1 Average and Instantaneous Velocity, Revisited

We begin this section by revisiting (again!) the notion of **average velocity** - a concept we first encountered in Example **??** in Section **??** and later revisited in Example **??** in Section **??**.

In this scenario, the position function[1] $s(t) = -5t^2 + 100t$, $0 \leq t \leq 20$ gives the height of a model rocket above the Moon's surface, in feet, t seconds after liftoff. The average rate of change of s over an interval is the **average velocity** of the rocket over that interval. The average velocity provides two pieces of information: the average speed of the rocket along with the rocket's direction. We formalized the average velocity in Definition **??** in Section **??**:

Definition. Let $s(t)$ be the position of an object at time t and t_0 a fixed time in the domain of s. The **average velocity** between time t and time t_0 for $t \neq t_0$ is given by

$$\overline{v}(t) = \frac{\Delta[s(t)]}{\Delta t} = \frac{s(t) - s(t_0)}{t - t_0}.$$

If we define the change in time, $\Delta t = t - t_0$, we get $t = t_0 + \Delta t$ which gives:

$$\overline{v}(\Delta t) = \frac{\Delta[s(t)]}{\Delta t} = \frac{s(t_0 + \Delta t) - s(t_0)}{\Delta t}, \quad \Delta t \neq 0.$$

The above formula measures the average velocity between time t_0 and time $t_0 + \Delta t$ as a function of Δt.

We now revisit Example **??** in Section **??** using this new formulation.[2]

Example 1. Let $s(t) = -5t^2 + 100t$, $0 \leq t \leq 20$ give the height of a model rocket above the Moon's surface, in feet, t seconds after liftoff.

1. Find, and simplify: $\overline{v}(\Delta t) = \dfrac{s(15 + \Delta t) - s(15)}{\Delta t}$, for $\Delta t \neq 0$.
2. Find and interpret $\overline{v}(-1)$.
3. Find and interpret $\lim\limits_{\Delta t \to 0} \overline{v}(\Delta t)$.
4. Graph $y = \overline{v}(\Delta t)$ and interpret your answer to part **??** graphically.

Explanation. 1. To find $\overline{v}(\Delta t)$, we first find $s(15 + \Delta t)$:

$$\begin{aligned} s(15 + \Delta t) &= -5(15 + \Delta t)^2 + 100(15 + \Delta t) \\ &= -5(225 + 30\Delta t + (\Delta t)^2) + 1500 + 100\Delta t \\ &= -5(\Delta t)^2 - 50\Delta t + 375 \end{aligned}$$

Since $s(15) = -5(15)^2 + 100(15) = 375$, we get:

$$\begin{aligned} \overline{v}(\Delta t) &= \frac{s(15 + \Delta t) - s(15)}{\Delta t} \\ &= \frac{(-5(\Delta t)^2 - 50\Delta t + 375) - 375}{\Delta t} \\ &= \frac{\Delta t(-5\Delta t - 50)}{\Delta t} \\ &= \frac{\cancel{\Delta t}(-5\Delta t - 50)}{\cancel{\Delta t}} \\ &= -5\Delta t - 50 \qquad\qquad \Delta t \neq 0 \end{aligned}$$

In addition to $\Delta t \neq 0$, the domain of s is restricted to $0 \leq t \leq 20$. Hence, we require $0 \leq 15 + \Delta t \leq 20$ or $-15 \leq \Delta t \leq 5$. Our final answer is $\overline{v}(\Delta t) = -5\Delta t - 50$, for $\Delta t \in [-15, 0) \cup (0, 5]$.

[1] So named because $s(t)$ provides information about **where** the rocket is at time t.

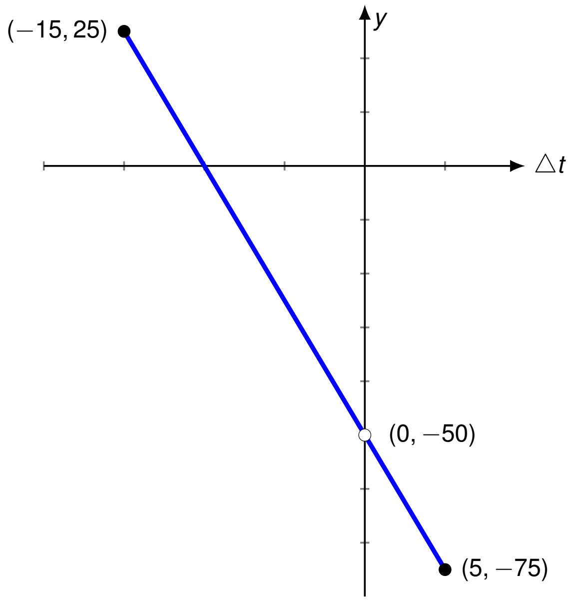
[2] Along with some of the new tools we learned in Section **??**.

2. We find $\overline{v}(-1) = -5(-1) - 50 = -45$. This means the average velocity over between time $t = 15 + (-1) = 14$ seconds and $t = 15$ seconds is -45 feet per second. This indicates the rocket is, on average, heading *downwards* at a rate of 45 feet per second.

3. Since $\overline{v}(\Delta t) = -5\Delta t - 50$, for all values of Δt near $\Delta t = 0$ (excluding $\Delta t = 0$), Theorem **??** applies. We get $\lim\limits_{\Delta t \to 0} \overline{v}(\Delta t) = \lim\limits_{\Delta t \to 0} -5\Delta t - 50 = -5(0) - 50 = -50$, where we have used the fact that the function $f(\Delta t) = -5\Delta t - 50$ is continuous to evaluate the limit.

   Recall from Example **??** that the limit value here, -50 is the so-called *instantaneous velocity* of the rocket *at* $t = 15$ seconds. That is, 15 seconds after lift-off, the rocket is heading back towards the surface of the moon at a rate of 50 feet per second.

4. Since the domain of $\overline{v}$ is $[-15, 0) \cup (0, 5]$, the graph of $y = \overline{v}(\Delta t) = -5\Delta t - 50$ is a line **segment** from $(-15, 25)$ to $(5, -75)$ with a hole at $(0, -50)$.



The reader is invited to compare Example **??** in Section **??** with Exercise **??** above. We obtain the exact same information because we are asking the *exact same* questions - they are just framed differently. We now take the time to formally define **instantaneous velocity**:

Definition 1. Let $s(t)$ be the position of an object at time t and t_0 a fixed time in the domain of s. The **instantaneous velocity** at t_0 is given by:

$$v(t_0) = \lim_{\Delta t \to 0} \frac{\Delta[s(t)]}{\Delta t} = \lim_{\Delta t \to 0} \frac{s(t_0 + \Delta t) - s(t_0)}{\Delta t},$$

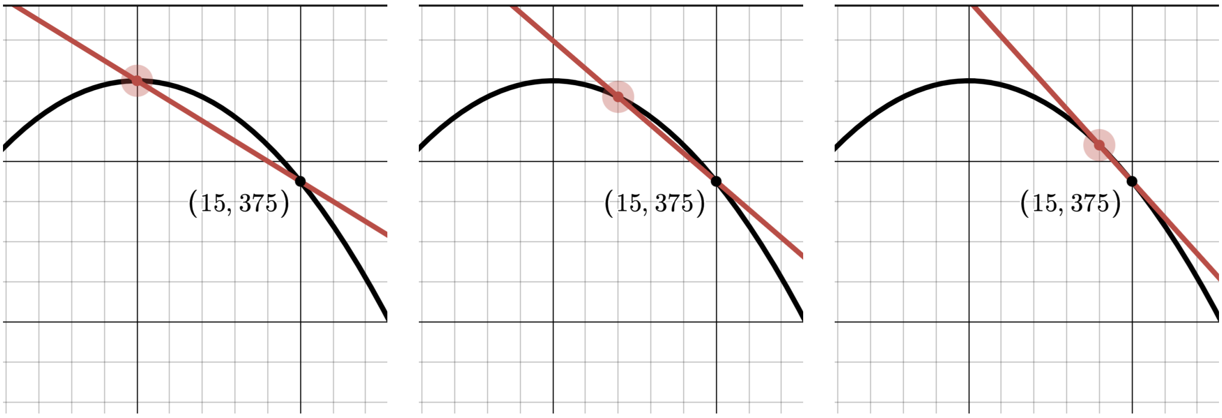
provided this limit exists.

Based on our work in Examples **??** and **??**, we have $v(15) = -50$. In both of those examples, we've seen what $v(15)$ means on the graph of $\overline{v}$, but there is a more important interpretation when we analyze the graph of s. Recall that the average velocity, and, more generally, average rates of change can be visualized as slopes of **secant lines**.[3]

Below is a sequence of secant lines along with the graph of $y = s(t)$. In each case, the secant line is graphed between $(15, s(15)) = (15, 375)$ and another point on the graph.[4] As the points on the parabola approach $(15, 375)$ the secant lines approach what is known as the **tangent line**.

[3] See Section **??** for a refresher, if needed.

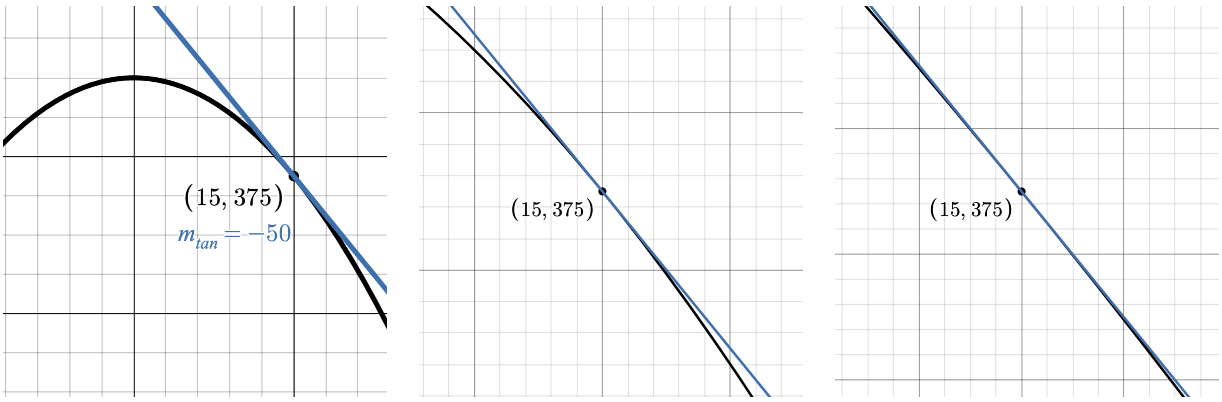
[4] For an interactive demonstration of this process, check out this desmos worksheet.



To find the equation of the tangent line in this case, which we'll call $L(t)$, we refer to the point-slope form of a line, Equation **??**:

$$L(t) = s(t_0) + m(t - t_0) = s(15) + v(15)(t - 15) = 375 - 50(t - 15) = -50t + 1125.$$

The tangent line can best be thought of as 'the best linear approximation' to the graph of $y = s(t)$ at $(15, 375)$. That is, if we zoom in near $(15, 375)$, the graph of $y = s(t)$ and this tangent line become indistinguishable. This geometric property of $y = s(t)$ is called **local linearity**[5] and is foundational to the analysis of functions. Below we graph $y = s(t) = -5t^2 + 100t$ along with $y = L(t) = -50t + 1125$ and observe the local linearity near $(15, 375)$.



The tangent line at $(15, 375)$. Zooming in near $(15, 375)$. Zooming in closer to $(15, 375)$.

Using desmos, we can more smoothly[6] visualize this process using the interactive below. By dragging the highlighted point along the graph of $y = s(t)$ closer to $(15, 375)$, we can observe the secant lines approaching the tangent line.

Desmos link: `https://www.desmos.com/calculator/ht4ljf0bsv`

Our next step is to generalize these notions to all functions.

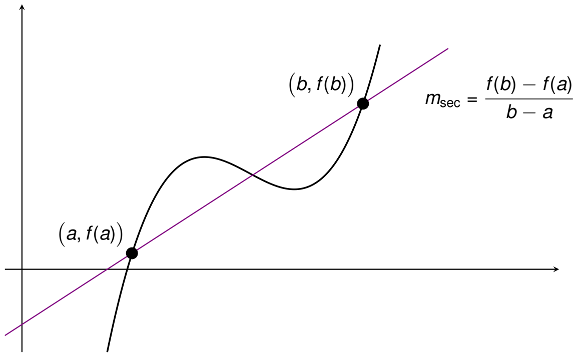
[5] a.k.a. **differentiability**
[6] math pun!

2 Difference Quotients and Derivatives

Recall in Section **??** the concept of the average rate of change of a function over the interval $[a, b]$ is the slope between the two points $(a, f(a))$ and $(b, f(b))$ and is given by

$$\frac{\Delta[f(x)]}{\Delta x} = \frac{f(b) - f(a)}{b - a}.$$

Geometrically, the average rate of change is the slope of the so-called **secant line** which 'cuts' through the graph of $y = f(x)$ at the points $(a, f(a))$ and $(b, f(b))$:



Consider a function f defined over an interval containing x and $x + h$ where $h \neq 0$. The average rate of change of f over the interval $[x, x + h]$ is thus given by the formula:[7]

$$\frac{\Delta[f(x)]}{\Delta x} = \frac{f(x + h) - f(x)}{h}, \quad h \neq 0.$$

The above is an example of what is traditionally called the **difference quotient** or **Newton quotient** of f, since it is the *quotient* of two *differences*, namely $\Delta[f(x)]$ and Δx. Another formula for the difference quotient (as seen in Section **??**) keeps with the notation Δx instead of h:

$$\frac{\Delta[f(x)]}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}, \quad \Delta x \neq 0.$$

It is important to understand that in this formulation of the difference quotient, the variables 'x' and 'Δx' are distinct - that is they do not combine as like terms.

Note that, regardless of which form the difference quotient takes, when h, Δx, or Δt is 0, the difference quotient returns the indeterminate form '$\frac{0}{0}$.' As we've seen with rational functions in Section **??**, when this happens, we can use a limit to help us **determine** the **indeterminate** form.

In Section **??**, taking the limit of average velocity as $\Delta t \to 0$ produced instantaneous velocity. More generally, taking the limit of the average rate of change as the denominator approaches 0 produces the **instantaneous rate of change** of the function at that point. The instantaneous rate of change of a function, called the **derivative** of the function, is defined below.

Definition 2. Given a function f defined on an open interval containing $x = a$, the **derivative** of f at a, denoted $f'(a)$, is defined as

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}, \quad \text{provided this limit exists.}$$

The number $f'(a)$ represents the **instantaneous rate of change** of f with respect to x at the input $x = a$. If $f'(a)$ exists, we say f is **differentiable** at $x = a$.

[7] assuming $h > 0$; otherwise, the interval is $[x + h, x]$. We get the same formula for the difference quotient either way.

Using the language of derivatives, Examples **??** and **??** have us computing $v(15) = s'(15)$. Moreover, since the derivative is a rate of change, it's important to note that the associated units of $f'(a)$ are $\dfrac{\text{units of } f(x)}{\text{units of } x}$. This tracks with the units of $v(15) = s'(15)$ being $\dfrac{\text{feet}}{\text{second}}$, a velocity.

As in Section **??**, $f'(a)$ represents the slope of the tangent line at the point $(a, f(a))$. We use this to formally define the **tangent line** below.

Definition 3. If f is differentiable at $x = a$, then $f'(a) = m_{\tan}$, the slope of the **tangent linetangent line** to $y = f(x)$ at $(a, f(a))$. The equation of the tangent line is therefore: $y = f'(a)(x - a) + f(a)$.

We put these definitions to good use in the following example.

Example 2. Let $f(x) = -x^2 + 3x - 1$.

1. Find the equation of the tangent line to $y = f(x)$ at $x = -2$. Check your answer graphically.
2. If f represents the temperature (in degrees Celsius) x hours after Noon on a particular day, interpret $f'(-2)$ in terms of time and temperature.

Explanation. 1. We first find $m_{\tan} = f'(-2)$ using Definition **??** with $a = -2$: $f'(-2) = \lim\limits_{h \to 0} \dfrac{f(-2+h) - f(-2)}{h}$.

First we find $f(-2 + h)$ and are careful to apply the exponent in the expression $-(-2 + h)^2$:

$$\begin{aligned} f(-2+h) &= -(-2+h)^2 + 3(-2+h) - 1 \\ &= -(4 - 4h + h^2) - 6 + 3h - 1 \\ &= -4 + 4h - h^2 - 6 + 3h - 1 \\ &= -h^2 + 7h - 11 \end{aligned}$$

Next, we find $f(-2) = -(-2)^2 + 3(-2) - 1 = -11$, so the difference quotient is:

$$\begin{aligned} \frac{f(-2+h) - f(-2)}{h} &= \frac{(-h^2 + 7h - 11) - (-11)}{h} \\ &= \frac{-h^2 + 7h}{h} && \text{simplify} \\ &= \frac{h(-h+7)}{h} && \text{factor} \\ &= \frac{\cancel{h}(-h+7)}{\cancel{h}} && \text{cancel} \\ &= -h + 7 \end{aligned}$$

Finally, we get $f'(-2) = \lim\limits_{h \to 0} \dfrac{f(-2+h) - f(-2)}{h} = \lim\limits_{h \to 0} (-h + 7) = -(0) + 7 = 7$.

Hence, the slope of the tangent line is $m_{\tan} = f'(-2) = 7$. The equation of the tangent line is:

$$\begin{aligned} y &= f'(-2)(x - (-2)) + f(-2) \\ &= 7(x + 2) - 11 \\ &= 7x + 14 - 11 \\ y &= 7x + 3 \end{aligned}$$

Graphing $y = 7x + 3$ and $y = f(x)$ near $(-2, -11)$ reveals the local linearity we would expect. Using the desmos graph below, we can zoom in on the point $(-2, -11)$ to our heart's content until the graph of $y = f(x)$ and the tangent line become indistinguishable.

Desmos link: `https://www.desmos.com/calculator/jhyxrweogw`

2. Since x represents the number of hours **after** Noon, $x = -2$ corresponds to 2 hours **before** Noon, or 10 AM. Since the units of $f(x)$ are degrees Celsius and the units of x are hours, the units of $f'(-2)$ are degrees Celsius per hour. Since $f'(-2)$ is positive, we know the slope is positive, so the temperature is increasing. Putting all this together, $f'(-2) = 7$ means that at 10 AM, the temperature is rising at a rate of 7 degrees Celsius per hour.

What if we wanted to find the equation of the tangent line to the graph of the function in Example **??** at $x = 0$? $x = 1$? $x = 5$? We'd ostensibly need to run through difference quotients and limit calculations for each and every input value: $x = 0$, $x = 1$, and $x = 5$. Or we could do a single limit with a generic 'x', simplify the difference quotient and take the limit once, and substitute in particular values of x:

Definition 4. Given a function f defined on an open interval, the **derivative** of f, denoted $f'(x)$, is the function

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

It is worth noting that if we set $h = \Delta x$, and consider the graph $y = f(x)$, we get:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x\to 0} \frac{\Delta[f(x)]}{\Delta x} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x},$$

which is why sometimes[8] the derivative is denoted $\dfrac{dy}{dx}$.

Example 3. Let $f(x) = -x^2 + 3x - 1$.

1. Find an expression for $f'(x)$.
2. Find $f'(-2)$ using your answer to part **??** and compare that to what you obtained in Example **??**.
3. Find the equation of the tangent line to the graph of $y = f(x)$ at $x = 0$. Check your answer graphically.
4. Solve $f'(x) = 0$ and interpret your answer graphically.

Explanation. 1. To find $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$ we first find $f(x+h)$:

$$\begin{aligned} f(x+h) &= -(x+h)^2 + 3(x+h) - 1 \\ &= -(x^2 + 2xh + h^2) + 3x + 3h - 1 \\ &= -x^2 - 2xh - h^2 + 3x + 3h - 1 \end{aligned}$$

The difference quotient simplifies as follows:

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{(-x^2 - 2xh - h^2 + 3x + 3h - 1) - (-x^2 + 3x - 1)}{h} \\ &= \frac{-x^2 - 2xh - h^2 + 3x + 3h - 1 + x^2 - 3x + 1}{h} \\ &= \frac{-2xh - h^2 + 3h}{h} && \text{simplify} \\ &= \frac{h(-2x - h + 3)}{h} && \text{factor} \\ &= \frac{\cancel{h}(-2x - h + 3)}{\cancel{h}} && \text{cancel} \\ &= -2x - h + 3 \end{aligned}$$

[8] This is the so-called Leibniz notation ...

Our last step is to take the limit: $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} (-2x - h + 3)$. Notice here that we have two variables, x and h, in the limit. Of these two variables, we are taking the limit on the h: $h \to 0$. As far as h is concerned, x may as well be just another constant like the '3'. Hence, $f'(x) = \lim\limits_{h \to 0} (-2x - h + 3) = -2x - 0 + 3 = -2x + 3$.

2. Evaluating our formula for $f'(x)$ at $x = -2$ gives $f'(-2) = -2(-2) + 3 = 7$ which matches with what we obtained in Example **??**.

3. The equation of the tangent line to the graph of $y = f(x)$ at $x = 0$ is $y = f'(0)(x - 0) + f(0)$. We have $f'(0) = 2(0) + 3 = 3$ and $f(0) = -(0)^2 + 3(0) - 1 = -1$. We get $y = 3(x - 0) + (-1)$ so $y = 3x - 1$. Our graph bears this out.

   Desmos link: `https://www.desmos.com/calculator/euggn9v5hr`

4. Solving $f'(x) = 0$ gives $-2x + 3 = 0$ so $x = \dfrac{3}{2}$. This means the slope of the tangent line at the point $\left(\dfrac{3}{2}, f\left(\dfrac{3}{2}\right)\right)$ is 0, so the tangent line there is horizontal. We find $f\left(\dfrac{3}{2}\right) = -\left(\dfrac{3}{2}\right)^2 + 3\left(\dfrac{3}{2}\right) - 1 = \ldots = \dfrac{5}{4}$. Hence, the tangent line at $\left(\dfrac{3}{2}, \dfrac{5}{4}\right)$ is $y = \dfrac{5}{4}$. Graphically, this checks out.

   Desmos link: `https://www.desmos.com/calculator/cqjid2sy7u`

The astute reader will note that the graph of $f(x) = -x^2 + 3x - 1$ in Example **??** is a parabola and finding where $f'(x) = 0$ lead us right back to the vertex. Using a derivative to find the vertex may seem a bit excessive given that we've algebraically derived a handy 'vertex formula' in Section **??**. However, as the functions we aim to analyze become more and more sophisticated, the tools we use to analyze them must also become more sophisticated. The derivative is one such tool that has a near universal application.[9]

Example 4.

1. For $f(x) = x^2 - x - 2$, find and simplify:
   (i) $f'(3)$
   (ii) The equation of the tangent line to the graph $y = f(x)$ at $(3, f(3))$.
   Check your answer graphically.
   (iii) $f'(x)$

2. For $g(x) = \dfrac{3}{2x+1}$, find and simplify:[10]
   (i) $g'(0)$
   (ii) The equation of the tangent line to the graph $y = g(x)$ at $(0, g(0))$.
   Check your answer graphically.
   (iii) $g'(x)$

3. $r(t) = \sqrt{t}$, find and simplify:[11]
   (i) $r'(9)$

[9] As we'll see in Section **??**.
[10] A review of Section **??** may be in order for this problem.
[11] A review of Section **??** may be in order for this problem.

(ii) The equation of the tangent line to the graph $y = r(t)$ at $(9, r(9))$.
Check your answer graphically.

(iii) $r'(t)$

Explanation. 1. (i) To find $f'(3) = \lim_{h\to 0} \dfrac{f(3+h) - f(3)}{h}$ we first simplify $f(3+h)$:

$$\begin{aligned} f(3+h) &= (3+h)^2 - (3+h) - 2 \\ &= 9 + 6h + h^2 - 3 - h - 2 \\ &= 4 + 5h + h^2 \end{aligned}$$

Since $f(3) = (3)^2 - 3 - 2 = 4$, we get for the difference quotient:

$$\begin{aligned} \frac{f(3+h) - f(3)}{h} &= \frac{(4 + 5h + h^2) - 4}{h} \\ &= \frac{5h + h^2}{h} \\ &= \frac{h(5+h)}{h} && \text{factor} \\ &= \frac{\cancel{h}(5+h)}{\cancel{h}} && \text{cancel} \\ &= 5 + h \end{aligned}$$

Hence, $f'(3) = \lim_{h\to 0}(5+h) = 5 + 0 = 5$.

(ii) The equation of the tangent line at $x = 3$ is: $y = f'(3)(x-3) + f(3) = 5(x-3) + 4$, or $y = 5x - 11$.
We check graphically below.

Desmos link: `https://www.desmos.com/calculator/qrzvujthok`

(iii) To find $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$, we first find $f(x+h)$:

$$\begin{aligned} f(x+h) &= (x+h)^2 - (x+h) - 2 \\ &= x^2 + 2xh + h^2 - x - h - 2. \end{aligned}$$

So the difference quotient is

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{\left(x^2 + 2xh + h^2 - x - h - 2\right) - \left(x^2 - x - 2\right)}{h} \\ &= \frac{x^2 + 2xh + h^2 - x - h - 2 - x^2 + x + 2}{h} \\ &= \frac{2xh + h^2 - h}{h} \\ &= \frac{h\,(2x + h - 1)}{h} && \text{factor} \\ &= \frac{\cancel{h}\,(2x + h - 1)}{\cancel{h}} && \textit{cancel} \\ &= 2x + h - 1. \end{aligned}$$

Hence, $f'(x) = \lim_{h\to 0}(2x + h - 1) = 2x + 0 - 1$ so $f'(x) = 2x - 1$. Note that using this formula, we get $f'(3) = 2(3) - 1 = 5$ which checks our answer above.

2. (i) Next we find $g'(0) = \lim_{h\to 0} \dfrac{g(0+h) - g(0)}{h} = \dfrac{g(h) - g(0)}{h}$.
Since $g(h) = \dfrac{3}{2h+1}$ and $g(0) = \dfrac{3}{2(0)+1} = 3$, our difference quotient contains a complex fraction. Thinking ahead, we need to (eventually) be able to cancel the factor 'h' from the denominator $\dfrac{g(h) - g(0)}{h}$, so we begin by simplifying the complex fraction and see where that takes us:

$$\begin{aligned}
\frac{g(0+h)-g(0)}{h} &= \frac{\frac{3}{2h+1}-3}{h} \\
&= \frac{\frac{3}{2h+1}-3}{h}\cdot\frac{(2h+1)}{(2h+1)} \\
&= \frac{3-3(2h+1)}{h(2h+1)} \\
&= \frac{3-6h-3}{h(2h+1)} \\
&= \frac{-6h}{h(2h+1)} \\
&= \frac{-6\cancel{h}}{\cancel{h}(2h+1)} && \text{cancel} \\
&= \frac{-6}{2h+1}.
\end{aligned}$$

We are now ready to take the limit:

$$g'(0) = \lim_{h\to 0}\frac{-6}{2h+1} = \frac{-6}{2(0)+1} = -6.$$

(ii) The equation of the tangent line when $x = 0$ is: $y = g'(0)(x-0)+g(0) = (-6)(x-0)+3$ or $y = -6x+3$, which checks graphically below.

Desmos link: `https://www.desmos.com/calculator/otiohnzjqy`

(iii) To find $g'(x)$, we first find $g(x+h)$:

$$\begin{aligned}
g(x+h) &= \frac{3}{2(x+h)+1} \\
&= \frac{3}{2x+2h+1}
\end{aligned}$$

Simplifying the difference quotient involves simplifying the resulting complex fraction, as above, keeping an eye out for an opportunity to cancel the factor 'h' from the denominator:

$$\begin{aligned}
\frac{g(x+h)-g(x)}{h} &= \frac{\frac{3}{2x+2h+1}-\frac{3}{2x+1}}{h} \\
&= \frac{\frac{3}{2x+2h+1}-\frac{3}{2x+1}}{h}\cdot\frac{(2x+2h+1)(2x+1)}{(2x+2h+1)(2x+1)} \\
&= \frac{3(2x+1)-3(2x+2h+1)}{h(2x+2h+1)(2x+1)} \\
&= \frac{6x+3-6x-6h-3}{h(2x+2h+1)(2x+1)} \\
&= \frac{-6h}{h(2x+2h+1)(2x+1)} \\
&= \frac{-6\cancel{h}}{\cancel{h}(2x+2h+1)(2x+1)} && \text{cancel} \\
&= \frac{-6}{(2x+2h+1)(2x+1)}.
\end{aligned}$$

Hence,

$$g'(x) = \lim_{h\to 0}\frac{-6}{(2x+2h+1)(2x+1)} = \frac{-6}{(2x+2(0)+1)(2x+1)} = -\frac{6}{(2x+1)^2}.$$

We check $g'(0) = -\dfrac{6}{(2(0)+1)^2} = \ldots = -6$, as required.

3. (i) To find $r'(9) = \lim_{h\to 0}\dfrac{r(9+h)-r(9)}{h}$, we start with $r(9+h) = \sqrt{9+h}$ and $r(9) = \sqrt{9} = 3$. Hence our difference quotient is:

$$\frac{r(9+h)-r(9)}{h} = \frac{\sqrt{9+h}-3}{h}.$$

In order for us to determine the limit as $h \to 0$, we need to somehow cancel the factor of h from the **denominator**. To do so, we set about **rationalizing the numerator** by multiplying both numerator and denominator by the conjugate[12]of the numerator, $\sqrt{9+h}-3$:

$$\begin{aligned}
\frac{r(9+h)-r(9)}{h} &= \frac{\sqrt{9+h}-3}{h} \\
&= \frac{\left(\sqrt{9+h}-3\right)}{h} \cdot \frac{\left(\sqrt{9+h}+3\right)}{\left(\sqrt{9+h}+3\right)} && \text{Multiply by the conjugate.} \\
&= \frac{\left(\sqrt{9+h}\right)^2-(3)^2}{h\left(\sqrt{9+h}+3\right)} && \text{Difference of Squares.} \\
&= \frac{(9+h)-9}{h\left(\sqrt{9+h}+3\right)} \\
&= \frac{h}{h\left(\sqrt{9+h}+3\right)} \\
&= \frac{\cancelto{1}{h}}{\cancel{h}\left(\sqrt{9+h}+3\right)} && \text{cancel} \\
&= \frac{1}{\sqrt{9+h}+3}
\end{aligned}$$

Hence,

$$r'(9) = \lim_{h\to 0}\frac{1}{\sqrt{9+h}+3} = \frac{1}{\sqrt{9+0}+3} = \frac{1}{6}.$$

(ii) The equation of the tangent line to $y = r(t)$ at $(9,3)$ is therefore: $y = r'(9)(t-9)+r(9) = \frac{1}{6}(t-9)+3$ or $y = \frac{1}{6}\,t+\frac{3}{2}$. The graph below confirms this.

Desmos link: `https://www.desmos.com/calculator/bvbce8baci`

(iii) As one might expect, we use this same strategy of rationalizing the numerator to simplify the difference quotient to find $r'(t)$:

$$\begin{aligned}
\frac{r(t+h)-r(t)}{h} &= \frac{\sqrt{t+h}-\sqrt{t}}{h} \\
&= \frac{\left(\sqrt{t+h}-\sqrt{t}\right)}{h} \cdot \frac{\left(\sqrt{t+h}+\sqrt{t}\right)}{\left(\sqrt{t+h}+\sqrt{t}\right)} && \text{Multiply by the conjugate.} \\
&= \frac{\left(\sqrt{t+h}\right)^2-\left(\sqrt{t}\right)^2}{h\left(\sqrt{t+h}+\sqrt{t}\right)} && \text{Difference of Squares.} \\
&= \frac{(t+h)-t}{h\left(\sqrt{t+h}+\sqrt{t}\right)} \\
&= \frac{h}{h\left(\sqrt{t+h}+\sqrt{t}\right)} \\
&= \frac{\cancelto{1}{h}}{\cancel{h}\left(\sqrt{t+h}+\sqrt{t}\right)} \\
&= \frac{1}{\sqrt{t+h}+\sqrt{t}}
\end{aligned}$$

[12] Again, see Section **??** for a review of these sorts of machinations.

We get

$$r'(t) = \lim_{h\to 0} \frac{1}{\sqrt{t+h}+\sqrt{t}} = \frac{1}{\sqrt{t+0}+\sqrt{t}} = \frac{1}{2\sqrt{t}}.$$

We check $r'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{2(3)} = \frac{1}{6}$ which matches our answer above.