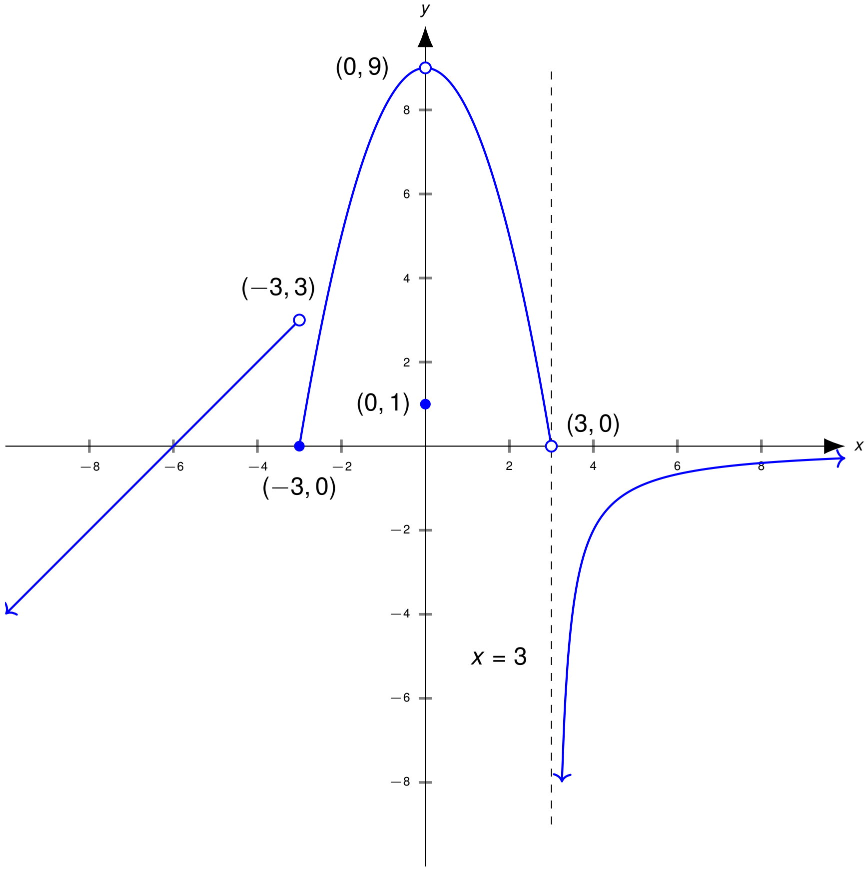
Exercises for Intro Limits

Problem 1 *Consider the complete graph of the function f below. Use the graph to find the indicated values. If a limit fails to exist, state that is the case or use the symbols '∞' or '$-\infty$' appropriately.*

(You can type "infty" to get the '∞' symbol.)

NOTE: *The graph has a vertical asymptote* $x = 3$.



1. $\lim_{x \to -3^-} f(x)$

   $\lim_{x \to -3^-} f(x) = 3$

2. $\lim_{x \to -3^+} f(x) = 0$

3. $\lim_{x \to -3} f(x)$

   $\lim_{x \to -3} f(x)$ *does not exist. (d.n.e.)*

4. $f(-3) = 0$

5. $\lim_{x \to 0} f(x)$

   $\lim_{x \to 0} f(x) = 9$

6. $f(0) = 1$

7. $\lim_{x \to 3^-} f(x) = 0$

8. $\lim_{x\to 3^+} f(x)$

   $\lim_{x\to 3^+} f(x) = -\infty$

Problem 2 1. *Explain why if* $\lim_{x\to a} f(x)$ *exist, then so do* $\lim_{x\to a^-} f(x)$ *and* $\lim_{x\to a^+} f(x)$.

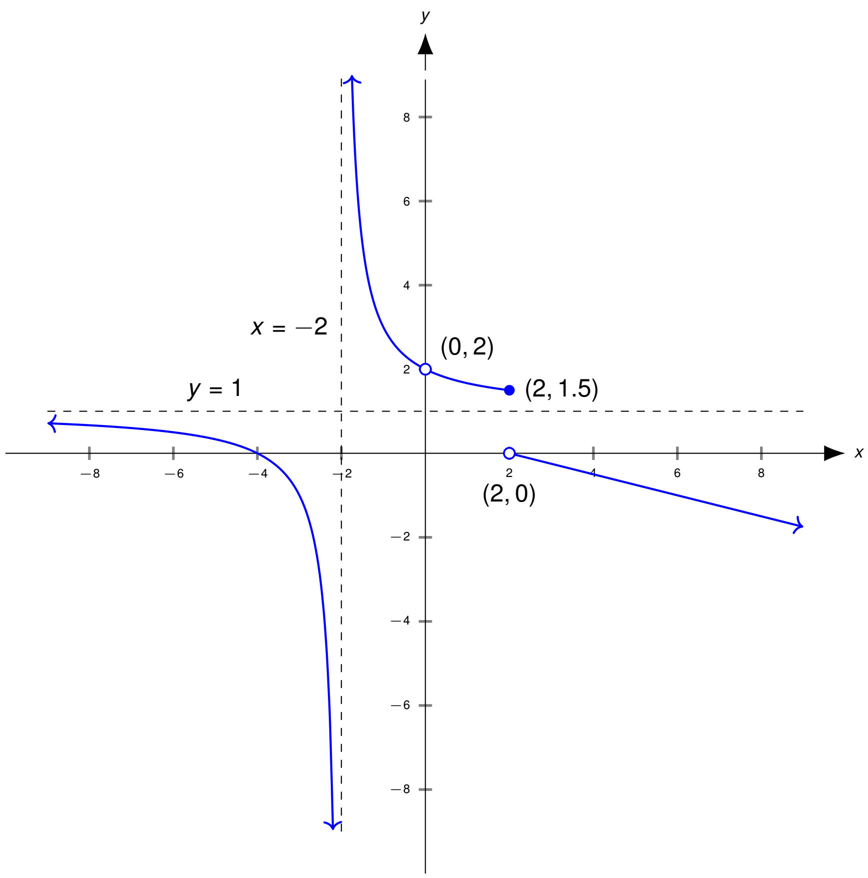
If $\lim_{x\to a} f(x)$ *exists, say* $\lim_{x\to a} f(x) = L$ *then the* $f(x)$ *values approach* L *as* $x \to a$ *from both directions. Hence both one-sided limits exist. In particular,* $\lim_{x\to a^-} f(x) = L$ *and* $\lim_{x\to a^+} f(x) = L$.

2. *Find an instance*[1] *where* $\lim_{x\to a^-} f(x)$ *and* $\lim_{x\to a^+} f(x)$ *both exist but* $\lim_{x\to a} f(x)$ *does not.*

   In Example **??**, $\lim_{x\to -1^-} f(x) = 0$ *and* $\lim_{x\to -1^+} f(x) = 4$ *both exist but* $\lim_{x\to -1} f(x)$ *does not because the two one-sided limits are not equal.*

Problem 3 *Consider the complete graph of the function g below. Use the graph to find the indicated values. If a limit fails to exist, state that is the case or use the symbols '∞' or '$-\infty$' appropriately.*

NOTE: *The graph has a vertical asymptote* $x = -2$ *and a horizontal asymptote* $y = 1$.



1. $\lim_{x\to -\infty} g(x)$

   $\lim_{x\to -\infty} g(x) = 1$

2. $\lim_{x\to -2^-} g(x) = -\infty$

[1] There are a few examples to be found in Example **??** . . .

3. $\lim_{x\to -2^+} g(x)$

   $\lim_{x\to -2^+} g(x) = \infty$

4. $\lim_{x\to \infty} g(x) = -\infty$

5. $\lim_{x\to 0} g(x)$

   $\lim_{x\to 0} g(x) = 2$

6. $\lim_{x\to 2^-} g(x) = 1.5$

7. $g(2)$

   $g(2) = 1.5$

8. $\lim_{x\to 2^+} g(x) = 0$

Question 4 *For Exercises **??** - **??**, find the limit analytically using Exercise **??** as a guide. If a limit fails to exist, state that is the case or use the symbols '∞' or '$-\infty$' appropriately.*

Problem 4.1 $\lim_{x\to 2} \frac{2x^2+x-3}{x^2-1}$

$\lim_{x\to 2} \frac{2x^2+x-3}{x^2-1} = \frac{7}{3}$

Problem 4.2 $\lim_{x\to 1} \frac{2x^2+x-3}{x^2-1} = \frac{5}{2}$

Problem 4.3 $\lim_{x\to -1} \frac{2x^2+x-3}{x^2-1}$

$\lim_{x\to -1} \frac{2x^2+x-3}{x^2-1}$ *does not exist (d.n.e.)*

Problem 4.4 $\lim_{x\to -1^+} \frac{2x^2+x-3}{x^2-1} = \infty$

Problem 4.5 $\lim_{x\to 3} \frac{x^2-3x}{x^2-x-6}$

$\lim_{x\to 3} \frac{x^2-3x}{x^2-x-6} = \frac{3}{5}$

Problem 4.6 $\lim_{x\to 3} \frac{x^2-6x}{x^2-6x+9} = -\infty$

Question 5 *For Exercises **??** - **??**, use the piecewise definition of absolute value, Definition **??** to help you find the limit analytically. If a limit fails to exist, state that is the case or use the symbols '∞' or '$-\infty$' appropriately.*

Problem 5.1 $\lim_{x\to 3^-} \frac{|3x-x^2|}{x-3} = -3$

Problem 5.2 $\lim_{x\to 2^+} \frac{|6-3x|}{x^2-4x+4}$

$\lim_{x\to 2^+} \frac{|6-3x|}{x^2-4x+4} = \infty$

Question 6 *For Exercises* **??** - **??***, simplify the complex fraction in order to help you find the limit analytically.*[2] *If a limit fails to exist, state that is the case or use the symbols '∞' or '$-\infty$' appropriately.*

Problem 6.1 $\lim\limits_{x\to 1} \dfrac{\frac{x}{x-2}+1}{x-1}$

$$\lim_{x\to 1} \frac{\frac{x}{x-2}+1}{x-1} = -2$$

Problem 6.2 $\lim\limits_{x\to 2} \dfrac{\frac{2x}{x+2}-1}{x-2} = \dfrac{1}{4}$

Problem 6.3 $\lim\limits_{h\to 0} \dfrac{\frac{1}{2(x+h)-1}-\frac{1}{2x-1}}{h}$

$$\lim_{h\to 0} \frac{\frac{1}{2(x+h)-1}-\frac{1}{2x-1}}{h} = -\frac{2}{(2x-1)^2}$$

Question 7 *For Exercises* **??** - **??***, rationalize the numerator of the fraction in order to help you find the limit analytically.*[3] *If a limit fails to exist, state that is the case or use the symbols '∞' or '$-\infty$' appropriately.*

Problem 7.1 $\lim\limits_{x\to -1} \dfrac{\sqrt{x+5}-2}{x+1}$

$$\lim_{x\to -1} \frac{\sqrt{x+5}-2}{x+1} = \frac{1}{4}$$

Problem 7.2 $\lim\limits_{h\to 0} \dfrac{\sqrt{4+h}-2}{h} = \dfrac{1}{4}$

Problem 7.3 $\lim\limits_{h\to 0} \dfrac{\sqrt{2x+2h-1}-\sqrt{2x-1}}{h}$

$$\lim_{h\to 0} \frac{\sqrt{2x+2h-1}-\sqrt{2x-1}}{h} = \frac{1}{\sqrt{2x-1}}$$

Question 8 *In Exercises* **??** - **??***, find the limit analytically. Use the symbols '∞' and '$-\infty$' as appropriate.*

Problem 8.1 $\lim\limits_{x\to \infty} \dfrac{3x-4}{2x+1}$

$$\lim_{x\to \infty} \frac{3x-4}{2x+1} = \frac{3}{2}$$

Problem 8.2 $\lim\limits_{x\to -\infty} \dfrac{1-2x}{x-5} = -2$

Problem 8.3 $\lim\limits_{x\to -\infty} \dfrac{2x-1}{x^2+4}$

$$\lim_{x\to -\infty} \frac{2x-1}{x^2+4} = 0$$

Problem 8.4 $\lim\limits_{x\to \infty} \dfrac{\sqrt{4x^2+x-1}}{1-x} = -2$

[2] A review of Section **??** may be in order.
[3] A review of Section **??** may be in order.

Problem 8.5 $\lim_{x\to-\infty} \frac{\sqrt{4x^2+x-1}}{1-x}$

$\lim_{x\to-\infty} \frac{\sqrt{4x^2+x-1}}{1-x} = 2$

Problem 8.6 $\lim_{x\to\infty} \frac{x^2+2x+3}{4-x} = -\infty$.

Problem 9 *Let* $f(x) = \frac{x}{\lfloor x \rfloor}$ *where '*$\lfloor x \rfloor$*' is the greatest integer (or floor) function.*[4]

Fill in the blanks below to help you analyze $\lim_{x\to 0} f(x)$.

1. *If* $-1 < x < 0$*, then* $\lfloor x \rfloor = -1$*. So we can rewrite* $f(x) = \frac{x}{\lfloor x \rfloor} = -x$.
2. *Using part (a), we can find* $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (-x) = 0$.
3. *If* $0 < x < 1$*, then* $\lfloor x \rfloor = 0$*, hence* $f(x) = \frac{x}{\lfloor x \rfloor}$ *does not exist as* $x \to 0^+$.
4. *Putting parts (b) and (c) together, we have that* $\lim_{x\to 0} f(x)$ *does not exist.*
5. *Graph* $f(x) = \frac{x}{\lfloor x \rfloor}$ *on desmos near* $x = 0$ *to confirm your answers.*

Problem 10 *With help from your classmates, sketch the graph of a function which satisfies all of the following criteria:*

- $\lim_{x\to-\infty} f(x) = \infty$
- $\lim_{x\to 4^-} f(x) = 6$
- $\lim_{x\to 4^+} f(x) = -\infty$
- $\lim_{x\to\infty} f(x) = 0$

Problem 11 *With help from your classmates, sketch the graph of a function* f *which satisfies all of the following criteria:*

- $\lim_{x\to-\infty} f(x) = 2$
- $\lim_{x\to 0^-} f(x) = \infty$
- $\lim_{x\to 0^+} f(x) = -\infty$
- $\lim_{x\to 2^-} f(x) = 3$
- $\lim_{x\to 2^+} f(x) = 0$
- $\lim_{x\to\infty} f(x) = -\infty$

[4] See Example **??** in Section **??** for review, if needed.

Problem 12 *A function is said to be* ***continuous from the left*** *at $x = a$ if $\lim_{x\to a^-} f(x) = f(a)$. Likewise, a function is said to be* ***continuous from the right*** *at $x = a$ if $\lim_{x\to a^+} f(x) = f(a)$.*

1. *Explain why $r(x) = \sqrt{5-x}$ is not continuous at $x = 5$. Is r continuous from the left at $x = 5$? From the right? Explain.*

   The function $r(x) = \sqrt{5-x}$ is not continuous at $x = 5$ since r is undefined if $x > 5$, so $\lim_{x\to 5^+} r(x)$ does not exist. However, r is continuous from the left at $x = 5$ since

   $$\lim_{x\to 5^-} r(x) = \lim_{x\to 5^-} \sqrt{5-x} = 0 = \sqrt{5-5} = r(5).$$

2. *Explain why the floor function[5] $F(x) = \lfloor x \rfloor$ is not continuous at $x = 117$. Is F continuous from the left at $x = 117$? From the right? Explain.*

   The function $F(x) = \lfloor x \rfloor$ is not continuous at $x = 117$ since $\lim_{x\to 117^-} \lfloor x \rfloor = 116$ but $\lim_{x\to 117^+} \lfloor x \rfloor = 117$. However, F is continuous from the right at $x = 117$ since $\lim_{x\to 117^+} \lfloor x \rfloor = 117 = \lfloor 117 \rfloor = F(117)$.

3. *If a function f is continuous at $x = a$, explain why f is continuous from both the left and the right at $x = a$. Is the converse true? That is, if f is continuous from both the left and the right at $x = a$, is f continuous at $x = a$?*

   If f is continuous at $x = a$, then $\lim_{x\to a} f(x) = f(a)$. This means $\lim_{x\to a^-} f(x) = f(a)$ and $\lim_{x\to a^+} f(x) = f(a)$, so f is continuous from both directions at $x = a$. The converse is also true since if $\lim_{x\to a^-} f(x) = f(a)$ and $\lim_{x\to a^+} f(x) = f(a)$, then $\lim_{x\to a} f(x) = f(a)$.

4. *Compare and contrast your answers in this exercise to those in Exercise* **??**.

   The difference between the scenario here and that in Exercise **??** *is that here, we know what each of the one-sided limits are: $f(a)$. They can't be different numbers like they could be in Example* **??**.

Problem 13 *Consider the table of values below:*

x	$f(x)$
-0.001	1.9
-0.0001	1.99
-0.00001	1.999
-0.000001	1.9999
0.000001	-10000
0.00001	-1000
0.0001	-100
0.001	-10

It turns out that $\lim_{x\to 0} f(x) = 117$. How is this possible assuming the data in the table is correct?

We are told nothing of the function on the interval $(-0.000001, 0.000001)$ so the function has plenty of opportunities to approach 117.

Problem 14 *In this exercise, we use Definition* **??** *to prove $\lim_{x\to\infty} x^3 = \infty$ and $\lim_{x\to-\infty} x^3 = -\infty$:*

1. *Solve the following inequalities:*

[5] See Example **??** in Section **??** for review, if needed.

- $x^3 > 1000$
  $x^3 > 1000$ *for* $x > 10$
- $x^3 > 100000$
  $x^3 > 1000000$ *for* $x > 100$
- $x^3 > 10^{99}$
  $x^3 > 10^{99}$ *for* $x > 10^{33}$
- $x^3 > N$
  $x^3 > N$ *for* $x > \sqrt[3]{N}$

2. *Show that for* $N > 0$*, if* $x > \sqrt[3]{N}$*, then* $x^3 > N$*. Write a sentence (or two!) which uses your work and Definition* **??** *to prove* $\lim_{x\to\infty} x^3 = \infty$*.*

   If $x > \sqrt[3]{N}$*, then* $x^3 > \left(\sqrt[3]{N}\right)^3 = N$*. Per Definition* **??***, given* $N > 0$*, choose* $M = \sqrt[3]{N}$*. If* $x > M$*, then* $x^3 > M^3 = N$*. Hence,* $\lim_{x\to\infty} x^3 = \infty$*.*

3. *Repeat a similar argument to prove* $\lim_{x\to-\infty} x^3 = -\infty$*.*

   Given $N < 0$*, choose* $M = \sqrt[3]{N}$*. If* $x < M$*, then* $x^3 < M^3 = \left(\sqrt[3]{N}\right)^3 = N$*. Hence,* $\lim_{x\to-\infty} x^3 = -\infty$*.*