The Shape of Graphs

We know if f is differentiable at $x = a$ then the graph of f is **locally linear** at $x = a$ and $f'(a)$ is the **slope** of the tangent line at the point $(a, f(a))$. In this section, we explore how local behavior near a point can be extrapolated to global behavior over an interval. First, we review Definition **??** from Section **??**:

Definition. Let f be a function defined on an interval I. Then f is said to be:

- **increasing** on I if, whenever $a < b$, then $f(a) < f(b)$. (i.e., as inputs increase, outputs **increase**.)

  NOTE: The graph of an increasing function **rises** as one moves from left to right.

- **decreasing** on I if, whenever $a < b$, then $f(a) > f(b)$. (i.e., as inputs increase, outputs **decrease**.)

  NOTE: The graph of a decreasing function **falls** as one moves from left to right.

- **constant** on I if $f(a) = f(b)$ for all a, b in I. (i.e., outputs don't change with inputs.)

  NOTE: The graph of a function that is constant over an interval is a horizontal line.

Suppose a function satisfies $f'(x) > 0$ for all x in an open interval[1] I. Then we know that not only is the graph of f locally linear on I, but the slopes of all of the tangent lines are positive. This means that all of the tangent lines are increasing so it stands to reason that the function f is likewise increasing on I. In other words, if a function is **locally** increasing on I, then it is **globally** increasing on I as well.

We can apply the same reasoning above to situations where $f'(x) < 0$ for all x in I, which implies f is decreasing on I or $f'(x) = 0$ on I, which implies f is constant on I. In Calculus, you'll learn this fact is a consequence of the Mean Value Theorem.[2] In this text, we'll just accept the following theorem is true and hope we've done enough hand-waving to deem it reasonable.

Theorem 1. Suppose f is differentiable on an open interval I:

- If $f'(x) > 0$ for all x in I, then f is increasing on I.
- If $f'(x) < 0$ for all x in I, then f is decreasing on I.
- If $f'(x) = 0$ for all x in I, then f is constant on I.

Thanks to GeoGebra, we can visualize Theorem **??**. In the first case below, we have two sections of a graph over which a function is increasing. By adjusting the slider, we can move the point across the graph, observe the corresponding tangent line, and see the value of the slope. Note that in both cases below, the slope of the tangent line, $f'(x)$, is always positive.

GeoGebra link: `https://www.geogebra.org/m/sztmmygz`

$f'(x) > 0$ for all x in I.

Next, we have two sections of a graph over which a function is decreasing. Here, we observe the slopes of the tangent lines are always negative.

GeoGebra link: `https://www.geogebra.org/m/zm2jxnpw`

$f'(x) < 0$ for all x in I.

Last, but not least, we have the graph of a constant function. Here the slopes are always 0.

[1] We've defined derivatives as two-sided limits, so an open interval here guarantees enough 'room' on either side of any given number to take such a limit.

[2] which Carl thinks is the actual 'Fundamental Theorem of Calculus' since it relates local and global behavior . . .

$f'(x) = 0$ for all x in I.

We can use Theorem **??** to help us determine the (open) intervals over which a function f is increasing, decreasing, and constant by making a sign diagram for the derivative f'.

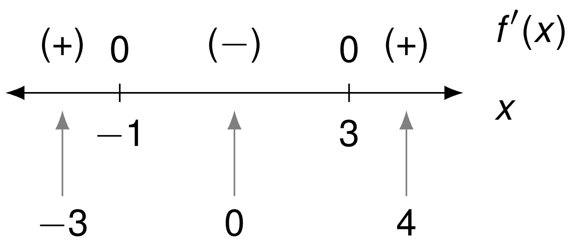
In order to avoid us having to go through the (somewhat lengthy) process of finding $f'(x)$ using Definition **??**, we'll just use some properties of derivatives from Calculus behind the scenes and present you with both a function and its derivative. It's time for an example.

Example 1. Let $f(x) = x^3 - 3x^2 - 9x + 5$. Use the fact that $f'(x) = 3x^2 - 6x - 9$ to find the open intervals over which f is increasing, decreasing, and constant. Check your answer graphically.

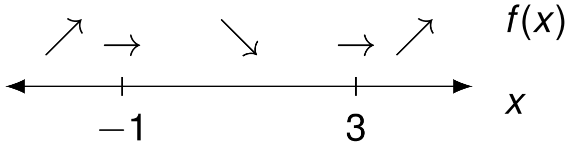
Explanation. To make use of Theorem **??**, we make a sign diagram for $f'(x)$. Since f' is a polynomial, f' is continuous so the per the Intermediate Value Theorem, Theorem **??**, f' will only change sign on either side of zeros. Hence, our first step is to solve $f'(x) = 0$.

We are given $f'(x) = 3x^2-6x-9$. Solving $f'(x) = 3x^2-6x-9 = 0$ gives $3(x^2-2x-3) = 0$ or $3(x-3)(x+1) = 0$. We get two solutions: $x = -1$ and $x = 3$ which divides the x-axis into three regions: $x < -1$, $-1 < x < 3$ and $x > 3$.

Next we select a test value in each of these three regions to determine the sign of $f'(x)$. For the interval $x < -1$, we select $x = -3$: $f'(-3) = 3(-3)^2 - 6(-3) - 9 = (+)$. For $-1 < x < 3$, we select $x = 0$: $f'(0) = 3(0)^2 - 6(0) - 9 = (-)$. Finally, for $x > 3$, we select $x = 4$: $f'(4) = 3(4)^2 - 6(4) - 9 = (+)$.



Next, we use Theorem **??** to interpret what the sign diagram for $f'(x)$ means for the graph of $y = f(x)$:



We find f is increasing on $(-\infty, -1)$ and again on $(3, \infty)$ while f is decreasing on $(-1, 3)$. At the points $x = -1$ and $x = 3$, we have $f'(x) = 0$ so the graph of f is locally flat there.

Since f changes from increasing just to the left of $x = -1$ to decreasing just to the right of $x = -1$, it stands to reason that f has a local maximum at $x = -1$. This is indeed the case and we find that the local maximum value is $f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 5 = 10$.

Similarly, since f changes from decreasing just to the left of $x = 3$ to increasing just to the right of $x = 3$, f has a local minimum at $x = 3$. The local minimum value is $f(3) = (3)^3 - 3(3)^2 - 9(3) + 5 = -22$.

A quick check using desmos confirms our results. Note here the portions of the graph over which f is increasing are highlighted in orange while the interval over which f is decreasing is highlighted in blue.

Desmos link: `https://www.desmos.com/calculator/8jwjfz66ye`

We generalize our observations about local extrema in the following result.

Theorem 2. The (First)[3] Derivative Test for Local Extema: Let f be continuous on an open interval I containing a critical number c.[4] If f is differentiable on I, except possibly at c, then

- If $f'(x)$ changes from (+) for $x < c$ to (−) for $x > c$, f has a local maximum at $x = c$.
- If $f'(x)$ changes from (−) for $x < c$ to (+) for $x > c$, f has a local minimum at $x = c$.
- If $f'(x)$ doesn't change sign going from $x < c$ to $x > c$, f does not have a local extremum at $x = c$.

Example 2. Let $f(x) = x^{4/3} - 4x^{1/3}$. Use the fact that $f'(x) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3}$ to help you find:

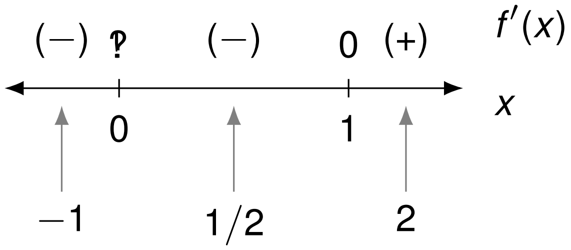
1. the open intervals over which f is increasing, decreasing, and constant.
2. the local extrema.

Explanation. 1. In order to make a sign diagram for $f'(x)$, we rewrite $f'(x)$ as a single fraction:

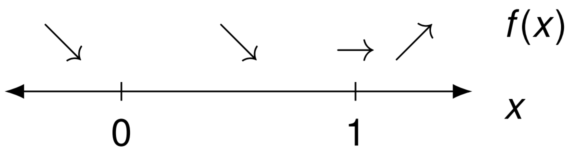
$$f'(x) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} = \frac{4x^{1/3}}{3} - \frac{4}{3x^{2/3}} = \frac{4x^{1/3}}{3} \cdot \frac{x^{2/3}}{x^{2/3}} - \frac{4}{3x^{2/3}} = \frac{4x}{3x^{2/3}} - \frac{4}{3x^{2/3}} = \frac{4x-4}{3x^{2/3}}.$$

Unlike the derivative in Example **??**, $f'(x) = \frac{4x-4}{3x^{2/3}}$ is undefined when $3x^{2/3} = 0$, that is, when $x = 0$, so we need to record this on our sign diagram with the customary '‽.'

Next, we solve $f'(x) = \frac{4x-4}{3x^{2/3}} = 0$ to get $4x - 4 = 0$ or $x = 1$. The usual machinations produces the sign diagram for $f'(x)$ below.



Using Theorem **??**, we get:



[3] Why 'First'? Stay tuned ...
[4] Recall this means $f'(c) = 0$ or $f'(c)$ does not exist.

We get f is decreasing for $x < 0$ as well as from $0 < x < 1$. Since 0 is in the domain of f, we splice the two intervals together so f is decreasing from $(-\infty, 1)$. We see f is increasing from $(1, \infty)$.

2. We note that f satisfies the conditions of Theorem **??** since f is continuous everywhere and f' exists for all $x \neq 0$. Since f changes from decreasing just to the left of $x = 1$ to increasing just to the right of $x = 1$, the graph of f has a local minimum at $x = 1$. The local minimum value is $f(1) = (1)^{4/3} - 4(1)^{1/3} = -3$.

   What is happening at $x = 0$? Since $f'(x)$ doesn't change sign on either side of 0, the graph of f doesn't have a local extremum there. The sign diagram indicates f is decreasing through that point. A quick check using desmos reveals 'unsual steepness' at $x = 0$, a phenomenon which is called a **vertical tangent**. This means the function locally resembles a vertical line.[5]

   We confirm our results with desmos. Once again, the portion of the graph over which f is decreasing is highlighted in blue while the portion of the graph over which f is increasing is highlighted in orange.

   Desmos link: `https://www.desmos.com/calculator/2n3wtmceu5`

1 Concavity and the Second Derivative

In section Section **??**, we introduced the notion of **concavity**. In that section, we described curves as being **concave up** over an interval if it resembles a portion of a '$\smile$' shape and **concave down** over an interval if resembles part of a '$\frown$' shape. Now that we've had some exposure to Calculus, we can more precisely define these notions.

Definition 1. Let f be a differentiable function on an open interval I. Then f is said to be:

- **concave up** on I if the tangent lines lie **below** the graph on I.
- **concave down** on I if the tangent lines lie **above** the graph on I.

If we take the time to study a generic concave up curve, the '$\smile$' shape can be divided into a decreasing and increasing arc. Using the GeoGebra interactive below, we notice that as we move the slider from left to right on the decreasing portion of the curve, the slopes are increasing towards 0.

Moving the slider on the second portion of the graph, we see the slopes are increasing from 0 as we move fron left to right.

GeoGebra link: `https://www.geogebra.org/m/p7wayh3n`

In both of these cases, the **slopes** of the tangent line are **increasing**.

Likewise, we can dissect a generic '$\frown$' shape curve into an increasing and decreasing arc. Here, as we move from left to right on the increasing portion, the slopes are decreasing to 0,

Moving the slider on the second portion of the graph from left to right, we find the slopes continuing to decrease from 0.

GeoGebra link: `https://www.geogebra.org/m/jv4awzcx`

Here, the **slopes** of the tangent line are **decreasing**.

We know from Theorem **??** that the derivative of a function can tell us where that function is increasing and decreasing. Since the function which gives us the slopes of tangent lines is the derivative, $f'(x)$, we could use the derivative of $f'(x)$ to determine where the slopes of the tangent lines were increasing and decreasing. This leads us to define the **second derivative**, $f''(x)$ as the derivative of $f'(x)$.

We present the following theorem without proof, but hopefully sufficiently motivated.

[5] See Example **??** for another such example and discussion.

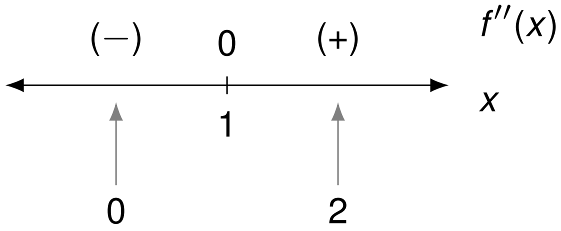
Theorem 3. Suppose f is twice differentiable on an open interval I:

- If $f''(x) > 0$ for all x in I, then **slopes** are **increasing** and f is **concave up** on I.
- If $f''(x) < 0$ for all x in I, then **slopes** are **decreasing** and f is **concave down** on I.

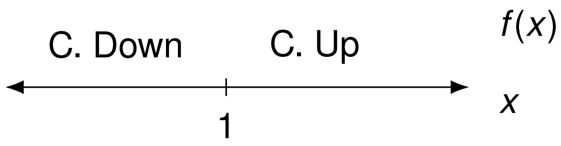
Example 3. Let $f(x) = x^3 - 3x^2 - 9x + 5$. Use the fact that $f''(x) = 6x - 6$ to find the intervals over which the graph of f is concave up and concave down.

Explanation. To analyze the concavity of the graph of f, we need to make a sign diagram for $f''(x)$.

Solving $f''(x) = 6x - 6 = 0$ gives $x = 1$. We find $f''(0) = 6(0) - 6 = (-)$ and $f''(2) = 6(2) - 6 = (+)$. This gives us our sign diagram for $f''(x)$ below.



Using Theroem **??**, we get the following:



We find f is concave down on $(-\infty, 1)$ and concave up on $(1, \infty)$.

At $x = 1$, the concavity changes. We find $f(1) = (1)^3 - 3(1)^2 - 9(1) + 5 = -6$ and we call the point $(1, -6)$ an **inflection point**. In this case since the concavity changes from concave down to concave up, the point $(1, -6)$ is the point on the graph of $y = f(x)$ were the slopes stop decreasing and start to increase.

A quick check using desmos confirms our results. Here the portion of the graph which is concave down is highlighted in blue whereas the portion of the graphh which is concave up is highlighted in orange.

Desmos link: `https://www.desmos.com/calculator/koqauksa1g`

Note that we can use concavity to help us distinguish local extrema.

For the function above, both $f'(-1) = 0$ and $f'(3) = 0$. Note that $f''(-1) < 0$ which means f is concave down there. This forces f to have a local maximum at $(-1, 6)$. Likewise, $f''(3) > 0$ which means f is concave up there. This forces f to have a local minimum at $(3, -22)$. We generalize this observation below.

Theorem 4. The Second[6] Derivative Test for Local Extrema: Suppose f is differentiable on an open interval I containing c and $f'(c) = 0$:

- If $f''(c) > 0$ then f has a local minimum at $x = c$.
- If $f''(c) < 0$ then f has a local maximum at $x = c$.

[6] Now you know why we titled Theorem **??** the 'First' Derivative Test for Local Extrema.

- If $f''(c) = 0$ then the test is inconclusive. f may or may not have a local extremum at $x = c$. (In this case, we would appeal to the first derivative test.)

Example 4. Let $f(x) = x^{4/3} - 4x^{1/3}$. Use the fact that $f''(x) = \frac{4}{9}x^{-2/3} + \frac{8}{9}x^{-5/3}$ to help you find:

1. the open intervals over which the graph of f is concave up and concave down.
2. the inflection points in the graph.

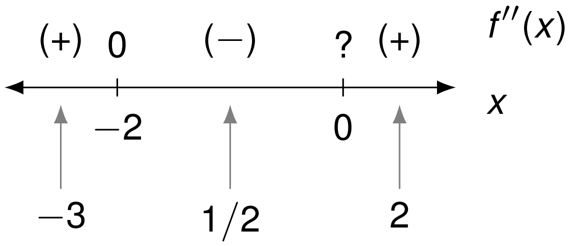
Explanation. 1. As in Example **??**, our first step is to rewrite $f''(x) = \frac{4}{9}x^{-2/3}+\frac{8}{9}x^{-5/3}$ as a single fraction:

$$f''(x) = \frac{4}{9}x^{-2/3} + \frac{8}{9}x^{-5/3} = \frac{4}{9x^{2/3}} + \frac{8}{9x^{5/3}} = \frac{4}{9x^{2/3}} \cdot \frac{x^{3/3}}{x^{3/3}} + \frac{8}{9x^{5/3}} = \frac{4x}{9x^{5/3}} + \frac{8}{9x^{5/3}} = \frac{4x+8}{9x^{5/3}}$$

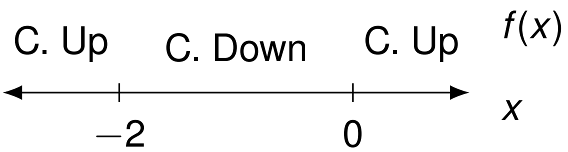
We see $f''(x) = \frac{4x+8}{9x^{5/3}}$ is undefined when $9x^{5/3} = 0$, that is, when $x = 0$.

Solving $f''(x) = \frac{4x+8}{9x^{5/3}} = 0$ gives $4x + 8 = 0$ so $x = -2$.

Going through the usual routine, we obtain our sign diagram for $f''(x)$.



Using Theorem **??** we get:



We see f is concave up on $(-\infty, -2)$ and again from $(0, \infty)$. f is concave down on $(-2, 0)$.

2. Since f changes concavity at both $x = -2$ and $x = 0$, we have inflection point at both of these values.

   We find: $f(-2) = (-2)^{4/3} - 4(-2)^{1/3} = 2(2)^{1/3} + 4(2)^{1/3} = 6(2)^{1/3}$. So $\left(-2, 6(2)^{1/3}\right)$ is one inflection point. When $x = 0$, $f(0) = (0)^{4/3} - 4(0)^{1/3} = 0$, so $(0, 0)$ is the other inflection point.

   Our graph below highlights the concave up portions of the graph in orange and the concave down portion of the graph in blue. While it's not apparent that the graph of $y = f(x)$ is concave up for $x < -2$, we invite the reader to zoom out until that characteristic is more pronounced.

   Desmos link: `https://www.desmos.com/calculator/rc9moeukik`

Our last example offers a twist on these sorts of curve-sketching problems.

Example 5. Below is the graph of the **derivative** of a function. Assume that both $\lim_{x\to-\infty} f'(x) = -\infty$ and $\lim_{x\to\infty} f'(x) = -\infty$.

Desmos link: https://www.desmos.com/calculator/kk5rp0t6nc

1. Use the graph of $y = f'(x)$ to determine the open intervals where f is increasing and decreasing.
   Find the x-coordinates of the local extrema.
2. Use the graph of $y = f'(x)$ make a sign diagram for $y = f''(x)$.
3. List the open intervals over which the graph of f is concave up and concave down.
   Find the x-coordinates of the inflection points.
4. Sketch a possible graph of $y = f(x)$.

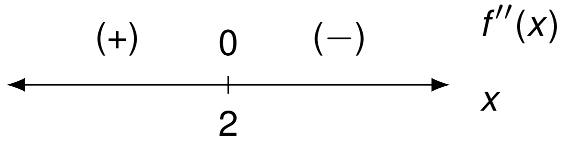
Explanation.

1. Recall from algebra, the solutions to $f'(x) < 0$ are the x-values where the graph of $y = f'(x)$ is below the x-axis. This happens on the intervals $(-\infty, 0)$ and $(4, \infty)$, so this means f is decreasing here.

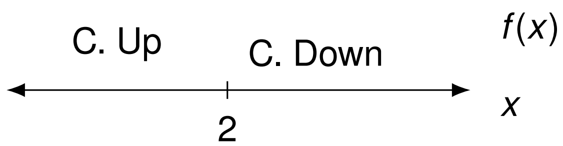
   Likewise, the solutions to $f'(x) > 0$ are the x-values where $y = f'(x)$ is above the x-axis. This happens on the interval $(0, 4)$, so f is increasing here.

   Since f goes from decreasing to the left of $x = 0$ to increasing to the right of $x = 0$, f has a local minimum at $x = 0$. Since f goes from increasing to the left of $x = 4$ to decreasing to the right of $x = 4$, f has a local maximum at $x = 4$.

2. Since $f''(x)$ is the derivative of $f'(x)$, we know $f''(x) > 0$ on $(-\infty, 2)$ since $f'(x)$ is increasing there. We see $f''(2) = 0$ since $f'(x)$ is locally flat at $(2, 4)$. Lastly, we see $f''(x) < 0$ on $(2, \infty)$ since $f'(x)$ is decreasing there. We put all this together in a sign diagram below.

   Hence, we know by Theorem **??**:

3. We have f is concave up on $(-\infty, 2)$ and concave down on $(2, \infty)$.
   Since f changes concavity at $x = 2$, there is an inflection point there.

4. A plausible graph of $y = f(x)$ is below. Note that we cannot determine any y-coordinates (why not?)

   Desmos link: https://www.desmos.com/calculator/idw1bg1sga

2 Exercises