}{y^2z}\right)$

Problem 2.5 $2\ln(x)-3\ln(y)-4\ln(z)$

$\ln\left(\frac{x^2}{y^3z^4}\right)$

Problem 2.6 $\log(x)-\frac{1}{3}\log(z)+\frac{1}{2}\log(y)$

$\log\left(\frac{x\sqrt{y}}{\sqrt[3]{z}}\right)$

Problem 2.7 $-\frac{1}{3}\ln(x)-\frac{1}{3}\ln(y)+\frac{1}{3}\ln(z)$

$\ln\left(\sqrt[3]{\frac{z}{xy}}\right)$

Problem 2.8 $\log_5(x)-3$

$\log_5\left(\frac{x}{125}\right)$

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Problem 2.9 $3 - \log(x)$

$\log\left(\dfrac{1000}{x}\right)$

Problem 2.10 $\log_7(x) + \log_7(x-3) - 2$

$\log_7\left(\dfrac{x(x-3)}{49}\right)$

Problem 2.11 $\ln(x) + \dfrac{1}{2}$

$\ln\left(x\sqrt{e}\right)$

Problem 2.12 $\log_2(x) + \log_4(x)$

$\log_2\left(x^{3/2}\right)$

Problem 2.13 $\log_2(x) + \log_4(x-1)$

$\log_2\left(x\sqrt{x-1}\right)$

Problem 2.14 $\log_2(x) + \log_{\frac{1}{2}}(x-1)$

$\log_2\left(\dfrac{x}{x-1}\right)$

Question 3 *In Exercises* **??** - **??***, use the appropriate change of base formula to convert the given expression to an expression with the indicated base.*

Problem 3.1 7^{x-1} *to base e*

$7^{x-1} = e^{(x-1)\ln(7)}$

Problem 3.2 $\log_3(x+2)$ *to base 10*

$\log_3(x+2) = \dfrac{\log(x+2)}{\log(3)}$

Problem 3.3 $\left(\dfrac{2}{3}\right)^x$ *to base e*

$\left(\dfrac{2}{3}\right)^x = e^{x\ln\left(\frac{2}{3}\right)}$

Problem 3.4 $\log(x^2+1)$ *to base e*

$\log(x^2+1) = \dfrac{\ln(x^2+1)}{\ln(10)}$

Question 4 *In Exercises* **??** - **??***, use the appropriate change of base formula to approximate the logarithm.*

Problem 4.1 $\log_3(12)$

$\log_3(12) \approx 2.26186$

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Problem 4.2 $\log_5(80)$

$\log_5(80) \approx 2.72271$

Problem 4.3 $\log_6(72)$

$\log_6(72) \approx 2.38685$

Problem 4.4 $\log_4\left(\frac{1}{10}\right)$

$\log_4\left(\frac{1}{10}\right) \approx -1.66096$

Problem 4.5 $\log_{\frac{3}{5}}(1000)$

$\log_{\frac{3}{5}}(1000) \approx -13.52273$

Problem 4.6 $\log_{\frac{2}{3}}(50)$

$\log_{\frac{2}{3}}(50) \approx -9.64824$

Problem 5 *In Example **??** number **??** in Section **??**, we obtained the solution $F(x) = \log_2(-x+4) - 3$ as one formula for the given graph by making a simplifying assumption that $b = -1$. This exercises explores if there are any other solutions for different choices of b.*

1. *Show $G(x) = \log_2(-2x+8) - 4$ also fits the data for the given graph.*
2. *Use properties of logarithms to show $G(x) = \log_2(-2x+8) - 4 = \log_2(-x+4) - 3 = F(x)$.*
3. *With help from your classmates, find solutions to Example **??** number **??** in Section **??** by assuming $b = -4$ and $b = -8$. In each case, use properties of logarithms to show the solutions reduce to $F(x) = \log_2(-x+4) - 3$.*
4. *Using properties of logarithms and the fact that the range of $\log_2(x)$ is all real numbers, show that any function of the form $f(x) = a\log_2(bx-h)+k$ where $a \neq 0$ can be rewritten as:*

$$f(x) = a\left(\log_2(bx-h) + \frac{k}{a}\right) = a(\log_2(bx-h) + \log_2(p)) = a\log_2(p(bx-h)) = a\log_2(pbx - ph),$$

where $\frac{k}{a} = \log_2(p)$ for some positive real number p. Relabeling, we get every function of the form $f(x) = a\log_2(bx-h)+k$ with four parameters (a, b, h, and k) can be rewritten as $f(x) = a\log_2(Bx - H)$, a formula with just three parameters: a, B, and H.

*Show every solution to Example **??** number **??** in Section **??** can be written in the form $f(x) = \log_2\left(-\frac{1}{8}x + \frac{1}{2}\right)$ and that, in particular, $F(x) = \log_2(-x+4) - 3 = \log_2\left(-\frac{1}{8}x + \frac{1}{2}\right) = f(x)$. Hence, there is really just one solution to Example **??** number **??** in Section **??**.*

Problem 6 *The Henderson-Hasselbalch Equation: Suppose HA represents a weak acid. Then we have a reversible chemical reaction*

$$HA \rightleftharpoons H^+ + A^-.$$

The acid disassociation constant, K_a, is given by

$$K_a = \frac{[H^+][A^-]}{[HA]} = [H^+]\frac{[A^-]}{[HA]},$$

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*where the square brackets denote the concentrations just as they did in Exercise **??** in Section **??**. The symbol pK_a is defined similarly to pH in that $pK_a = -\log(K_a)$. Using the definition of pH from Exercise **??** and the properties of logarithms, derive the Henderson-Hasselbalch Equation:*

$$pH = pK_a + \log \frac{[A^-]}{[HA]}$$

Problem 7 *Compare and contrast the graphs of $y = \ln(x^2)$ and $y = 2\ln(x)$.*

Problem 8 *Prove the Quotient Rule and Power Rule for Logarithms.*

Problem 9 *Give numerical examples to show that, in general,*

1. $\log_b(x + y) \neq \log_b(x) + \log_b(y)$
2. $\log_b(x - y) \neq \log_b(x) - \log_b(y)$
1. $\log_b\left(\frac{x}{y}\right) \neq \frac{\log_b(x)}{\log_b(y)}$

Problem 10 *Research the history of logarithms including the origin of the word 'logarithm' itself. Why is the abbreviation of natural log 'ln' and not 'nl'?*

Problem 11 *There is a scene in the movie 'Apollo 13' in which several people at Mission Control use slide rules to verify a computation. Was that scene accurate? Look for other pop culture references to logarithms and slide rules.*

Problem 12 1. (i) *Use properties of logarithm functions to show that if $f(x) = \ln(x)$, then*

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\ln\left(1 + \frac{h}{x}\right)}{h}$$

(ii) *Numerically and graphically investigate the limit: $\lim_{h\to 0} \frac{\ln\left(1 + \frac{h}{x}\right)}{h}$ for various positive numbers x to convince yourself that $\lim_{h\to 0} \frac{\ln\left(1 + \frac{h}{x}\right)}{h} = \frac{1}{x}$*

(iii) *Use parts **??** and **??** to show to show that the derivative of $\ln(x)$ is . . . $\frac{1}{x}$.*

(iv) *Write the equations of the tangent lines to $y = \ln(x)$ at the following points. Check your answers graphically. Compare your answers the tangent lines at the corresponding points in Exercise **??** in Section **??**.*

i. $(1, 0)$ *ii.* $(e, 1)$ *iii.* $\left(e^{-1}, -1\right)$

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