LOGARITHMIC FUNCTIONS

In Section **??**, we saw exponential functions $f(x) = b^x$ are are one-to-one which means they are invertible. In this section, we explore their inverses, the *logarithmic functions* which are called 'logs' for short.

Definition 1. For the exponential function $f(x) = b^x$, $f^{-1}(x) = \log_b(x)$ is called the **base *b* logarithm function**. We read '$\log_b(x)$' as 'log base b of x.'

We have special notations for the common base, $b = 10$, and the natural base, $b = e$.

Definition 2.

- The **common logarithm** of a real number x is $\log_{10}(x)$ and is usually written $\log(x)$.
- The **natural logarithm** of a real number x is $\log_e(x)$ and is usually written $\ln(x)$.

Since logs are defined as the inverses of exponential functions, we can use Theorems **??** and **??** to tell us about logarithmic functions. For example, we know that the domain of a log function is the range of an exponential function, namely $(0, \infty)$, and that the range of a log function is the domain of an exponential function, namely $(-\infty, \infty)$.

Moreover, since we know the basic shapes of $y = f(x) = b^x$ for the different cases of b, we can obtain the graph of $y = f^{-1}(x) = \log_b(x)$ by reflecting the graph of f across the line $y = x$. The y-intercept $(0, 1)$ on the graph of f corresponds to an x-intercept of $(1, 0)$ on the graph of f^{-1}. The horizontal asymptotes $y = 0$ on the graphs of the exponential functions become vertical asymptotes $x = 0$ on the log graphs.

Below we use desmos to graph $f(x) = 2^x$ along with $y = \log_2(x)$. Note the corresponding points and asymptotes and the symmetry across the line $y = x$.

Desmos link: `https://www.desmos.com/calculator/qatyujujep`

Next, we graph $f(x) = \left(\frac{1}{2}\right)^x$ along with its inverse $y = \log_{\frac{1}{2}}(x)$.

Desmos link: `https://www.desmos.com/calculator/cwoz62jn3g`

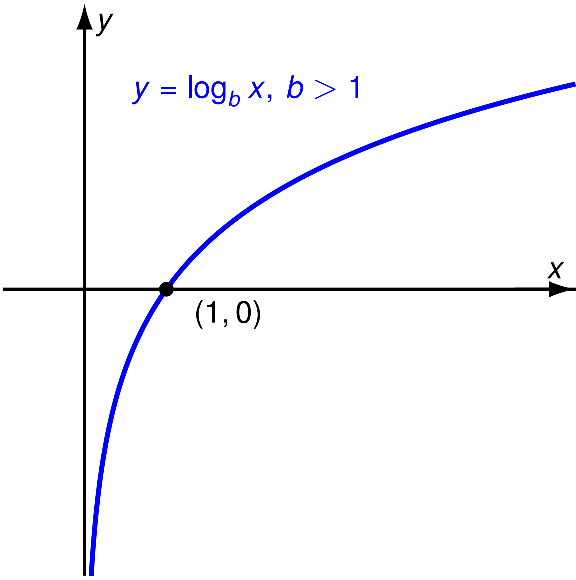
Procedurally, logarithmic functions 'undo' the exponential functions. Consider the function $f(x) = 2^x$. When we evaluate $f(3) = 2^3 = 8$, the input 3 becomes the exponent on the base 2 to produce the real number 8. The function $f^{-1}(x) = \log_2(x)$ then takes the number 8 as its input and returns the exponent 3 as its output. In symbols, $\log_2(8) = 3$.

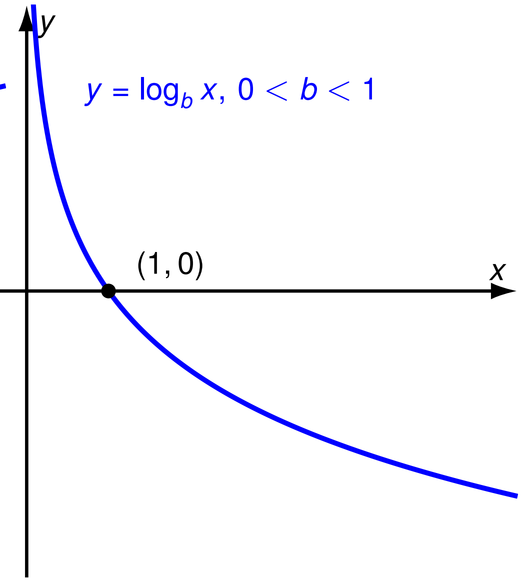
More generally, $\log_2(x)$ is the exponent you put on 2 to get x. Thus, $\log_2(16) = 4$, because $2^4 = 16$. The following theorem summarizes the basic properties of logarithmic functions, all of which come from the fact that they are inverses of exponential functions.

Theorem 1. Properties of Logarithmic Functions: Suppose $f(x) = \log_b(x)$.

- The domain of f is $(0, \infty)$ and the range of f is $(-\infty, \infty)$.
- $(1, 0)$ is on the graph of f and $x = 0$ is a vertical asymptote of the graph of f.
- f is one-to-one, continuous and smooth
- $b^a = c$ if and only if $\log_b(c) = a$. That is, $\log_b(c)$ is the exponent you put on b to obtain c.
- $\log_b\left(b^x\right) = x$ for all real numbers x and $b^{\log_b(x)} = x$ for all $x > 0$

- If $b > 1$:
  - f is always increasing
  - $\lim_{x \to 0^+} f(x) = -\infty$
  - $\lim_{x \to \infty} f(x) = \infty$
  - The graph of f resembles:
- If $0 < b < 1$:
  - f is always decreasing
  - $\lim_{x \to 0^+} f(x) = \infty$
  - $\lim_{x \to \infty} f(x) = -\infty$
  - The graph of f resembles:





As we have mentioned, Theorem **??** is a consequence of Theorems **??** and **??**. However, it is worth the reader's time to understand Theorem **??** from an exponent perspective.

As an example, we know that the domain of $g(x) = \log_2(x)$ is $(0, \infty)$. Why? Because the range of $f(x) = 2^x$ is $(0, \infty)$. In a way, this says everything, but at the same time, it doesn't.

To really *understand* why the domain of $g(x) = \log_2(x)$ is $(0, \infty)$, consider trying to compute $\log_2(-1)$. We are searching for the exponent we put on 2 to give us -1. In other words, we are looking for x that satisfies $2^x = -1$. There is no such real number, since all powers of 2 are positive.

While what we have said is exactly the same thing as saying 'the domain of $g(x) = \log_2(x)$ is $(0, \infty)$ because the range of $f(x) = 2^x$ is $(0, \infty)$', we feel it is in a student's best interest to understand the statements in Theorem **??** at this level instead of just merely memorizing the facts.

Our first example gives us practice computing logarithms as well as constructing basic graphs.

Example 1.

1. Simplify the following.

   (i) $\log_3(81)$

   (ii) $\log_2\left(\frac{1}{8}\right)$

   (iii) $\log_{\sqrt{5}}(25)$

   (iv) $\ln\left(\sqrt[3]{e^2}\right)$

   (v) $\log(0.001)$

   (vi) $2^{\log_2(8)}$

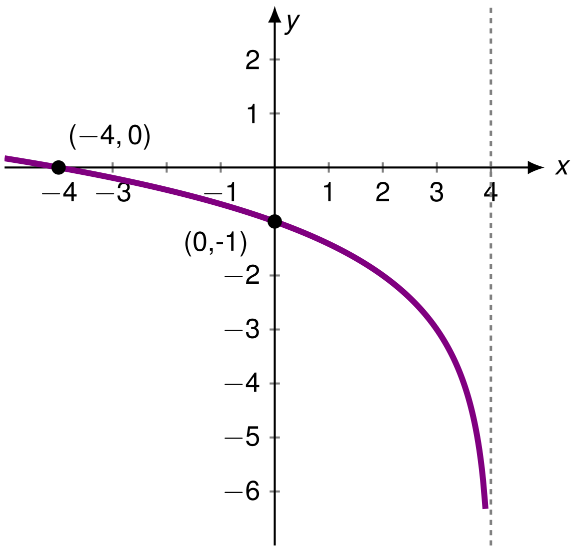
(vii) $117^{-\log_{117}(6)}$

2. Graph the following functions by starting with a basic logarithmic function and using transformations, Theorem **??**. Track at least three points and the vertical asymptote through the transformations.

   (i) $F(x) = \log_{\frac{1}{3}}\left(\frac{x}{2}\right) + 1$

   (ii) $G(t) = -\ln(2-t)$

3. Find a formula for the graph of the function below. Assume the base of the logarithm is 2.



Explanation. 1. (i) The number $\log_3(81)$ is the exponent we put on 3 to get 81. As such, we want to write 81 as a power of 3. We find $81 = 3^4$, so that $\log_3(81) = 4$.

(ii) To find $\log_2\left(\frac{1}{8}\right)$, we need rewrite $\frac{1}{8}$ as a power of 2. We find $\frac{1}{8} = \frac{1}{2^3} = 2^{-3}$, so $\log_2\left(\frac{1}{8}\right) = -3$.

(iii) To determine $\log_{\sqrt{5}}(25)$, we need to express 25 as a power of $\sqrt{5}$. We know $25 = 5^2$, and $5 = \left(\sqrt{5}\right)^2$, so we have $25 = \left(\left(\sqrt{5}\right)^2\right)^2 = \left(\sqrt{5}\right)^4$. We get $\log_{\sqrt{5}}(25) = 4$.

(iv) First, recall that the notation $\ln\left(\sqrt[3]{e^2}\right)$ means $\log_e\left(\sqrt[3]{e^2}\right)$, so we are looking for the exponent to put on e to obtain $\sqrt[3]{e^2}$. Rewriting $\sqrt[3]{e^2} = e^{2/3}$, we find $\ln\left(\sqrt[3]{e^2}\right) = \ln\left(e^{2/3}\right) = \frac{2}{3}$.

(v) Rewriting $\log(0.001)$ as $\log_{10}(0.001)$, we see that we need to write 0.001 as a power of 10. We have $0.001 = \frac{1}{1000} = \frac{1}{10^3} = 10^{-3}$. Hence, $\log(0.001) = \log\left(10^{-3}\right) = -3$.

(vi) We can use Theorem **??** directly to simplify $2^{\log_2(8)} = 8$.
We can also understand this problem by first finding $\log_2(8)$. By definition, $\log_2(8)$ is the exponent we put on 2 to get 8. Since $8 = 2^3$, we have $\log_2(8) = 3$.
We now substitute to find $2^{\log_2(8)} = 2^3 = 8$.

(vii) From Theorem **??**, we know $117^{\log_{117}(6)} = 6$,[1] but we cannot directly apply this formula to the expression $117^{-\log_{117}(6)}$ without first using a property of exponents. (Can you see why?)
Rather, we find: $117^{-\log_{117}(6)} = \frac{1}{117^{\log_{117}(6)}} = \frac{1}{6}$.

[1] It is worth a moment of your time to think your way through why $117^{\log_{117}(6)} = 6$. By definition, $\log_{117}(6)$ is the exponent we put on 117 to get 6. What are we doing with this exponent? We are putting it on 117, so we get 6.

2. (i) To graph $F(x) = \log_{\frac{1}{3}}\left(\frac{x}{2}\right) + 1$ we start with the graph of $f(x) = \log_{\frac{1}{3}}(x)$. and use Theorem **??**.

First we choose some 'control points' on the graph of $f(x) = \log_{\frac{1}{3}}(x)$. Since we are instructed to track three points (and the vertical asymptote, $x = 0$) through the transformations, we choose the points corresponding to powers of $\frac{1}{3}$: $\left(\frac{1}{3}, 1\right)$, $(1, 0)$, and $(3, -1)$, respectively.

Next, we note $F(x) = \log_{\frac{1}{3}}\left(\frac{x}{2}\right)+1 = f\left(\frac{x}{2}\right)+1$. Per Theorem **??**, we first multiply the x-coordinates of the points on the graph of $y = f(x)$ by 2, horizontally expanding the graph by a factor of 2. Next, we add 1 to the y-coordinates of each point on this new graph, vertically shifting the graph up 1.

Looking at each point, we get $\left(\frac{1}{3}, 1\right) \to \left(\frac{2}{3}, 1\right) \to \left(\frac{2}{3}, 2\right)$, $(1, 0) \to (2, 0) \to (2, 1)$, and $(3, -1) \to (6, -1) \to (6, 0)$. The horizontal asymptote, $x = 0$ remains unchanged under the horizontal stretch and the vertical shift.

The GeoGebra interactive below helps us visualize each of these transformation steps starting with the graph of $f(x) = \log_{\frac{1}{3}}(x)$. When we select each step, GeoGebra not only tracks our three control points through the transformation, it also displays the resulting graph.

GeoGebra link: `https://www.geogebra.org/m/cg8cjnue`

As always we can check our answer by verifying each of the points $\left(\frac{2}{3}, 2\right)$, $(2, 1)$, , and $(6, 0)$, is on the graph of $F(x) = \log_{\frac{1}{3}}\left(\frac{x}{2}\right) + 1$ by checking $F\left(\frac{2}{3}\right) = 2$, $F(2) = 1$, and $F(6) = 0$. We can check the end behavior as well, that is, $\lim\limits_{x\to 0^+} F(x) = \infty$ and as $\lim\limits_{x\to\infty} F(x) \to -\infty$. We leave these calculations to the reader.

(ii) Since the base of $G(t) = -\ln(2 - t)$ is e, we start with the graph of $g(t) = \ln(t)$. As usual, since e is an irrational number, we use the approximation $e \approx 2.718$ when plotting points, but label points using exact coordinates in terms of e.

We choose points corresponding to powers of e on the graph of $g(t) = \ln(t)$: $(e^{-1}, -1) \approx (0.368, -1)$, $(1, 0)$, and $(e, 1) \approx (2.718, 1)$, respectively.

Since $G(t) = -\ln(2 - t) = -\ln(-t + 2) = -g(-t + 2)$, Theorem **??** instructs us to first subtract 2 from each of the t-coordinates of the points on the graph of $g(t) = \ln(t)$, shifting the graph to the left two units.

Next, we multiply (divide) the t-coordinates of points on this new graph by -1 which reflects the graph across the y-axis. Lastly, we multiply each of the y-coordinates of this second graph by -1, reflecting it across the t-axis.

Tracking points, we have $(e^{-1}, -1) \to (e^{-1} - 2, -1) \to (-e^{-1} + 2, -1) \to (-e^{-1} + 2, 1) \approx (1.632, 1)$, $(1, 0) \to (-1, 0) \to (1, 0) \to (1, 0)$, and $(e, 1) \to (e - 2, 1) \to (-e + 2, 1) \to (-e + 2, -1) \approx (-0.718, -1)$. The vertical asymptote is affected by the horizontal shift and the reflection about the y-axis only: $t = 0 \to t = -2 \to t = 2$.

Once again, we have a GeoGebra interactive to help us visualize the step-by-step transformation of the graph of $g(t) = \ln(t)$ to the graph of $G(t) = -\ln(-t + 2)$. As usual, we can check our answer by verifying the indicated points do, in fact, lie on the graph of $y = G(t)$ along with checking the behavior as $t \to -\infty$ and $t \to 2^-$.

GeoGebra link: `https://www.geogebra.org/m/e6xe62cw`

3. Since we are told to assume the base of the exponential function is 2, we assume the function $F(x)$ is the result of the transforming the graph of $f(x) = \log_2(x)$ using Theorem **??**. This means we are tasked with finding values for a, b, h, and k so that $F(x) = af(bx - h) + k = a\log_2(bx - h) + k$.

We invite the reader to explore some possibilities for these values using the GeoGebra interactive below. Input values for each of a, b, h, and k and see if you can get the transformed graph of $y = \log_2(x)$ to match the data given in the target graph. As usual, we can see the impact of the individual parameters as we move step-by-step through the transformation.

GeoGebra link: https://www.geogebra.org/m/tdy7cegd

We now proceed to tackle this problem analytically.

Since the vertical asymptote to the graph of $y = f(x) = \log_2(x)$ is $x = 0$ and the vertical asymptote to the graph $y = F(x)$ is $x = 4$, we know we have a horizontal shift of 4 units. Moreover, since the curve approaches the vertical asymptote from the *left*, we also know we have a reflection about the y-axis, so $b < 0$. Since the recipe in Theorem **??** instructs us to perform the horizontal shift *before* the reflection across the y-axis, we take $h = -4$ and assume for simplicity $b = -1$ so $F(x) = a\log_2(-x + 4) + k$.

To determine a and k, we make use of the two points on the graph. Since $(-4, 0)$ is on the graph of F, $F(-4) = a\log_2(-(-4) + 4) + k = 0$. This reduces to $a\log_2(8) + k = 0$ or $3a + k = 0$. Next, we use the point $(0, -1)$ to get $F(0) = a\log_2(-(0) + 4) + k = -1$. This reduces to $a\log_2(4) + k = -1$ or $2a + k = -1$. From $3a + k = 0$, we get $k = -3a$ which when substituted into $2a + k = -1$ gives $2a + (-3a) = -1$ or $a = 1$. Hence, $k = -3a = -3(1) = -3$.

Putting all of this work together we find $F(x) = \log_2(-x + 4) - 3$. As always, we can check our answer by verifying $F(-4) = 0$, $F(0) = -1$, $\lim\limits_{x\to-\infty} F(x) = \infty$, and $\lim\limits_{x\to4^-} F(x) = -\infty$. We leave these details to the reader.[2]

Up until this point, restrictions on the domains of functions came from avoiding division by zero and keeping negative numbers from beneath even indexed radicals. With the introduction of logs, we now have another restriction. Since the domain of $f(x) = \log_b(x)$ is $(0, \infty)$, the argument of the logarithm[3] must be strictly positive.

Example 2. Find the domain of each function analytically and check your answer using a graphing utility.

1. $f(x) = 2\log(3 - x) - 1$
2. $g(x) = \ln\left(\dfrac{x}{x-1}\right)$

Explanation. 1. We set $3 - x > 0$ to obtain $x < 3$, or $(-\infty, 3)$. Using the desmos interactive below, we can enter $f(x) = 2\log(3 - x) - 1$ to confirm our answer graphically.

Desmos link: https://www.desmos.com/calculator/qzvdurrrzh

Note that in this case, we can graph f using transformations, which we do so here for extra practice.

Taking a cue from Theorem **??**, we rewrite $f(x) = 2\log_{10}(-x + 3) - 1$ and view this function as a transformed version of $h(x) = \log_{10}(x)$.

To graph $y = \log(x) = \log_{10}(x)$, We select three points to track corresponding to powers of 10: $(0.1, -1)$, $(1, 0)$ and $(10, 1)$, along with the vertical asymptote $x = 0$.

Since $f(x) = 2h(-x + 3) - 1$, Theorem **??** tells us that to obtain the destinations of these points, we first subtract 3 from the x-coordinates (shifting the graph left 3 units), then divide (multiply) by the x-coordinates by -1 (causing a reflection across the y-axis).

Next, we multiply the y-coordinates by 2 which results in a vertical stretch by a factor of 2, then we finish by subtracting 1 from the y-coordinates which shifts the graph down 1 unit.

Tracking points, we find: $(0.1, -1) \to (-2.9, -1) \to (2.9, -1) \to (2.9, -2) \to (2.9, -3)$, $(1, 0) \to (-2, 0) \to (2, 0) \to (2, 0) \to (2, -1)$, and $(10, 1) \to (7, 1) \to (-7, 1) \to (-7, 2) \to (-7, 1)$. The vertical shift and reflection about the y-axis affects the vertical asymptote: $x = 0 \to x = -3 \to x = 3$.

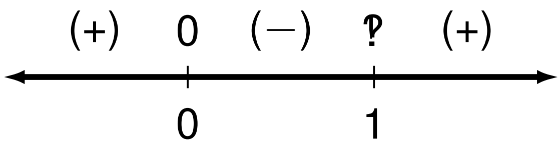
Plotting these three points along with the vertical asymptote produces the graph of f as seen above.

[2] As with Exercise **??** in Section **??**, we may well wonder if our solution to this problem is the *only* solution since we made a simplifying assumption that $b = -1$. We leave this for a thoughtful discussion in Exercise **??** in Section **??**.

[3] that is, what's 'inside' the logarithm

2. To find the domain of g, we need to solve the inequality $\dfrac{x}{x-1} > 0$ using a sign diagram.[4]

   If we define $r(x) = \dfrac{x}{x-1}$, we find r is undefined at $x = 1$ and $r(x) = 0$ when $x = 0$. Choosing some test values, we generate the sign diagram below.



   We find $\dfrac{x}{x-1} > 0$ on $(-\infty, 0) \cup (1, \infty)$ which is the domain of g. As above, we can enter $g(x) = \ln\left(\dfrac{x}{x-1}\right)$ into the desmos interactive below to confirm our answer graphically.

   Desmos link: https://www.desmos.com/calculator/qzvdurrrzh

   We can tell from the graph of g that it is not the result of Section **??** transformations being applied to the graph $y = \ln(x)$, (do you see why?) so barring a more detailed analysis using Calculus, producing a graph using a graphing utility is the best we can do.

   One thing worthy of note, however, is the end behavior of g. The graph suggests that $\lim_{x\to-\infty} g(x) = 0$ and $\lim_{x\to\infty} g(x) = 0$.

   We can verify this analytically. Using results from Chapter **??** and continuity, we know that $\lim_{x\to-\infty} \dfrac{x}{x+1} = 1$ and $\lim_{x\to\infty} \dfrac{x}{x+1} = 1$. Hence, it stands to reason that

$$\lim_{x\to-\infty} g(x) = \lim_{x\to-\infty} \ln\left(\frac{x}{x-1}\right) = \ln(1) = 0$$

   and, likewise, $\lim_{x\to-\infty} g(x) = \ln(1) = 0$.

While logarithms have some interesting applications of their own which you'll explore in the exercises, their primary use to us will be to undo exponential functions. (This is, after all, how they were defined.) Our last example reviews not only the major topics of this section, but reviews the salient points from Section **??**.

Example 3. Let $f(x) = 2^{x-1} - 3$.

1. Graph f using transformations and state the domain and range of f.
2. Explain why f is invertible and find a formula for $f^{-1}(x)$.
3. Graph f^{-1} using transformations and state the domain and range of f^{-1}.
4. Verify $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f and $\left(f \circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1}.
5. Graph f and f^{-1} on the same set of axes and check for symmetry about the line $y = x$.
6. Use f or f^{-1} to solve the following equations. Check your answers algebraically.

   (i) $2^{x-1} - 3 = 4$

   (ii) $\log_2(t+3) + 1 = 0$

[4] See Section **??** for a review of this process, if needed.

Explanation. 1. To graph $f(x) = 2^{x-1} - 3$ using Theorem **??**, we first identify $g(x) = 2^x$ and note $f(x) = g(x-1) - 3$. Choosing the 'control points' of $\left(-1, \frac{1}{2}\right)$, $(0, 1)$ and $(1, 2)$ on the graph of g along with the horizontal asymptote $y = 0$, we implement the algorithm set forth in Theorem **??**.

First, we first add 1 to the x-coordinates of the points on the graph of g which shifts the the graph of g to the right one unit. Next, we subtract 3 from each of the y-coordinates on this new graph, shifting the graph down 3 units to get the graph of f.

Looking point-by-point, we have $\left(-1, \frac{1}{2}\right) \to \left(0, \frac{1}{2}\right) \to \left(0, -\frac{5}{2}\right)$, $(0, 1) \to (1, 1) \to (1, -2)$, and, finally, $(1, 2) \to (2, 2) \to (2, -1)$. The horizontal asymptote is affected only by the vertical shift, $y = 0 \to y = -3$.

The desmos interactive below graphs both the starting function $g(x) = 2^x$ and the transformed function $f(x) = g(x-1) - 3 = 2^{x-1} - 3$. Here instead of tracking the key points and graph through each transformation discretely, we are presented with a full array of parameters from Theorem **??** with which to continuously transform the graph of g into the graph.

Rewriting $f(x) = g(x-1) - 3 = (1)g((1)x - 1) + (-3)$, we see the two parameters which need adjusting are h and k.

Using the sliders, adjust h from 0 to 1. This moves the graph of g gradually to the right 1 unit. Adjusting k from 0 to -3 moves the graph down 3 units. The key points move along with the graph so we have a nice graphical check of our work above.

Desmos link: `https://www.desmos.com/calculator/rbhejvkamm`

From the graph of f, we get the domain is $(-\infty, \infty)$ and the range is $(-3, \infty)$.

2. The graph of f passes the Horizontal Line Test so f is one-to-one, hence invertible.

   To find a formula for $f^{-1}(x)$, we normally set $y = f(x)$, interchange the x and y, then proceed to solve for y. Doing so in this situation leads us to the equation $x = 2^{y-1} - 3$. We have yet to discuss how to solve this kind of equation, so we will attempt to find the formula for f^{-1} procedurally.

   Thinking of f as a process, the formula $f(x) = 2^{x-1} - 3$ takes an input x and applies the steps: first subtract 1. Second put the result of the first step as the exponent on 2. Last, subtract 3 from the result of the second step.

   Clearly, to undo subtracting 1, we will add 1, and similarly we undo subtracting 3 by adding 3. How do we undo the second step? The answer is we use the logarithm.

   By definition, $\log_2(x)$ undoes exponentiation by 2. Hence, f^{-1} should: first, add 3. Second, take the logarithm base 2 of the result of the first step. Lastly, add 1 to the result of the second ste. In symbols, $f^{-1}(x) = \log_2(x + 3) + 1$.

3. To graph $f^{-1}(x) = \log_2(x + 3) + 1$ using Theorem **??**, we start with $g(x) = \log_2(x)$ and track the points $\left(\frac{1}{2}, -1\right)$, $(1, 0)$ and $(2, 1)$ on the graph of j along with the vertical asymptote $x = 0$ through the transformations.

   Since $f^{-1}(x) = g(x + 3) + 1$, we first subtract 3 from each of the x-coordinates of each of the points on the graph of $y = g(x)$ shifting the graph of g to the left three units. We then add 1 to each of the y-coordinates of the points on this new graph, shifting the graph up one unit.

   Tracking points, we get $\left(\frac{1}{2}, -1\right) \to \left(-\frac{5}{2}, -1\right) \to \left(-\frac{5}{2}, 0\right)$, $(1, 0) \to (-2, 0) \to (-2, 1)$, and $(2, 1) \to (-1, 1) \to (-1, 2)$.

   The vertical asymptote is only affected by the horizontal shift, so we have $x = 0 \to x = -3$.

   Once again, we can use the desmos interactive below to continuously transform $g(x) = \log_2(x)$ into $f^{-1}(x) = g(x+3)+1 = (1)g((1)x - (-3)) + 1$. Adjusting the slider for h from 0 to -3 gradually moves the graph of g to the left 3 units while adjusting k from 0 to 1 then slowly moves the graph up 1 unit. The key points move to the locations we predicted above.

Desmos link: https://www.desmos.com/calculator/zwbbtbbeke

From the graph, we get the domain of f^{-1} is $(-3, \infty)$, which matches the range of f, and the range of f^{-1} is $(-\infty, \infty)$, which matches the domain of f, in accordance with Theorem **??**.

4. We now verify that $f(x) = 2^{x-1} - 3$ and $f^{-1}(x) = \log_2(x+3) + 1$ satisfy the composition requirement for inverses.

   First, we simplify $(f^{-1} \circ f)(x)$. Here we assume x can be any real number.

$$\begin{aligned}
\left(f^{-1} \circ f\right)(x) &= f^{-1}(f(x)) \\
&= f^{-1}\left(2^{x-1} - 3\right) \\
&= \log_2\left(\left[2^{x-1} - 3\right] + 3\right) + 1 \\
&= \log_2\left(2^{x-1}\right) + 1 \\
&= (x - 1) + 1 && \text{since } \log_2(2^u) = u. \\
&= x \checkmark
\end{aligned}$$

   Next, we check $(f \circ f^{-1})(x)$. Here, we restrict our attention to $x > -3$.

$$\begin{aligned}
\left(f \circ f^{-1}\right)(x) &= f\left(f^{-1}(x)\right) \\
&= f\left(\log_2(x+3) + 1\right) \\
&= 2^{\left(\log_2(x+3)+1\right)-1} - 3 \\
&= 2^{\log_2(x+3)} - 3 \\
&= (x + 3) - 3 && \text{since } 2^{\log_2(u)} = u. \\
&= x \checkmark
\end{aligned}$$

5. Last, but certainly not least, we graph $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes and observe the symmetry about the line $y = x$.

Desmos link: https://www.desmos.com/calculator/ac7xyiqiqh

1. Viewing $2^{x-1} - 3 = 4$ as $f(x) = 4$, we apply f^{-1} to 'undo' f to get $f^{-1}(f(x)) = f^{-1}(4)$, which reduces to $x = f^{-1}(4)$. Since we have shown (algebraically and graphically!) that $f^{-1}(x) = \log_2(x+3) + 1$, we get $x = f^{-1}(4) = \log_2(4+3) + 1 = \log_2(7) + 1$.

   Alternatively, we know from Theorem **??** that $f(x) = 4$ is equivalent to $x = f^{-1}(4)$ directly.

   Note that since, by definition, $2^{\log_2(7)} = 7$, $2^{(\log_2(7)+1)-1} - 3 = 2^{\log_2(7)} - 3 = 7 - 3 = 4$, as required.

2. Since we may think of the equation $\log_2(t+3) + 1 = 0$ as $f^{-1}(t) = 0$, we can solve this equation by applying f to both sides to get $f(f^{-1}(t)) = f(0)$ or $t = 2^{0-1} - 3 = \dfrac{1}{2} - 3 = -\dfrac{5}{2}$.

   Since $\log_2(2^{-1}) = -1$, we get $\log_2\left(-\dfrac{5}{2} + 3\right) + 1 = \log_2\left(\dfrac{1}{2}\right) + 1 = \log_2(2^{-1}) - 1 + 1 = 0$, as required.