Exercises for The Shape of Graphs

Question 1 *In Exercises* **??** - **??**, *use the given function f and its (first) derivative f' to help you find:*

- *the open intervals over which f is increasing, decreasing, and constant.*
- *the local extrema.*

Check your answers using a graphing utility.

Problem 1.1 $f(x) = 2x^3 - 3x^2 - 12x + 1$, $f'(x) = 6x^2 - 6x - 12$

increasing: $(-\infty, -1)$, $(2, \infty)$

decreasing: $(-1, 2)$

local max: $(-1, 8)$

local min: $(2, -19)$

Problem 1.2 $f(x) = \dfrac{10x}{x^2+1}$, $f'(x) = \dfrac{10-10x^2}{\left(x^2+1\right)^2}$

increasing: $(-1, 1)$

decreasing: $(-\infty, -1)$, $(1, \infty)$

local max: $(1, 5)$

local min: $(-1, -5)$

Problem 1.3 $f(x) = x\sqrt[3]{x-2}$, $f'(x) = \dfrac{4x-6}{3(x-2)^{\frac{2}{3}}}$

increasing: $\left(\dfrac{3}{2}, \infty\right)$

decreasing: $\left(-\infty, \dfrac{3}{2}\right)$

local (absolute) min: $\left(\dfrac{3}{2}, -\dfrac{3}{2\sqrt[3]{2}}\right)$

Question 2 *In Exercises* **??** - **??**, *use the given function f and its second derivative f'' to help you find:*

- *the open intervals over which the graph of f is concave up and concave down.*
- *the inflection points in the graph.*

Check your answers using a graphing utility.

Problem 2.1 $f(x) = 2x^3 - 3x^2 - 12x + 1$, $f''(x) = 12x - 6$

concave up: $\left(\dfrac{1}{2}, \infty\right)$

concave down: $\left(-\infty, \dfrac{1}{2}\right)$

inflection point: $\left(\dfrac{1}{2}, \dfrac{11}{2}\right)$

Problem 2.2 $f(x) = \dfrac{10x}{x^2+1}$, $f''(x) = \dfrac{20x^3-60x}{\left(x^2+1\right)^3}$

concave up: $\left(-\sqrt{3}, 0\right)$, $\left(\sqrt{3}, \infty\right)$

concave down: $\left(-\infty, -\sqrt{3}\right)$, $\left(0, \sqrt{3}\right)$

inflection points: $\left(-\sqrt{3}, -\dfrac{5\sqrt{3}}{2}\right)$, $(0,0)$, $\left(\sqrt{3}, \dfrac{5\sqrt{3}}{2}\right)$

Problem 2.3 $f(x) = x\sqrt[3]{x-2}$, $f''(x) = \dfrac{4(x-3)}{9(x-2)^{\frac{5}{3}}}$

concave up: $(-\infty, 2)$, $(3, \infty)$

concave down: $(2, 3)$

inflection points: $(2, 0)$, $(3, 3)$.

Problem 3 *If $a \neq 0$, we showed in Exercise* **??** *in Section* **??** *that if $f(x) = ax^2 + bx + c$, then $f'(x) = 2ax + b$. Solving $f'(x) = 0$ produced $x = -\dfrac{b}{2a}$, the x-coordinate of the vertex of the parabola $y = f(x)$. This Exercise shows this is part of a pattern.*

1. *If $a \neq 0$, show the x-coordinate of the x-intercept of the graph of $y = ax + b$ is $x = -\dfrac{b}{a} = -\dfrac{b}{1\,a}$.*

   To find the x-intercept, we set $ax + b = 0$ and get $x = -\dfrac{b}{a}$ provided $a \neq 0$.

2. *If $a \neq 0$, for $f(x) = ax^3 + bx^2 + cx + d$ it turns out that $f''(x) = 6ax + 2b$. Show $x = -\dfrac{b}{3a}$ is the x-coordinate of the inflection point of the graph of $y = f(x)$.*

   Solving $f''(x) = 6ax + 2b = 0$, we get $x = -\dfrac{2b}{6a} = -\dfrac{b}{3a}$, provided $a \neq 0$.

   The graph of $f''(x) = 6ax + 2b$ is a line so we know on one side of $x = -\dfrac{b}{3a}$, $f''(x) > 0$ and on the other side, $f''(x) < 0$.

   Hence, the graph of the original function $y = f(x)$ changes concavity at $x = -\dfrac{b}{3a}$.

Problem 4 *In Exercise* **??** *in Section* **??***, we observed that average cost appeared to be minimized when average cost was approximately equal to marginal cost. In this Exercise, we use Calculus and the tools from this section to show this.*

Recall if $C(x)$ is the cost to produce x items, the ***average cost*** *is defined as $\overline{C}(x) = \dfrac{C(x)}{x}$, $x > 0$, is the cost per item.*

1. *It turns out that $\overline{C}'(x) = \dfrac{x\,C'(x) - C(x)}{x^2}$. Show $\overline{C}'(x) = 0$ when $C'(x) = \overline{C}(x)$.*

   To solve $\overline{C}'(x) = \dfrac{x\,C'(x) - C(x)}{x^2} = 0$, we set the numerator, $x\,C'(x) - C(x) = 0$. We get $x\,C'(x) = C(x)$ so $C'(x) = \dfrac{C(x)}{x} = \overline{C}(x)$.

2. *It turns out that $\overline{C}''(x) = \dfrac{x^2\,C''(x) - 2x\,C'(x) + 2C(x)}{x^3}$.*

   Show we can rewrite this as: $\overline{C}''(x) = \dfrac{x\,C''(x) - 2\,C'(x) + 2\overline{C}(x)}{x^2}$.

Divide both numerator and denominator of $\overline{C}''(x) = \dfrac{x^2\,C''(x) - 2x\,C'(x) + 2C(x)}{x^3}$ *by* x*:*

$\overline{C}''(x) = \dfrac{x\,C''(x) - 2\,C'(x) + 2\frac{C(x)}{x}}{x^2}$ *and substitute* $\dfrac{C(x)}{x} = \overline{C}(x)$.

3. *Show that when* $C'(x) = \overline{C}(x)$*, then* $\overline{C}''(x) = \dfrac{C''(x)}{x}$.

   If $C'(x) = \overline{C}(x)$*, then:*

$$\overline{C}''(x) = \frac{x\,C''(x) - 2\,C'(x) + 2\overline{C}(x)}{x^2} = \frac{x\,C''(x) - 2\overline{C}(x) + 2\overline{C}(x)}{x^2} = \frac{x\,C''(x)}{x^2} = \frac{C''(x)}{x}.$$

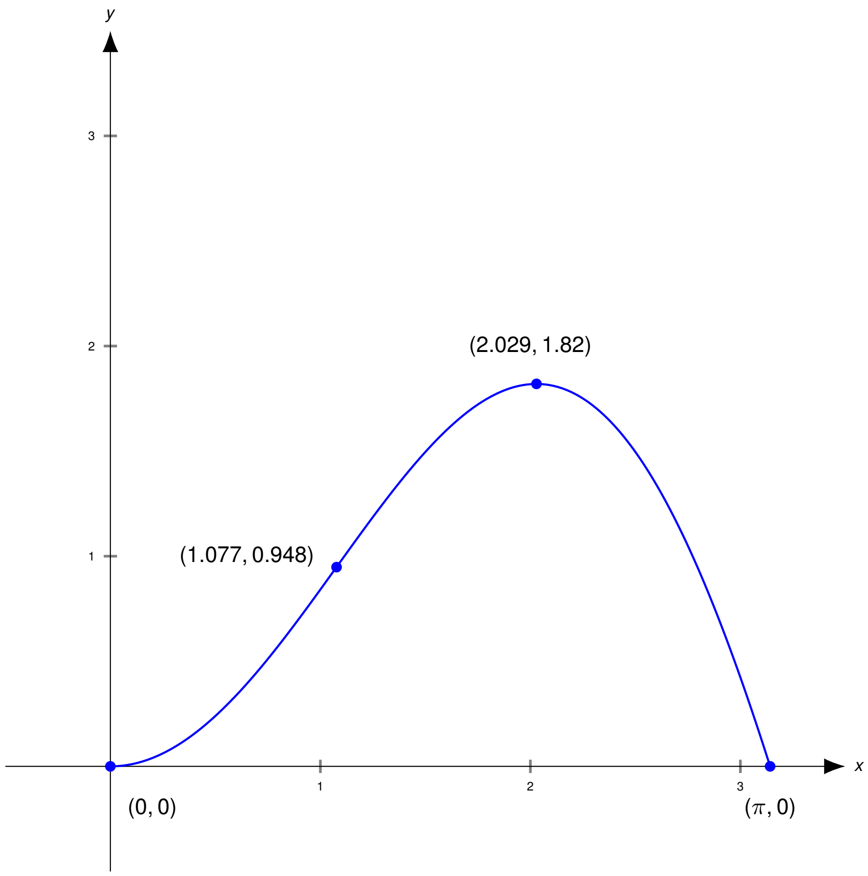
4. *It is usually assumed in most economic settings that for cost functions,* $C''(x) > 0$. *(Can you think of reasons why?) Use this and your results from parts* **??** *and* **??** *to prove that a minimum is produced when* $C'(x) = \overline{C}(x)$.

   NOTE: *In Exercise* **??** *in Section* **??**, *we saw how* $C'(x)$ *can be used to approximate the marginal cost,* $MC(x)$, *so we have established that in order to minimize average cost, we should look where the average cost matches the marginal cost.*

   When $C'(x) = \overline{C}(x)$*, we have that* $\overline{C}'(x) = 0$ *and* $\overline{C}''(x) > 0$. *By the Second Derivative Test for Local Extrema, Theorem* **??**, *the average cost* $\overline{C}(x)$ *has a minimum when* $C'(x) = \overline{C}(x)$

Problem 5 *The complete graph of* $y = f(x)$ *is shown below. Use the graph to answer the following questions.*

NOTE: *Assume* $(2.029, 1.82)$ *is a local maximum and that* $(1.077, 0.948)$ *is an inflection point.*



1. *Determine the x-values where:*

   (i) $f(x) = 0$.
   $x = 0, \pi$

   (ii) $f'(x) = 0$.
   $x = 2.029$.
   NOTE: $f'_+(0) = 0$, *too.*[1]

2. *List the open intervals over which:*

   (i) $f(x) > 0$.
   $(0, \pi)$

   (ii) $f(x) < 0$.
   None.

   (iii) $f'(x) > 0$.
   $(0, 2.029)$

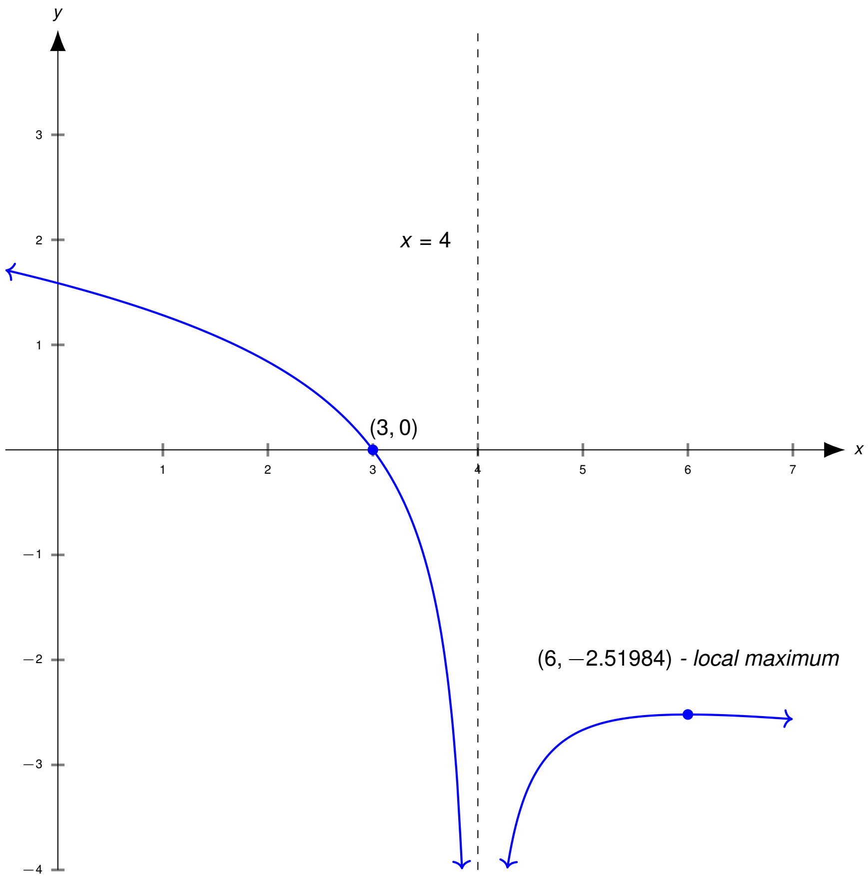
   (iv) $f'(x) < 0$.
   $(2.029, \pi)$

   (v) $f''(x) > 0$.
   $(0, 1.077)$

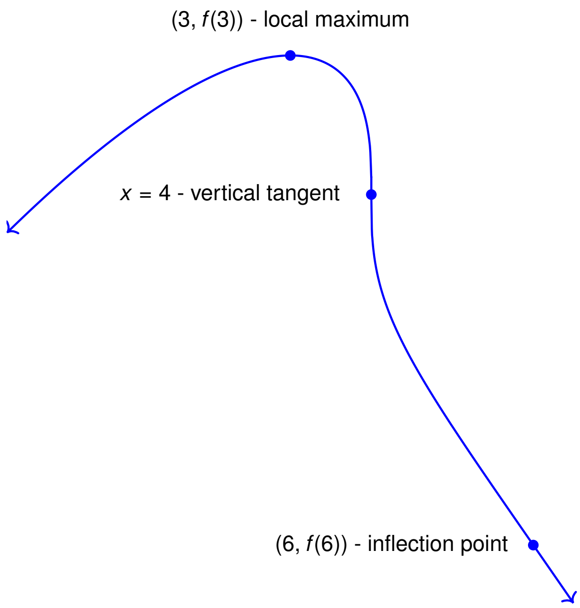
   (vi) $f''(x) < 0$. $(1.077, \pi)$

Problem 6 *Below is the graph of $y = f'(x)$ for a **continuous** function f.*
*Using Example **??** as a guide, sketch a probable graph of $y = f(x)$.*

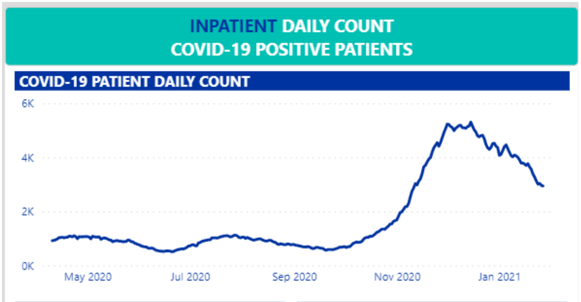
[1] See problem **??** in Section **??**.



Answers vary. Key features:



Problem 7 *The graph below was taken from https://ohiohospitals.org/covid19data on January 28th, 2021:*



With help from your classmate, highlight and label one segment on the graph which (roughly) represents the following scenarios.

For brevity, we'll use 'patients' to mean 'inpatient COVID positive patients' and 'the rate of change' to mean 'the rate of change of inpatient COVID positive patients with respect to time.'

1. *The number of patients is **decreasing** and the rate of change is **decreasing**. (Label this 'a.')*
2. *The number of patients is **decreasing** and the rate of change is **increasing**. (Label this 'b.')*
3. *The number of patients is **increasing** and the rate of change is **increasing**. (Label this 'c.')*
4. *The number of patients is **increasing** and the rate of change is **decreasing**. (Label this 'd.')*
5. *Discuss with your classmates what the phrase 'flatten the curve' could mean in terms of first and second derivatives. (See below for an illustration.)*