Exponential Functions

Of all of the functions we study in this text, exponential functions are possibly the ones which impact everyday life the most. This section introduces us to these functions while the rest of the chapter will more thoroughly explore their properties.

Up to this point, we have dealt with functions which involve terms like x^3, $x^{\frac{3}{2}}$, or x^{π} - in other words, terms of the form x^p where the base of the term, x, varies but the exponent of each term, p, remains constant.

In this chapter, we study functions of the form $f(x) = b^x$ where the base b is a constant and the exponent x is the variable. We start our exploration of these functions with the time-honored classic, $f(x) = 2^x$.

We make a table of function values and plot the corresponding points on a graph furnished by desmos below.

x	$f(x)$	$(x, f(x))$
-3	$2^{-3} = \frac{1}{8}$	$\left(-3, \frac{1}{8}\right)$
-2	$2^{-2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$
-1	$2^{-1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$
0	$2^0 = 1$	$(0, 1)$
1	$2^1 = 2$	$(1, 2)$
2	$2^2 = 4$	$(2, 4)$
3	$2^3 = 8$	$(3, 8)$

Desmos link: `https://www.desmos.com/calculator/w9kupo8qsr`

A few remarks about the graph of $f(x) = 2^x$ are in order. As $x \to -\infty$ and takes on values like $x = -100$ or $x = -1000$, the function $f(x) = 2^x$ takes on values like $f(-100) = 2^{-100} = \frac{1}{2^{100}}$ or $f(-1000) = 2^{-1000} = \frac{1}{2^{1000}}$. In other words, as $x \to -\infty$, $2^x \approx \frac{1}{\text{very big } (+)} \approx$ very small $(+)$ That is, as $x \to -\infty$, $2^x \to 0^+$, so $\lim_{x\to-\infty} 2^x = 0$. This produces the x-axis, $y = 0$ as a horizontal asymptote to the graph as $x \to -\infty$. We invite the reader to see this behavior graphically by scrolling towards the left on the Desmos graph above.

On the flip side, as $x \to \infty$, we find $f(100) = 2^{100}$, $f(1000) = 2^{1000}$, and so on, thus $\lim_{x\to\infty} 2^x = \infty$. We can see this graphically by scrolling towards the right (and up!) on the Desmos graph above.

Note that the smooth connected appearance of the graph of $f(x) = 2^x$ suggests that not only is f defined for all real numbers,[1] but also that f is *continuous*. Moreover, we are assuming $f(x) = 2^x$ is increasing: that is, if $a < b$, then $2^a < 2^b$. While these facts may appear obvious, the proofs of these properties are best left to Calculus. For us, we assume these properties in order to state the domain of f is $(-\infty, \infty)$ and the range of f is $(0, \infty)$. Moreover, since f is increasing, f is one-to-one, hence invertible.

Suppose we wish to study the family of functions $f(x) = b^x$. Which bases b make sense to study? We find that we run into difficulty if $b < 0$. For example, if $b = -2$, then the function $f(x) = (-2)^x$ has trouble, for instance, at $x = \frac{1}{2}$ since $(-2)^{1/2} = \sqrt{-2}$ is not a real number. In general, if x is any rational number with an even denominator,[2] then $(-2)^x$ is not defined, so we must restrict our attention to bases $b \geq 0$.

What about $b = 0$? The function $f(x) = 0^x$ is undefined for $x \leq 0$ because we cannot divide by 0 and 0^0 is an indeterminate form. For $x > 0$, $0^x = 0$ so the function $f(x) = 0^x$ is the same as the function $f(x) = 0$, $x > 0$. Since we know everything about this function, we ignore this case.

The only other base we exclude is $b = 1$, since the function $f(x) = 1^x = 1$ for all real numbers x, since, once again, a function we have already studied. We are now ready for our definition of exponential functions.

[1] See the discussion of real number exponents in Section **??**.

[2] or, as we defined real number exponents in Section **??**, if x is an irrational number . . .

Definition 1. An **exponential function** is the function of the form

$$f(x) = b^x$$

where b is a real number, $b > 0$, $b \neq 1$. The domain of an exponential function $(-\infty, \infty)$.

NOTE: More specifically, $f(x) = b^x$ is called the '*base b exponential function*.'

We leave it to the reader to verify[3] that if $b > 1$, then the exponential function $f(x) = b^x$ will share the same basic shape and characteristics as $f(x) = 2^x$.

What if $0 < b < 1$? Consider $g(x) = \left(\frac{1}{2}\right)^x$. We could certainly build a table of values and connect the points, or we could take a step back and note that $g(x) = \left(\frac{1}{2}\right)^x = \left(2^{-1}\right)^x = 2^{-x} = f(-x)$, where $f(x) = 2^x$. Per Section **??**, the graph of $f(-x)$ is obtained from the graph of $f(x)$ by reflecting it across the y-axis as seen in the graph below.

GeoGebra link: `https://www.geogebra.org/m/u97db2hf`

We see that the domain and range of g match that of f, namely $(-\infty, \infty)$ and $(0, \infty)$, respectively. Like f, g is also one-to-one. Whereas f is always increasing, g is always decreasing. As a result, as $\lim\limits_{x \to -\infty} g(x) = \infty$, and on the flip side, $\lim\limits_{x \to \infty} g(x) = 0$. (More specifically, $x \to \infty$, $g(x) \to 0^+$.) It shouldn't be too surprising that for all choices of the base $0 < b < 1$, the graph of $y = b^x$ behaves similarly to the graph of g.

We summarize the basic properties of exponential functions in the following theorem.

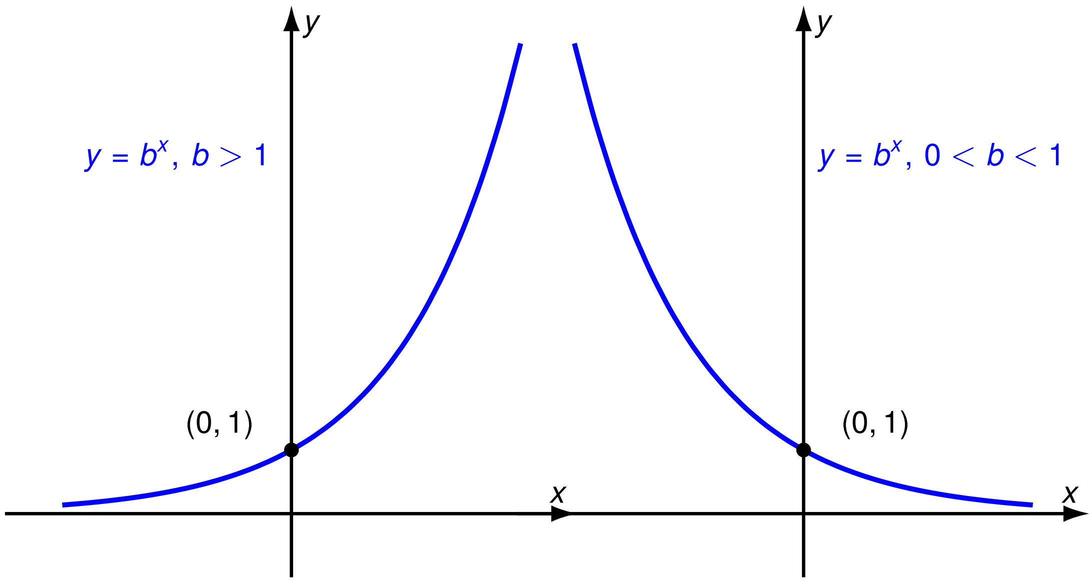
Theorem 1. Properties of Exponential Functions: Suppose $f(x) = b^x$.

- The domain of f is $(-\infty, \infty)$ and the range of f is $(0, \infty)$.
- $f(x) > 0$ for all x.
- $(0, 1)$ is on the graph of f and $y = 0$ is a horizontal asymptote to the graph of f.
- f is one-to-one, continuous and smooth[4]

[3] Meaning, graph some more examples on your own.
[4] Recall that this means the graph of f has no sharp turns or corners.

- If $b > 1$:
  - f is always increasing
  - $\lim_{x \to -\infty} f(x) = 0$
  - $\lim_{x \to \infty} f(x) = \infty$
  - The graph of f resembles:

- If $0 < b < 1$:
  - f is always decreasing
  - $\lim_{x \to -\infty} f(x) = \infty$
  - $\lim_{x \to \infty} f(x) = 0$
  - The graph of f resembles:



Exponential functions also inherit the basic properties of exponents from Theorem **??**. We formalize these below and use them as needed in the coming examples.

Theorem 2. (Algebraic Properties of Exponential Functions) Let $f(x) = b^x$ be an exponential function ($b > 0$, $b \neq 1$) and let u and w be real numbers.

- **Product Rule:** $f(u + w) = f(u)f(w)$. In other words, $b^{u+w} = b^u b^w$
- **Quotient Rule:** $f(u - w) = \dfrac{f(u)}{f(w)}$. In other words, $b^{u-w} = \dfrac{b^u}{b^w}$
- **Power Rule:** $(f(u))^w = f(uw)$. In other words, $\left(b^u\right)^w = b^{uw}$

In addition to base 2 which is important to computer scientists,[5] two other bases are used more often than not in scientific and economic circles. The first is base 10. Base 10 is called the '**common base**' and is important in the study of intensity (sound intensity, earthquake intensity, acidity, etc.)

The second base is an irrational number, e. Like $\sqrt{2}$ or π, the decimal expansion of e neither terminates nor repeats, so we represent this number by the letter 'e.' A decimal approximation of e is $e \approx 2.718$, so the function $f(x) = e^x$ is an increasing exponential function.

The number e is called the '**natural base**' for lots of reasons, one of which is that it 'naturally' arises in the study of growth functions in Calculus. We will more formally discuss the origins of e in Section **??**.

It is time for an example.

Example 1.

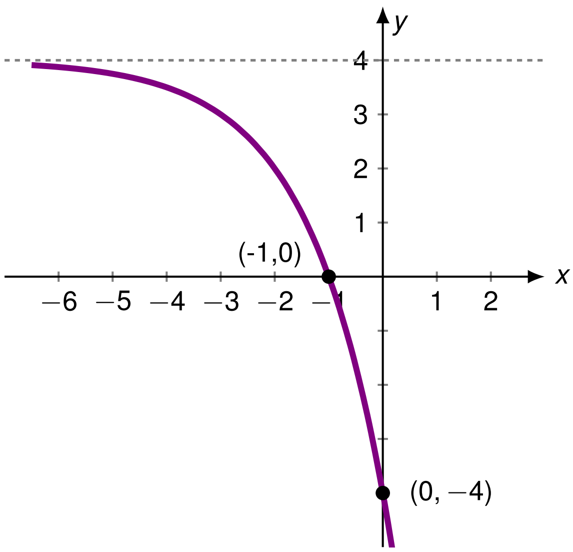
[5] The digital world is comprised of bytes which take on one of *two* values: 0 or 'off' and 1 or 'on.'

1. Graph the following functions by starting with a basic exponential function and using transformations, Theorem **??**. Track at least three points and the horizontal asymptote through the transformations.

   (i) $F(x) = 2\left(\frac{1}{3}\right)^{x-1}$

   (ii) $G(t) = 2 - e^{-t}$

2. Find a formula for the graph of the function below. Assume the base of the exponential is 2.



Explanation. 1. (i) Since the base of the exponent in $F(x) = 2\left(\frac{1}{3}\right)^{x-1}$ is $\frac{1}{3}$, we start with the graph of $f(x) = \left(\frac{1}{3}\right)^{x}$.

To use Theorem **??**, we first need to choose some 'control points' on the graph of $f(x) = \left(\frac{1}{3}\right)^{x}$. Since we are instructed to track three points (and the horizontal asymptote, $y = 0$) through the transformations, we choose the points corresponding to $x = -1$, $x = 0$, and $x = 1$: $(-1, 3)$, $(0, 1)$, and $\left(1, \frac{1}{3}\right)$, respectively.

Next, we need determine how to modify $f(x) = \left(\frac{1}{3}\right)^{x}$ to obtain $F(x) = 2\left(\frac{1}{3}\right)^{x-1}$. The key is to recognize the argument, or 'inside' of the function is the exponent and the 'outside' is anything outside the base of $\frac{1}{3}$. Using these principles as a guide, we find $F(x) = 2f(x-1)$.

Per Theorem **??**, we first add 1 to the x-coordinates of the points on the graph of $y = f(x)$, shifting the graph to the right 1 unit. Next, multiply the y-coordinates of each point on this new graph by 2, vertically stretching the graph by a factor of 2.

Looking point by point, we have $(-1, 3) \to (0, 3) \to (0, 6)$, $(0, 1) \to (1, 1) \to (1, 2)$, and $\left(1, \frac{1}{3}\right) \to \left(2, \frac{1}{3}\right) \to \left(2, \frac{2}{3}\right)$. The horizontal asymptote, $y = 0$ remains unchanged under the horizontal shift and the vertical stretch since $2 \cdot 0 = 0$.

The GeoGebra interactive below helps us visualize each of these transformation steps starting with the graph of $f(x) = \left(\frac{1}{3}\right)^{x}$. When we select each step, GeoGebra not only tracks our three control points through the transformation, it also displays the resulting graph.

GeoGebra link: https://www.geogebra.org/m/x88t5ypf

As always we can check our answer by verifying each of the points $(0,6)$, $(1,2)$, $\left(2,\dfrac{2}{3}\right)$ is on the graph of $F(x) = 2\left(\dfrac{1}{3}\right)^{x-1}$ by checking $F(0) = 6$, $F(1) = 2$, and $F(2) = \dfrac{2}{3}$.

We can check the end behavior as well, that is, $\lim\limits_{x\to-\infty} F(x) = \infty$ and $\lim\limits_{x\to\infty} F(x) = 0$. We leave these calculations to the reader.

(ii) Since the base of the exponential in $G(t) = 2 - e^{-t}$ is e, we start with the graph of $g(t) = e^{t}$.

Note that since e is an irrational number, we will use the approximation $e \approx 2.718$ when *plotting* points. However, when it comes to tracking and labeling said points, we do so with *exact* coordinates, that is, in terms of e.

We choose points corresponding to $t = -1$, $t = 0$, and $t = 1$: $(-1, e^{-1}) \approx (-1, 0.368)$, $(0,1)$, and $(1, e) \approx (1, 2.718)$, respectively.

Next, we need to determine how the formula for $G(t) = 2 - e^{-t}$ can be obtained from the formula $g(t) = e^{t}$. Rewriting $G(t) = -e^{-t} + 2$, we find $G(t) = -g(-t) + 2$.

Following Theorem **??**, we first multiply the t-coordinates of the graph of $y = g(t)$ by -1, effecting a reflection across the y-axis. Next, we multiply each of the y-coordinates by -1 which reflects the graph about the t-axis. Finally, we add 2 to each of the y-coordinates of the graph from the second step which shifts the graph up 2 units.

Tracking points, we have $(-1, e^{-1}) \to (1, e^{-1}) \to (1, -e^{-1}) \to (1, -e^{-1} + 2) \approx (1, 1.632)$, $(0,1) \to (0,1) \to (0,-1) \to (0,1)$, and $(1,e) \to (-1,e) \to (-1,-e) \to (-1,-e+2) \approx (-1,-0.718)$. The horizontal asymptote is unchanged by the reflections, but is shifted up 2 units $y = 0 \to y = 2$.

Once again, we have a GeoGebra interactive to help us visualize the step-by-step transformation of the graph of $g(t) = e^{t}$ to the graph of $G(t) = -e^{-t} + 2$. As usual, we can check our answer by verifying the indicated points do, in fact, lie on the graph of $y = G(t)$ along with checking end behavior. We leave these details to the reader.

GeoGebra link: https://www.geogebra.org/m/wfqfd3fs

2. Since we are told to assume the base of the exponential function is 2, we assume the function $F(x)$ is the result of the transforming the graph of $f(x) = 2^{x}$ using Theorem **??**. This means we are tasked with finding values for a, b, h, and k so that $F(x) = af(bx - h) + k = a \cdot 2^{bx-h} + k$.

   We invite the reader to explore some possibilities for these values using the GeoGebra interactive below. Input values for each of a, b, h, and k and see if you can get the transformed graph of $y = 2^{x}$ to match the data given in the target graph. As usual, we can see the impact of the individual parameters as we move step-by-step through the transformation.

   GeoGebra link: https://www.geogebra.org/m/w72snfyg

   Now we turn our attention to solving this problem analytically.

   Since the horizontal asymptote to the graph of $y = f(x) = 2^{x}$ is $y = 0$ and the horizontal asymptote to the graph $y = F(x)$ is $y = 4$, we know the vertical shift is 4 units up, so $k = 4$.

   Next, looking at how the graph of F approaches the vertical asymptote, it stands to reason the graph of $f(x) = 2^{x}$ undergoes a reflection across x-axis, meaning $a < 0$. For simplicity, we assume $a = -1$ and set see if we can find values for b and h that go along with this choice.

   Since $(-1,0)$ and $(0,-4)$ on the graph of $F(x) = -2^{bx-h} + 4$, we know $F(-1) = 0$ and $F(0) = -4$. From $F(-1) = 0$, we have $-2^{-b-h} + 4 = 0$ or $2^{-b-h} = 4 = 2^{2}$. Hence, $-b - h = 2$ is one solution.[6]

[6] This is the *only* solution. Since $f(x) = 2^{x}$, the equation $2^{-b-h} = 2^{2}$ is equivalent to the functional equation $f(-b-h) = f(2)$. Since f is one-to-one, we know this is true *only* when $-b - h = 2$.

Next, using $F(0) = -4$, we get $-2^{-h} + 4 = -4$ or $2^{-h} = 8 = 2^3$. From this, we have $-h = 3$ so $h = -3$. Putting this together with $-b - h = 2$, we get $-b + 3 = 2$ so $b = 1$.

Hence, one solution to the problem is $F(x) = -2^{x+3} + 4$. To check our answer, we leave it to the reader verify $F(-1) = 0$, $F(0) = -4$, $\lim_{x \to -\infty} F(x) = 4$, $\lim_{x \to \infty} F(x) = -\infty$ analytically as well as checking these values graphically using the GeoGebra interactive above.

Since we made a simplifying assumption ($a = -1$), we may well wonder if our solution is the *only* solution. Indeed, we started with what amounts to three pieces of information and set out to determine the value of four constants. We leave this for a thoughtful discussion in Exercise **??**.

Our next example showcases an important application of exponential functions: economic depreciation.

Example 2. The value of a car can be modeled by $V(t) = 25(0.8)^t$, where $t \geq 0$ is number of years the car is owned and $V(t)$ is the value in thousands of dollars.

1. Find and interpret $V(0)$, $V(1)$, and $V(2)$.
2. Find and interpret the average rate of change of V over the intervals $[0, 1]$ and $[0, 2]$ and $[1, 2]$.
3. Find and interpret $\dfrac{V(1)}{V(0)}$, $\dfrac{V(2)}{V(1)}$ and $\dfrac{V(2)}{V(0)}$.
4. For $t \geq 0$, find and interpret $\dfrac{V(t+1)}{V(t)}$ and $\dfrac{V(t+k)}{V(t)}$.
5. Find and interpret $\dfrac{V(1) - V(0)}{V(0)}$, $\dfrac{V(2) - V(1)}{V(1)}$, and $\dfrac{V(2) - V(0)}{V(0)}$.
6. For $t \geq 0$, find and interpret $\dfrac{V(t+1) - V(t)}{V(t)}$ and $\dfrac{V(t+k) - V(t)}{V(t)}$.
7. Graph $y = V(t)$ starting with the graph of $y = V(t)$ and using transformations.
8. Interpret the horizontal asymptote of the graph of $y = V(t)$.
9. Using a graphing utility, determine how long it takes for the car to depreciate to (a) one half its original value and (b) one quarter of its original value. Round your answers to the nearest hundredth.

Explanation.

1. We find $V(0) = 25(0.8)^0 = 25 \cdot 1 = 25$, $V(1) = 25(0.8)^1 = 25 \cdot 0.8 = 20$ and $V(2) = 25(0.8)^2 = 25 \cdot 0.64 = 16$. Since t represents the number of years the car has been owned, $t = 0$ corresponds to the purchase price of the car. Since $V(t)$ returns the value of the car in *thousands* of dollars, $V(0) = 25$ means the car is worth \$25,000 when first purchased. Likewise, $V(1) = 20$ and $V(2) = 16$ means the car is worth \$20,000 after one year of ownership and \$16,000 after two years, respectively.

2. Recall to find the average rate of change of V over an interval $[a, b]$, we compute: $\dfrac{V(b) - V(a)}{b - a}$. For the interval $[0, 1]$, we find $\dfrac{V(1) - V(0)}{1 - 0} = \dfrac{20 - 25}{1} = -5$, which means over the course of the first year of ownership, the value of the car depreciated, on average, at a rate of \$5000 per year.

   For the interval $[0, 1]$, we compute $\dfrac{V(2) - V(0)}{2 - 0} = \dfrac{16 - 25}{2} = -4.5$, which means over the course of the first two years of ownership, the car lost, on average, \$4500 per year in value.

   Finally, we find for the interval $[1, 2]$, $\dfrac{V(2) - V(1)}{2 - 1} = \dfrac{16 - 20}{1} = -4$, meaning the car lost, on average, \$4000 in value per year between the first and second years.

   Notice that the car lost more value over the first year (\$5000) than it did the second year (\$4000), and these losses average out to the average yearly loss over the first two years (\$4500 per year.)[7]

[7] It turns out for any function f, the average rate of change over the interval $[x, x + 2]$ is the average of the average rates of change of f over $[x, x + 1]$ and $[x + 1, x + 2]$. See Exercise **??**.

3. We compute: $\dfrac{V(1)}{V(0)} = \dfrac{20}{25} = 0.8$, $\dfrac{V(2)}{V(1)} = \dfrac{16}{20} = 0.8$, and $\dfrac{V(2)}{V(0)} = \dfrac{16}{25} = 0.64$.

   The ratio $\dfrac{V(1)}{V(0)} = 0.8$ can be rewritten as $V(1) = 0.8V(0)$ which means that the value of the car after 1 year, $V(1)$ is 0.8 times, or 80% the initial value of the car, $V(0)$.

   Similarly, the ratio $\dfrac{V(2)}{V(1)} = 0.8$ rewritten as $V(2) = 0.8V(1)$ means the value of the car after 2 years, $V(2)$ is 0.8 times, or 80% the value of the car after one year, $V(1)$.

   Finally, the ratio $\dfrac{V(2)}{V(0)} = 0.64$, or $V(2) = 0.64V(0)$ means the value of the car after 2 years, $V(2)$ is 0.64 times, or 64% of the initial value of the car, $V(0)$.

   Note that this last result tracks with the previous answers. Since $V(1) = 0.8V(0)$ and $V(2) = 0.8V(1)$, we get $V(2) = 0.8V(1) = 0.8(0.8V(0)) = 0.64V(0)$. Also note it is no coincidence that the base of the exponential, 0.8 has shown up in these calculations, as we'll see in the next problem.

4. Using properties of exponents, we find

$$\frac{V(t+1)}{V(t)} = \frac{25(0.8)^{t+1}}{25(0.8)^{t}} = (0.8)^{t+1-t} = 0.8$$

   Rewriting, we have $V(t+1) = 0.8V(t)$. This means after one year, the value of the car $V(t+1)$ is only 80% of the value it was a year ago, $V(t)$.

   Similarly, we find

$$\frac{V(t+k)}{V(t)} = \frac{25(0.8)^{t+k}}{25(0.8)^{t}} = (0.8)^{t+k-t} = (0.8)^{k}$$

   which, rewritten, says $V(t+k) = V(t)(0.8)^k$. This means in k years' time, the value of the car $V(t+k)$ is only $(0.8)^k$ times what it was worth k years ago, $V(t)$.

   These results shouldn't be too surprising. Verbally, the function $V(t) = 25(0.8)^t$ says to multiply 25 by 0.8 multiplied by itself t times. Therefore, for each additional year, we are multiplying the value of the car by an additional factor of 0.8.

5. We compute $\dfrac{V(1)-V(0)}{V(0)} = \dfrac{20-25}{25} = -0.2$, $\dfrac{V(2)-V(1)}{V(1)} = \dfrac{16-20}{20} = -0.2$, and $\dfrac{V(2)-V(0)}{V(0)} = \dfrac{16-25}{25} = -0.36$.

   The ratio $\dfrac{V(1)-V(0)}{V(0)}$ computes the ratio of *difference* in the value of the car after the first year of ownership, $V(1)-V(0)$, to the initial value, $V(0)$. We find this to be -0.2 or a 20% decrease in value. This makes sense since we know from our answer to number **??**, the value of the car after 1 year, $V(1)$ is 80% of the initial value, $V(0)$. Indeed:

$$\frac{V(1)-V(0)}{V(0)} = \frac{V(1)}{V(0)} - \frac{V(0)}{V(0)} = \frac{V(1)}{V(0)} - 1,$$

   and since $\dfrac{V(1)}{V(0)} = 0.8$, we get $\dfrac{V(1)-V(0)}{V(0)} = 1 - 0.8 = -0.2$.

   Likewise, the ratio $\dfrac{V(2)-V(1)}{V(1)} = -0.2$ means the value of the car has lost 20% of its value over the course of the second year of ownership.

   Finally, the ratio $\dfrac{V(2)-V(0)}{V(0)} = -0.36$ means that over the first two years of ownership, the car value has depreciated 36% of its initial purchase price. Again, this tracks with the result of number **??** which tells us that after two years, the car is only worth 64% of its initial purchase price.

6. Using properties of fractions and exponents, we get:

$$\frac{V(t+1)-V(t)}{V(t)} = \frac{25(0.8)^{t+1}-25(0.8)^{t}}{25(0.8)^{t}} = \frac{25(0.8)^{t+1}}{25(0.8)^{t}} - \frac{25(0.8)^{t}}{25(0.8)^{t}} = 0.8-1=-0.2,$$

   so after one year, the value of the car $V(t+1)$ has lost 20% of the value it was a year ago, $V(t)$.

   Similarly, we find:

$$\frac{V(t+k)-V(t)}{V(t)} = \frac{25(0.8)^{t+k}-25(0.8)^{t}}{25(0.8)^{t}} = \frac{25(0.8)^{t+1}}{25(0.8)^{t}} - \frac{25(0.8)^{t}}{25(0.8)^{t}} = (0.8)^{k}-1,$$

   so after k years' time, the value of the car $V(t)$ has decreased by $((0.8)^k - 1)\cdot 100\%$ of the value k years ago, $V(t)$.

7. To graph $y = 25(0.8)^t$, we start with the basic exponential function $f(t) = (0.8)^t$. Since the base $b = 0.8$ satisfies $0 < b < 1$, the graph of $y = f(t)$ is decreasing. We plot the y-intercept $(0,1)$ and two other points, $(-1,1.25)$ and $(1,0.8)$, and label the horizontal asymptote $y = 0$.

   To obtain the graph of $y = 25(0.8)^t = 25f(t)$, we multiply all of the y values in the graph by 25 (including the y value of the horizontal asymptote) in accordance with Theorem **??** to obtain the points $(-1,31.25)$, $(0,25)$ and $(1,20)$. The horizontal asymptote remains the same, since $25\cdot 0 = 0$. Finally, we restrict the domain to $[0,\infty)$ to fit with the applied domain given to us.

   Desmos link: `https://www.desmos.com/calculator/jejkmuwpx0`

8. We see from the graph of V that its horizontal asymptote is $y = 0$. This means as the car gets older, its value diminishes to 0.

9. We know the value of the car, brand new, is \$25,000, so when we are asked to find when the car depreciates to one half and one quarter of this value, we are trying to find when the value of the car dips to \$12,500 and \$6,125, respectively. Since $V(t)$ is measured in *thousands* of dollars, we this translates to solving the equations $V(t) = 12.5$ and $V(t) = 6.125$.

   Since we have yet to develop any analytic means to solve equations like $25(0.8)^t = 12.5$ (since t is in the exponent here), we are forced to approximate solutions to this equation numerically[8] or use a graphing utility. The GeoGebra interactive below displays the graph of $y = V(t)$ as well as a prompt for us to enter the fraction of the original value of the car that we're interested in.

   GeoGebra link: `https://www.geogebra.org/m/nwavgwu6`

   For example, entering the fraction '$\frac{1}{2}$' into the prompt triggers the display of the horizontal line $y = \left(\frac{1}{2}\right)(25) = 12.5$ along with the intersection point $(3.106\ldots, 12.5)$. This means the car depreciates to '$\frac{1}{2}$' (half) its initial value in (approximately) 3.11 years.

   Similarly, entering the fraction '$\frac{1}{4}$' into the prompt results in the graph of the horizontal line $y = \left(\frac{1}{4}\right)(25) = 6.25$ along with the intersection point $(6.212\ldots, 6.25)$. This means it takes approximately 6.21 years for the car to depreciate to '$\frac{1}{4}$' (one quarter) of its original value.[9]

■

[8] Since exponential functions are continuous we could use the Bisection Method to solve $f(t) = 25(0.8)^t - 12.5 = 0$. See the discussion on page **??** in Section **??** for more details.

[9] It turns out that it takes exactly twice as long for the car to depreciate to one-quarter of its initial value as it takes to depreciate to half its initial value. Does this pattern extend to the fractions of one-eighth and one-sixteenth? Can you see why?

Some remarks about Example **??** are in order. First the function in the previous example is called a 'decay curve'. Increasing exponential functions are used to model 'growth curves' and we shall see several different examples of those in Section **??**.

Second, as seen in numbers **??** and **??**, $V(t+1) = 0.8V(t)$. That is to say, the function V has a *constant unit multiplier*, in this case, 0.8 because to obtain the function value $V(t+1)$, we *multiply* the function value $V(t)$ by b. It is not coincidence that the multiplier here is the base of the exponential, 0.8.

Indeed, exponential functions of the form $f(x) = a \cdot b^x$ have a constant unit multiplier, b. To see this, note

$$\frac{f(x+1)}{f(x)} = \frac{a \cdot b^{x+1}}{a \cdot b^x} = b^1 = b.$$

Hence $f(x+1) = f(x) \cdot b$. This will prove useful to us in Section **??** when making decisions about whether or not a data set represents exponential growth or decay.

More generally, one can show (see Exercise **??**) for any real number x_0 that $f(x_0 + \Delta x) = f(x_0)b^{\Delta x}$. That is, to obtain $f(x_0 + \Delta x)$ from $f(x_0)$, we *multiply* by Δx factors of the constant unit multiplier, b. This is at the heart of what it means to be an exponential function.

If this discussion seems familiar, it should. For linear functions, $f(x) = mx + b$, we can obtain the slope m by computing $f(x+1) - f(x)$. To see this, note $f(x+1) - f(x) = (m(x+1) + b) - (mx + b) = m$ so that $f(x+1) = f(x) + m$. In this way, we see that the slope m is the constant unit *addend* in that in order to obtain $f(x+1)$, we *add* m to the function value $f(x)$.

This notion is solidified in the point-slope form of a linear function, Equation **??**. For for any real numbers x and x_0, we have $f(x) = f(x_0) + m(x - x_0)$. If we let $x = x_0 + \Delta x$, we get $f(x_0 + \Delta x) = f(x_0) + m\Delta x$. In other words, to obtain $f(x_0 + \Delta x)$ from $f(x_0)$, we *add* m times Δx.

Taking inspiration from linear functions, we define the 'point-base' form of an exponential function below.

Definition 2. The **point-base form** of the exponential function $f(x) = b^x$ is

$$f(x) = f(x_0)b^{x-x_0}$$

Just as the point-slope form of a linear function is helpful in building linear models, the point-base form of an exponential function will prove useful in building exponential models.

Next, while we saw in Example **??** number **??**, exponential functions, unlike linear functions, do not have a constant rate of change. However, in numbers **??** and **??**, we see that in some cases, they do have a constant *relative* rate of change. We define this notion below.

Definition 3. Let f be a function defined on the interval $[a, b]$ where $f(a) \neq 0$.

The **relative rate of change** of f over $[a, b]$ is defined as:

$$\frac{\Delta[f(x)]}{f(a)} = \frac{f(b) - f(a)}{f(a)}.$$

For exponential functions of the form $f(x) = a \cdot b^x$, we compute the relative rate of change over the interval $[x, x+1]$ and find it is constant:

$$\frac{f(x+1) - f(x)}{f(x)} = \frac{f(x+1)}{f(x)} - \frac{f(x)}{f(x)} = b - 1,$$

where we are using the fact that $\dfrac{f(x+1)}{f(x)} = b$.

One way to interpret this result is when comparing $f(x)$ to $f(x+1)$, the exponential function grows (if $b > 1$) or decays (if $b < 1$) by $(b-1) \cdot 100\%$. In our example, $V(t) = 25(0.8)^t$ so $b = 0.8$ and, as we saw, the relative rate of change from $V(t)$ to $V(t+1)$ was $0.8 - 1 = -0.2$, meaning the value of the car over the course of one year depreciates by 20%.

We close this section with another important application of exponential functions, Newton's Law of Cooling.

Example 3. According to Newton's Law of Cooling[10] the temperature of coffee $T(t)$ (in degrees Fahrenheit) t minutes after it is served can be modeled by $T(t) = 70 + 90e^{-0.1t}$.

1. Find and interpret $T(0)$.
2. Sketch the graph of $y = T(t)$ using transformations.
3. Find and interpret $\lim\limits_{t\to\infty} T(t)$.

Explanation.

1. Since $T(0) = 70 + 90e^{-0.1(0)} = 160$, the temperature of the coffee when it is served is 160°F.
2. To graph $y = T(t)$ using transformations, we start with the basic function, $f(t) = e^t$. As in Example **??**, we track the points $(-1, e^{-1}) \approx (-1, 0.368)$, $(0, 1)$, and $(1, e) \approx (1, 2.718)$, along with the horizontal asymptote $y = 0$ through each of transformations.

   To use Theorem **??**, we rewrite $T(t) = 70 + 90e^{-0.1t} = 90e^{-0.1t} + 70 = 90f(-0.1t) + 70$. Following Theorem **??**, we first divide the t-coordinates of each point on the graph of $y = f(t)$ by -0.1 which results in a horizontal expansion by a factor of 10 as well as a reflection about the y-axis.

   Next, we multiply the y-values of the points on this new graph by 90 which effects a vertical stretch by a factor of 90. Last but not least, we add 70 to all of the y-coordinates of the points on this second graph, which shifts the graph upwards 70 units.

   Tracking points, we have $(-1, e^{-1}) \to (10, e^{-1}) \to (10, 90e^{-1}) \to (10, 90e^{-1} + 70) \approx (10, 103.112)$, $(0, 1) \to (0, 1) \to (0, 90) \to (0, 160)$, and $(1, e) \to (-10, e) \to (-10, 90e) \to (-10, 90e + 70) \approx (-10, 314.62)$. The horizontal asymptote $y = 0$ is unaffected by the horizontal expansion, reflection about the y-axis, and the vertical stretch. The vertical shift moves the horizontal asymptote up 70 units, $y = 0 \to y = 70$.

   As usual, GeoGebra helps us visualize transforming $y = e^t$ into $y = T(t)$ by tracking the key points through each step of the transformation and displaying the corresponding graph.

   GeoGebra link: `https://www.geogebra.org/m/maxqgrzd`

   The final step would be to impose the applied domain to $t \geq 0$ so we are looking at the portion of the graph which is to the right of the y-axis.
3. We can determine $\lim\limits_{t\to\infty} T(t)$ two ways. First, we can employ the 'number sense' developed in Chapter **??**.

   That is, as $t \to \infty$, We get $T(t) = 70 + 90e^{-0.1t} \approx 70 + 90e^{\text{very big }(-)}$. Since $e > 1$, $e^{\text{very big }(-)} \approx$ very small (+) The larger t becomes, the smaller $e^{-0.1t}$ becomes, so the term $90e^{-0.1t} \approx$ very small (+). Hence, $T(t) = 70 + 90e^{-0.1t} \approx 70 +$ very small (+) ≈ 70.

   Alternatively, we can look to the graph of $y = T(t)$. We know the horizontal asymptote is $y = 70$ which means as $t \to \infty$, $T(t) \approx 70$.

   In either case, we find that as time goes by, the temperature of the coffee is cooling to 70° Fahrenheit, ostensibly room temperature.

[10] We will discuss this in greater detail in Section **??**.