Equations and Inequalities involving Exponential Functions

In this section we will develop techniques for solving equations involving exponential functions. Consider the equation $2^x = 128$. After a moment's calculation, we find $128 = 2^7$, so we have $2^x = 2^7$. The one-to-one property of exponential functions, detailed in Theorem **??**, tells us that $2^x = 2^7$ if and only if $x = 7$. This means that not only is $x = 7$ a solution to $2^x = 2^7$, it is the *only* solution.

Now suppose we change the problem ever so slightly to $2^x = 129$. We could use one of the inverse properties of exponentials and logarithms listed in Theorem **??** to write $129 = 2^{\log_2(129)}$. We'd then have $2^x = 2^{\log_2(129)}$, which means our solution is $x = \log_2(129)$.

After all, the definition of $\log_2(129)$ is 'the exponent we put on 2 to get 129.' Indeed we could have obtained this solution directly by rewriting the equation $2^x = 129$ in its logarithmic form $\log_2(129) = x$. Either way, in order to get a reasonable decimal approximation to this number, we'd use the change of base formula, Theorem **??**, to give us something more calculator friendly. Typically this means we convert our answer to base 10 or base e, and we choose the latter: $\log_2(129) = \dfrac{\ln(129)}{\ln(2)} \approx 7.011$.

Still another way to obtain this answer is to 'take the natural log' of both sides of the equation. Since $f(x) = \ln(x)$ is a *function*, as long as two quantities are equal, their natural logs are equal.[1]

We then use the Power Rule to write the exponent x as a factor then divide both sides by the constant $\ln(2)$ to obtain our answer.[2]

$$
\begin{array}{rcll}
2^x & = & 129 & \\
\ln\left(2^x\right) & = & \ln(129) & \text{Take the natural log of both sides.} \\
x\ln(2) & = & \ln(129) & \text{Power Rule} \\
x & = & \dfrac{\ln(129)}{\ln(2)} &
\end{array}
$$

We summarize our two strategies for solving equations featuring exponential functions below.

Steps for Solving an Equation involving Exponential Functions

1. Isolate the exponential function.
2. (i) If convenient, express both sides with a common base and equate the exponents.
   (ii) Otherwise, take the natural log of both sides of the equation and use the Power Rule.

Example 1. Solve the following equations. Check your answer using a graphing utility.

1. $2^{3x} = 16^{1-x}$
2. $2000 = 1000 \cdot 3^{-0.1t}$
3. $9 \cdot 3^x = 7^{2x}$
4. $75 = \dfrac{100}{1 + 3e^{-2t}}$
5. $25^x = 5^x + 6$
6. $\dfrac{e^x - e^{-x}}{2} = 5$

Explanation. 1. Since 16 is a power of 2, we can rewrite $2^{3x} = 16^{1-x}$ as $2^{3x} = \left(2^4\right)^{1-x}$. Using properties of exponents, we get $2^{3x} = 2^{4(1-x)}$.

Using the one-to-one property of exponential functions, we get $3x = 4(1-x)$ which gives $x = \dfrac{4}{7}$.

Using desmos, we graph $f(x) = 2^{3x}$ and $g(x) = 16^{1-x}$ and see that they intersect at $x \approx 0.571 \approx \dfrac{4}{7}$.

[1] This is also the 'if' part of the statement $\log_b(u) = \log_b(w)$ if and only if $u = w$ in Theorem **??**.

[2] Please resist the temptation to divide both sides by 'ln' instead of $\ln(2)$. Just like it wouldn't make sense to divide both sides by the square root symbol '$\sqrt{\ }$' when solving $x\sqrt{2} = 5$, it makes no sense to divide by 'ln'.

Desmos link: https://www.desmos.com/calculator/vgz48cail0

2. We begin solving $2000 = 1000 \cdot 3^{-0.1t}$ by dividing both sides by 1000 to isolate the exponential which yields $3^{-0.1t} = 2$.

   Since it is inconvenient to write 2 as a power of 3, we use the natural log to get $\ln\left(3^{-0.1t}\right) = \ln(2)$.

   Using the Power Rule, we get $-0.1t\ln(3) = \ln(2)$, so we divide both sides by $-0.1\ln(3)$ and obtain $t = -\dfrac{\ln(2)}{0.1\ln(3)} = -\dfrac{10\ln(2)}{\ln(3)}$.

   Using desmos, we find the graphs of $f(x) = 2000$ and $g(x) = 1000 \cdot 3^{-0.1x}$ intersect at $x \approx -6.309 \approx -\dfrac{10\ln(2)}{\ln(3)}$.

   Desmos link: https://www.desmos.com/calculator/pv2dugdpjn

3. We first note that we can rewrite the equation $9 \cdot 3^x = 7^{2x}$ as $3^2 \cdot 3^x = 7^{2x}$ to obtain $3^{x+2} = 7^{2x}$.

   Since it is not convenient to express both sides as a power of 3 (or 7 for that matter) we use the natural log: $\ln\left(3^{x+2}\right) = \ln\left(7^{2x}\right)$.

   The power rule gives $(x+2)\ln(3) = 2x\ln(7)$. Even though this equation appears very complicated, keep in mind that $\ln(3)$ and $\ln(7)$ are just constants.

   The equation $(x+2)\ln(3) = 2x\ln(7)$ is actually a linear equation (do you see why?) and as such we gather all of the terms with x on one side, and the constants on the other. We then divide both sides by the coefficient of x, which we obtain by factoring.

$$\begin{aligned}
(x+2)\ln(3) &= 2x\ln(7) \\
x\ln(3) + 2\ln(3) &= 2x\ln(7) \\
2\ln(3) &= 2x\ln(7) - x\ln(3) \\
2\ln(3) &= x(2\ln(7) - \ln(3)) \quad \text{Factor.} \\
x &= \frac{2\ln(3)}{2\ln(7) - \ln(3)}
\end{aligned}$$

   Using desmos, we see the graphs of $f(x) = 9 \cdot 3^x$ and $g(x) = 7^{2x}$ intersect at $x \approx 0.787 \approx \dfrac{2\ln(3)}{2\ln(7) - \ln(3)}$.

   Desmos link: https://www.desmos.com/calculator/tvahxyxai6

4. Our objective in solving $75 = \dfrac{100}{1 + 3e^{-2t}}$ is to first isolate the exponential.

   To that end, we clear denominators and get $75\left(1 + 3e^{-2t}\right) = 100$, or $75 + 225e^{-2t} = 100$. We get $225e^{-2t} = 25$, so finally, $e^{-2t} = \dfrac{1}{9}$.

   Taking the natural log of both sides gives $\ln\left(e^{-2t}\right) = \ln\left(\dfrac{1}{9}\right)$. Since natural log is log base e, $\ln\left(e^{-2t}\right) = -2t$. Likewise, we use the Power Rule to rewrite $\ln\left(\dfrac{1}{9}\right) = -\ln(9)$.

   Putting these two steps together, we simplify $\ln\left(e^{-2t}\right) = \ln\left(\dfrac{1}{9}\right)$ to $-2t = -\ln(9)$. We arrive at our solution, $t = \dfrac{\ln(9)}{2}$ which simplifies to $t = \ln(3)$. (Can you explain why?)

   To check, we see the graphs of $f(x) = 75$ and $g(x) = \dfrac{100}{1 + 3e^{-2x}}$, intersect at $x \approx 1.099 \approx \ln(3)$. As usual, our graph below is courtesy of desmos.

Desmos link: https://www.desmos.com/calculator/kqpadz2v1h

5. We start solving $25^x = 5^x + 6$ by rewriting $25 = 5^2$ so that we have $\left(5^2\right)^x = 5^x + 6$, or $5^{2x} = 5^x + 6$.

   Even though we have a common base, having two terms on the right hand side of the equation foils our plan of equating exponents or taking logs.

   If we stare at this long enough, we notice that we have three terms with the exponent on one term exactly twice that of another. To our surprise and delight, we have a 'quadratic in disguise'.

   Letting $u = 5^x$, we have $u^2 = \left(5^x\right)^2 = 5^{2x}$ so the equation $5^{2x} = 5^x + 6$ becomes $u^2 = u + 6$. Solving this as $u^2 - u - 6 = 0$ gives $u = -2$ or $u = 3$. Since $u = 5^x$, we have $5^x = -2$ or $5^x = 3$.

   Since $5^x = -2$ has no real solution,[3] we focus on $5^x = 3$. Since it isn't convenient to express 3 as a power of 5, we take natural logs and get $\ln\left(5^x\right) = \ln(3)$ so that $x\ln(5) = \ln(3)$ or $x = \dfrac{\ln(3)}{\ln(5)}$.

   Desmos shows the graphgs of $f(x) = 25^x$ and $g(x) = 5^x + 6$ intersect at $x \approx 0.683 \approx \dfrac{\ln(3)}{\ln(5)}$.

   Desmos link: https://www.desmos.com/calculator/vxcygzddiq

6. Clearing the denominator in $\dfrac{e^x - e^{-x}}{2} = 5$ gives $e^x - e^{-x} = 10$, at which point we pause to consider how to proceed. Rewriting $e^{-x} = \dfrac{1}{e^x}$, we see we have another denominator to clear: $e^x - \dfrac{1}{e^x} = 10$.

   Doing so gives $e^{2x} - 1 = 10e^x$, which, once again fits the criteria of being a 'quadratic in disguise.'

   If we let $u = e^x$, then $u^2 = e^{2x}$ so the equation $e^{2x} - 1 = 10e^x$ can be viewed as $u^2 - 1 = 10u$. Solving $u^2 - 10u - 1 = 0$ using the quadratic formula gives $u = 5 \pm \sqrt{26}$.

   From this, we have $e^x = 5 \pm \sqrt{26}$. Since $5 - \sqrt{26} < 0$, we get no real solution to $e^x = 5 - \sqrt{26}$ (why not?) but for $e^x = 5 + \sqrt{26}$, we take natural logs to obtain $x = \ln\left(5 + \sqrt{26}\right)$.

   Graphing $f(x) = \dfrac{e^x - e^{-x}}{2}$ and $g(x) = 5$ using desmos, we find an intersection at $x \approx 2.312 \approx \ln\left(5 + \sqrt{26}\right)$.

   Desmos link: https://www.desmos.com/calculator/u0x0hbap10

Note that verifying our solutions to the equations in Example **??** *analytically* holds great educational value, since it reviews many of the properties of logarithms and exponents in tandem.

For example, to verify our solution to $2000 = 1000 \cdot 3^{-0.1t}$, we substitute $t = -\dfrac{10\ln(2)}{\ln(3)}$ and check:

$$\begin{array}{rcll}
2000 & \stackrel{?}{=} & 1000 \cdot 3^{-0.1\left(-\frac{10\ln(2)}{\ln(3)}\right)} & \\
2000 & \stackrel{?}{=} & 1000 \cdot 3^{\frac{\ln(2)}{\ln(3)}} & \\
2000 & \stackrel{?}{=} & 1000 \cdot 3^{\log_3(2)} & \text{Change of Base} \\
2000 & \stackrel{?}{=} & 1000 \cdot 2 & \text{Inverse Property} \\
2000 & \stackrel{\checkmark}{=} & 2000 &
\end{array}$$

We strongly encourage the reader to check the remaining equations analytically as well.[4]

Since exponential functions are continuous on their domains, the Intermediate Value Theorem **??** applies. This allows us to solve inequalities using sign diagrams as demonstrated below.

[3] Why not?

[4] Who knows? Maybe your instructor will offer up a bonus point (or two!)

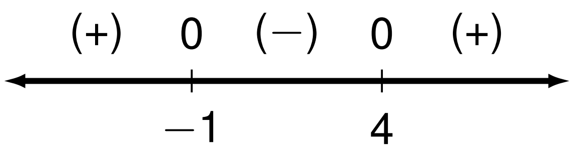
Example 2. Solve the following inequalities. Check your answer graphically.

1. $2^{x^2-3x} - 16 \geq 0$
2. $\dfrac{e^x}{e^x - 4} \leq 3$
3. $te^{2t} < 4t$

Explanation. 1. Since we already have 0 on one side of the inequality, we set $r(x) = 2^{x^2-3x} - 16$.

The domain of r is all real numbers, so to construct our sign diagram, we need to find the zeros of r.

Setting $r(x) = 0$ gives $2^{x^2-3x} - 16 = 0$ or $2^{x^2-3x} = 16$. Since $16 = 2^4$ we have $2^{x^2-3x} = 2^4$. By the one-to-one property of exponential functions, $x^2 - 3x = 4$ which gives $x = 4$ and $x = -1$. We construct a sign diagram below.



From the sign diagram, we see $r(x) \geq 0$ on $(-\infty, -1] \cup [4, \infty)$, which is our solution.

Graphing $r(x) = 2^{x^2-3x} - 16$ using desmos, we find it is on or above the line $y = 0$ (the x-axis) precisely on the intervals $(-\infty, -1]$ and $[4, \infty)$ which checks our answer.

Desmos link: https://www.desmos.com/calculator/olb42m5yup

2. The first step we need to take to solve $\dfrac{e^x}{e^x - 4} \leq 3$ is to get 0 on one side of the inequality. To that end, we subtract 3 from both sides and get a common denominator

$$\begin{aligned} \frac{e^x}{e^x - 4} &\leq 3 \\ \frac{e^x}{e^x - 4} - 3 &\leq 0 \\ \frac{e^x}{e^x - 4} - \frac{3\left(e^x - 4\right)}{e^x - 4} &\leq 0 \quad \text{Common denomintors.} \\ \frac{12 - 2e^x}{e^x - 4} &\leq 0 \end{aligned}$$

We set $r(x) = \dfrac{12 - 2e^x}{e^x - 4}$ and we note that r is undefined when its denominator $e^x - 4 = 0$, or when $e^x = 4$. Solving this gives $x = \ln(4)$, so the domain of r is $(-\infty, \ln(4)) \cup (\ln(4), \infty)$.

To find the zeros of r, we solve $r(x) = 0$ and obtain $12 - 2e^x = 0$. We find $e^x = 6$, or $x = \ln(6)$.

When we build our sign diagram, finding test values may be a little tricky since we need to check values around $\ln(4)$ and $\ln(6)$.

Recall that the function $\ln(x)$ is increasing[5] which means $\ln(3) < \ln(4) < \ln(5) < \ln(6) < \ln(7)$.

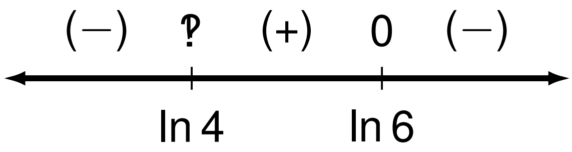
To determine the sign of $r\left(\ln(3)\right)$, we remember that $e^{\ln(3)} = 3$ and get

$$r\left(\ln(3)\right) = \frac{12 - 2e^{\ln(3)}}{e^{\ln(3)} - 4} = \frac{12 - 2(3)}{3 - 4} = -6.$$

We determine the signs of $r\left(\ln(5)\right)$ and $r\left(\ln(7)\right)$ similarly.[6]

[5] This is because the base of $\ln(x)$ is $e > 1$. If the base b were in the interval $0 < b < 1$, then $\log_b(x)$ would decreasing.

[6] We could, of course, use the calculator, but what fun would that be?



From the sign diagram, we find our answer to be $(-\infty, \ln(4)) \cup [\ln(6), \infty)$.

Using desmos, we find the graph of $f(x) = \dfrac{e^x}{e^x - 4}$ is below the graph of $g(x) = 3$ on $(-\infty, \ln(4)) \cup (\ln(6), \infty)$, and they intersect at $x \approx 1.792 \approx \ln(6)$.

Desmos link: `https://www.desmos.com/calculator/7zustjcohd`

3. As before, we start solving $te^{2t} < 4t$ by getting 0 on one side of the inequality, $te^{2t} - 4t < 0$.

   We set $r(t) = te^{2t} - 4t$ and since there are no denominators, even-indexed radicals, or logs, the domain of r is all real numbers.

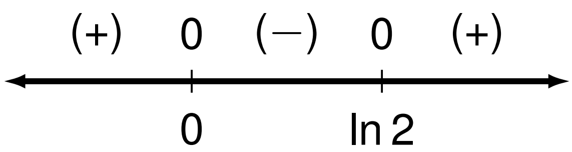
   Setting $r(t) = 0$ produces $te^{2t} - 4t = 0$. We factor to get $t\left(e^{2t} - 4\right) = 0$ which gives $t = 0$ or $e^{2t} - 4 = 0$.

   To solve the latter, we isolate the exponential and take logs to get $2t = \ln(4)$, or $t = \dfrac{\ln(4)}{2}$ which simplifies to $t = \ln(2)$. (Can you see why?)

   As in the previous example, we need to be careful about choosing test values. Since $\ln(1) = 0$, we choose $\ln\left(\dfrac{1}{2}\right)$, $\ln\left(\dfrac{3}{2}\right)$ and $\ln(3)$. Evaluating,[7] we get

$$\begin{aligned}
r\left(\ln\left(\frac{1}{2}\right)\right) &= \ln\left(\frac{1}{2}\right) e^{2\ln\left(\frac{1}{2}\right)} - 4\ln\left(\frac{1}{2}\right) \\
&= \ln\left(\frac{1}{2}\right) e^{\ln\left(\frac{1}{2}\right)^2} - 4\ln\left(\frac{1}{2}\right) && \text{Power Rule} \\
&= \ln\left(\frac{1}{2}\right) e^{\ln\left(\frac{1}{4}\right)} - 4\ln\left(\frac{1}{2}\right) \\
&= \frac{1}{4}\ln\left(\frac{1}{2}\right) - 4\ln\left(\frac{1}{2}\right) = -\frac{15}{4}\ln\left(\frac{1}{2}\right)
\end{aligned}$$

   Since $\dfrac{1}{2} < 1$, $\ln\left(\dfrac{1}{2}\right) < 0$ and we get $r(\ln\left(\dfrac{1}{2}\right))$ is $(+)$. Proceeding similarly, we find $r\left(\ln\left(\dfrac{3}{2}\right)\right) < 0$ and $r(\ln(3)) > 0$.



   Our solution corresponds to $r(t) < 0$ which occurs on $(0, \ln(2))$.

   Desmos confirms that the graph of $f(t) = te^{2t}$ is below the graph of $g(t) = 4t$ on $(0, \ln(2))$.[8]

   Desmos link: `https://www.desmos.com/calculator/t1dr2ragzl`

[7] A calculator can be used at this point. As usual, we proceed without apologies, with the analytical method.

[8] Note: $\ln(2) \approx 0.693$.

We note here that while sign diagrams will *always* work for solving inequalities involving exponential functions, as we've seen previously, there are circumstances in which we can short-cut this method.

For example, consider number **??** from Example **??** above: $2^{x^2-3x} - 16 \geq 0$. Since the base $2 > 1$, $\log_2(x)$ is an *increasing* function meaning it preserves inequalities.

We can use this to our advantage in this case and eliminate the exponential from the inequality altogether:

$$\begin{array}{rcll} 2^{x^2-3x} - 16 & \geq & 0 & \\ 2^{x^2-3x} & \geq & 16 & \\ \log_2\left(2^{x^2-3x}\right) & \geq & \log_2(16) & f(x) = \log_2(x) \text{ is increasing so if } b \geq a, \log_2(b) \geq \log_2(a). \\ x^2 - 3x & \geq & 4 & \end{array}$$

Hence, we've reduced our given inequality to $x^2 - 3x \geq 4$. As seen in Section **??**, we can solve this inequality by completing the square, graphing, or a sign diagram, whichever strikes the reader's fancy.

Our next example is a follow-up to Example **??** in Section **??**.

Example 3. Recall from Example **??** the temperature of coffee T (in degrees Fahrenheit) t minutes after it is served can be modeled by $T(t) = 70 + 90e^{-0.1t}$. When will the coffee be warmer than 100°F?

Explanation. We need to find when $T(t) > 100$, that is, we need to solve $70 + 90e^{-0.1t} > 100$.

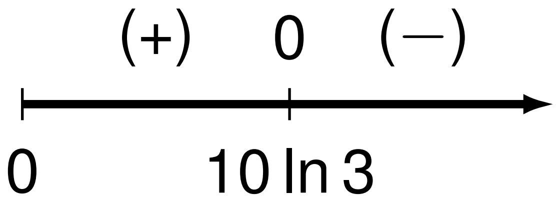
To use a sign diagram, we need to get 0 on one side of the inequality. Subtracting 100 from both sides of $70 + 90e^{-0.1t} > 100$ produces $90e^{-0.1t} - 30 > 0$.

Identifying $r(t) = 90e^{-0.1t} - 30$, we note from the context of the problem the domain of r is $[0, \infty)$, so to build the sign diagram, we proceed to find the zeros of r.

Solving $90e^{-0.1t} - 30 = 0$ results in $e^{-0.1t} = \dfrac{1}{3}$ so that $t = -10\ln\left(\dfrac{1}{3}\right)$ which reduces to $t = 10\ln(3)$.

If we wish to avoid using the calculator to choose test values, we note that $f(x) = \ln(x)$ is increasing. As a result, since $1 < 3$, $0 = \ln(1) < \ln(3)$ which proves $10\ln(3) > 0$. Hence, we may choose $t = 0$ as a test value in $[0, 10\ln(3))$. Since $3 < 4$, $\ln(3) < \ln(4)$, so $10\ln(3) < 10\ln(4)$. Hence, we may choose $10\ln(4)$ as test value for the interval $(10\ln(3), \infty)$.

We find $r(0) > 0$ and $r(10\ln(4)) < 0$ which gives the sign diagram below. We see $r(t) > 0$ on $[0, 10\ln(3))$.



Using desmos, we graph $y = T(t)$ along with with the horizontal line $y = 100$. We see the graph of T is above the horizontal line to the left of the intersection point, which we leave to the reader to show is $(10\ln(3), 100)$.

Desmos link: `https://www.desmos.com/calculator/ijdrhxov9w`

Hence, the coffee is warmer than 100°F up to $10\ln(3) \approx 11$ minutes after it is served, or, said differently, it takes approximately 11 minutes for the coffee to cool to under 100°F.

We note that, once again, we can short-cut the sign diagram in Example **??** to solve $70 + 90e^{-0.1t} > 100$. Since $\ln(x)$ is increasing, it preserves inequality. This means we can solve this inequality as follows.

$$\begin{aligned}
70 + 90e^{-0.1t} &> 100 \\
90e^{-0.1t} &> 30 \\
e^{-0.1t} &> \frac{1}{3} \\
\ln\left(e^{-0.1t}\right) &> \ln\left(\frac{1}{3}\right) && f(x) = \ln(x) \text{ is increasing so if } b \geq a, \ln(b) \geq \ln(a). \\
-0.1t &> -\ln(3) && \ln\left(\frac{1}{3}\right) = \ln\left(3^{-1}\right) = -\ln(3). \\
t &< \frac{-\ln(3)}{-0.1} = 10\ln(3)
\end{aligned}$$

Since we are given $t \geq 0$, we arrive at the same answer $0 \leq t < 10\ln(3)$ or $[0, 10\ln(3))$.

Note the importance, once again, of having a base larger than 1 so that the corresponding logarithmic function is *increasing*. We can still adapt this strategy to exponential functions whose base is less than 1, but we need to remember the corresponding logarithmic function is *decreasing* so it *reverses* inequalities.

Example 4. The function $f(x) = \dfrac{5e^x}{e^x + 1}$ is one-to-one.

1. Find a formula for $f^{-1}(x)$.
2. Solve $\dfrac{5e^x}{e^x + 1} = 4$.

Explanation. 1. We start by writing $y = f(x)$, and interchange the roles of x and y. To solve for y, we first clear denominators and then isolate the exponential function.

$$\begin{aligned}
y &= \frac{5e^x}{e^x + 1} \\
x &= \frac{5e^y}{e^y + 1} && \text{Switch } x \text{ and } y \\
x\left(e^y + 1\right) &= 5e^y \\
xe^y + x &= 5e^y \\
x &= 5e^y - xe^y \\
x &= e^y(5 - x) \\
e^y &= \frac{x}{5 - x} \\
\ln\left(e^y\right) &= \ln\left(\frac{x}{5 - x}\right) \\
y &= \ln\left(\frac{x}{5 - x}\right)
\end{aligned}$$

We claim $f^{-1}(x) = \ln\left(\dfrac{x}{5 - x}\right)$. To verify this analytically, we would need to verify the compositions $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f and that $\left(f \circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1}. We leave this, as well as a graphical check, to the reader in Exercise **??**.

2. We recognize the equation $\dfrac{5e^x}{e^x + 1} = 4$ as $f(x) = 4$. Hence, our solution is $x = f^{-1}(4) = \ln\left(\dfrac{4}{5 - 4}\right) = \ln(4)$.

   We can check this fairly quickly algebraically. Using $e^{\ln(4)} = 4$, we find $\dfrac{5e^{\ln(4)}}{e^{\ln(4)} + 1} = \dfrac{5(4)}{4 + 1} = \dfrac{20}{5} = 4$.

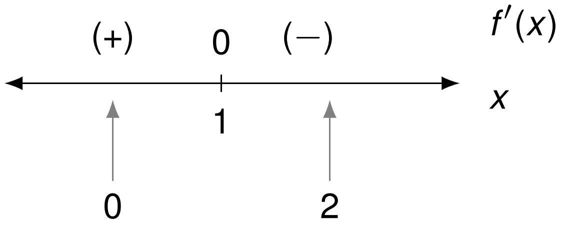
Our last example uses the tools of this section along with those developed in Section **??**.

Example 5. Let $f(x) = 3xe^{-x}$.

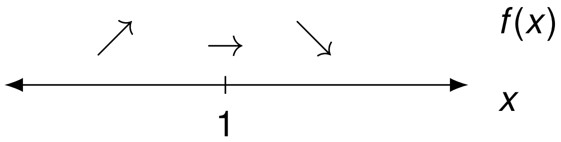
1. Given that $f'(x) = 3e^{-x} - 3xe^{-x}$, find the open intervals over which f is increasing and decreasing.
2. Find the local extrema.[9]
3. Given that $f''(x) = 3xe^{-x} - 6e^{-x}$, find the open intervals over which the graph of f is concave up and concave down.
4. Locate the inflection points.
5. Check your answers using a graphing utility.

Explanation. 1. To determine where f is increasing and decreasing, we need to make a sign diagram for $f'(x)$. Since the domain of f' is all real numbers, we just need to find the zeros of f'.

Solving $f'(x) = 3e^{-x} - 3xe^{-x} = 0$ gives $3e^{-x}(1-x) = 0$ so $3e^{-x} = 0$, which has no solution, or $1 - x = 0$ so $x = 1$. We make a sign diagram for $f'(x)$ below.

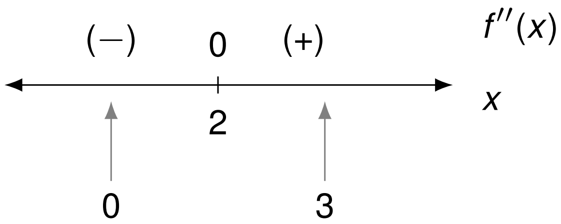


We get f is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$.

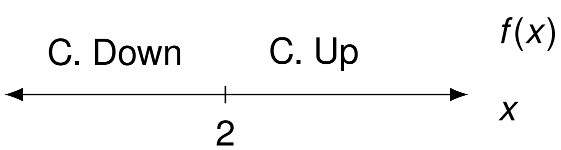


2. Using the sign diagram for $f'(x)$, we see we have a local maximum at $(1, f(1)) = \left(1, 3e^{-1}\right)$.
3. Once again, the domain of f'' is all real numbers, so our first step in constructing a sign diagram for $f''(x)$ is to find the zeros.

   Solving $f''(x) = 3xe^{-x} - 6e^{-x} = 0$ gives $3e^{-x}(x-2) = 0$ so either $3e^{-x} = 0$, which has no solution, or $x - 2 = 0$, so $x = 2$. We make the sign diagram for $f''(x)$ below.



We see the graph of f is concave up on $(-\infty, 2)$ and concave down on $(2, \infty)$.



[9] Recall this means the local maximums and minimums.

4. Since f changes concavity at $x = 2$, we have an inflection point at $(2, f(2)) = \left(2, 6e^{-2}\right)$.

5. Below is our check usingh desmos.

Desmos link: `https://www.desmos.com/calculator/ekrm7o7ecc`

Note that the graph of $f(x) = 3xe^{-x}$ produced by desmos in Example **??** suggests the graph of f has a horizontal asymptote $y = 0$ as $x \to \infty$.

Indeed, it is the case that $\lim\limits_{x\to\infty} f(x) = 0$, however if we try to reason this analytically, we get another instance of an indeterminate form.[10] As $x \to \infty$, $3x \to \infty$ but $e^{-x} \to 0$. Hence, as $x \to \infty$, we get the indeterminate form '$\infty \cdot 0$.' Depending on how quickly the first factor approaches '∞' and how quickly the second factor approaches '0', we could end up with '∞', '0,' or some number in between.[11]

We'll explore more of this phenomenon in Exercise **??**.[12] For now, we take it as true that exponential functions dominate polynomial functions so in the above indeterminate form, the factor e^{-x} determines[13] the end behavior of f, so $\lim\limits_{x\to\infty} f(x) = 0$.

[10] Previously, we've seen the indeterminate form '$\frac{0}{0}$.'

[11] It is important here to understand that the factor e^{-x} which results in the '0' in the indeterminate form '$\infty \cdot 0$' is **approaching** 0 and is not actually **equal** to 0. An incorrect reasoning that the form $\infty \cdot 0 \to 0$ in this case is that 'anything times 0 is 0.' Again, we'll have more to say about this in the Exercises.

[12] With formal proofs found in Calculus . . .

[13] Less formally, the factor $e^{-x} \to 0$ 'faster' than the factor $3x \to \infty$ which forces the product $3xe^{-x} \to 0$.