Exercises for Graphs of Sine and Cosine

Question 1 *In Exercises* **??** *-* **??***, graph one cycle of the given function. State the period, amplitude, phase shift and vertical shift of the function.*

Problem 1.1 $f(t) = 3\sin(t)$

Use the Desmos graph below with the settings $f(t) = \sin(t),\ h = 0,\ b = 1,\ a = 3,\ k = 0$

Desmos link: https://www.desmos.com/calculator/zuvihdpo5z

Period: 2π
Amplitude: 3
Phase Shift: 0
Vertical Shift: 0

Problem 1.2 $g(t) = \sin(3t)$

Use the Desmos graph below with the settings $f(t) = \sin(t),\ h = 0,\ b = 3,\ a = 1,\ k = 0$

Desmos link: https://www.desmos.com/calculator/zuvihdpo5z

Period: $\frac{2\pi}{3}$
Amplitude: 1
Phase Shift: 0
Vertical Shift: 0

Problem 1.3 $h(t) = -2\cos(t)$

Use the Desmos graph below with the settings $f(t) = \cos(t),\ h = 0,\ b = 1,\ a = -2,\ k = 0$

Desmos link: https://www.desmos.com/calculator/ymaw6hfngw

Period: 2π
Amplitude: 2
Phase Shift: 0
Vertical Shift: 0

Problem 1.4 $f(t) = \cos\left(t - \frac{\pi}{2}\right)$

Use the Desmos graph below with the settings $f(t) = \cos(t),\ h = \frac{\pi}{2},\ b = 1,\ a = 1,\ k = 0$

Desmos link: https://www.desmos.com/calculator/ymaw6hfngw

Period: 2π
Amplitude: 1
Phase Shift: $\frac{\pi}{2}$
Vertical Shift: 0

Problem 1.5 $g(t) = -\sin\left(t + \frac{\pi}{3}\right)$

Use the Desmos graph below with the settings $f(t) = \sin(t),\ h = -\frac{\pi}{3},\ b = 1,\ a = -1,\ k = 0$

Desmos link: https://www.desmos.com/calculator/zuvihdpo5z

Period: 2π
Amplitude: 1
Phase Shift: $-\frac{\pi}{3}$
Vertical Shift: 0

Problem 1.6 $h(t) = \sin(2t - \pi)$

Use the Desmos graph below with the settings $f(t) = \sin(t),\ h = \pi,\ b = 2,\ a = 1,\ k = 0$

Desmos link: https://www.desmos.com/calculator/zuvihdpo5z

Period: π
Amplitude: 1
Phase Shift: $\frac{\pi}{2}$
Vertical Shift: 0

Problem 1.7 $f(t) = -\frac{1}{3}\cos\left(\frac{1}{2}t + \frac{\pi}{3}\right)$

Use the Desmos graph below with the settings $f(t) = \cos(t),\ h = -\frac{\pi}{3},\ b = \frac{1}{2},\ a = -\frac{1}{3},\ k = 0$

Desmos link: https://www.desmos.com/calculator/ymaw6hfngw

Period: 4π
Amplitude: $\frac{1}{3}$
Phase Shift: $-\frac{2\pi}{3}$
Vertical Shift: 0

Problem 1.8 $g(t) = \cos(3t - 2\pi) + 4$

Use the Desmos graph below with the settings $f(t) = \cos(t),\ h = 2\pi,\ b = 3,\ a = 1,\ k = 4$

Desmos link: https://www.desmos.com/calculator/ymaw6hfngw

Period: $\frac{2\pi}{3}$
Amplitude: 1
Phase Shift: $\frac{2\pi}{3}$
Vertical Shift: 4

Problem 1.9 $h(t) = \sin\left(-t - \frac{\pi}{4}\right) - 2$

Use the Desmos graph below with the settings $f(t) = \sin(t),\ h = \frac{\pi}{4},\ b = -1,\ a = 1,\ k = -2$

Desmos link: https://www.desmos.com/calculator/zuvihdpo5z

Period: 2π
Amplitude: 1
Phase Shift: $-\frac{\pi}{4}$ *(You need to use* $y = -\sin\left(t + \frac{\pi}{4}\right) - 2$ *to find this.)*[1]
Vertical Shift: -2

Problem 1.10 $f(t) = \frac{2}{3}\cos\left(\frac{\pi}{2} - 4t\right) + 1$

Use the Desmos graph below with the settings $f(t) = \cos(t),\ h = -\frac{\pi}{2},\ b = -4,\ a = \frac{2}{3},\ k = 1$

Desmos link: `https://www.desmos.com/calculator/ymaw6hfngw`

Period: $\frac{\pi}{2}$
Amplitude: $\frac{2}{3}$
Phase Shift: $\frac{\pi}{8}$ *(You need to usey =* $\frac{2}{3}\cos\left(4t - \frac{\pi}{2}\right) + 1$ *to find this.)*[2]
Vertical Shift: 1

Problem 1.11 $g(t) = -\frac{3}{2}\cos\left(2t + \frac{\pi}{3}\right) - \frac{1}{2}$

Use the Desmos graph below with the settings $f(t) = \cos(t),\ h = -\frac{\pi}{3},\ b = 2,\ a = -\frac{3}{2},\ k = -\frac{1}{2}$

Desmos link: `https://www.desmos.com/calculator/ymaw6hfngw`

Period: π
Amplitude: $\frac{3}{2}$
Phase Shift: $-\frac{\pi}{6}$
Vertical Shift: $-\frac{1}{2}$

Problem 1.12 $h(t) = 4\sin(-2\pi t + \pi)$

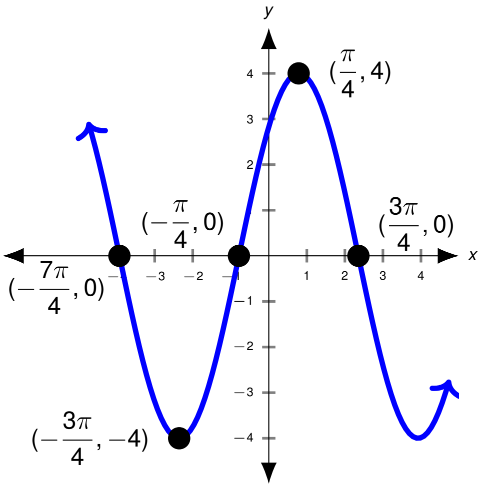
Use the Desmos graph below with the settings $f(t) = \sin(t),\ h = -\pi,\ b = -2\pi,\ a = 4,\ k = 0$

Desmos link: `https://www.desmos.com/calculator/zuvihdpo5z`

Period: 1
Amplitude: 4
Phase Shift: $\frac{1}{2}$ *(You need to use* $h(t) = -4\sin(2\pi t - \pi)$ *to find this.)*[3]
Vertical Shift: 0

Question 2 *In Exercises* **??** - **??***, a sinusoid is graphed. Find a formula for the sinusoid in the form* $S(t) = A\sin(\omega t + \phi) + B$ *and* $C(t) = A\cos(\omega t + \phi) + B$*. Select* ω *so* $\omega > 0$.

[1] Two cycles of the graph are shown to illustrate the discrepancy discussed on page **??**.
[2] Again, we graph two cycles to illustrate the discrepancy discussed on page **??**.
[3] This will be the last time we graph two cycles to illustrate the discrepancy discussed on page **??**.

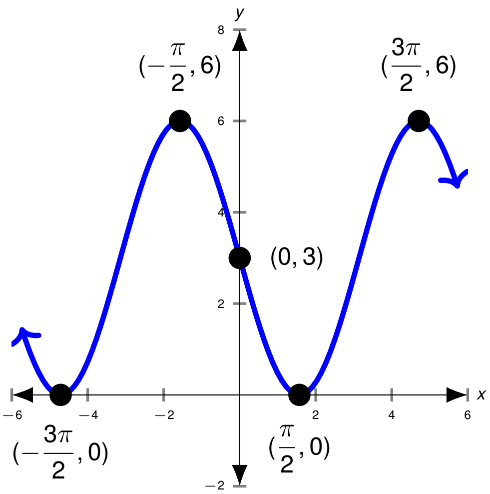


Problem 2.1

$S(t) = 4\sin\left(t + \frac{\pi}{4}\right)$, $C(t) = 4\cos\left(t - \frac{\pi}{4}\right)$

Use the Desmos graph below to check that both of your formulas agree.

Desmos link: https://www.desmos.com/calculator/6etbg1ifrh

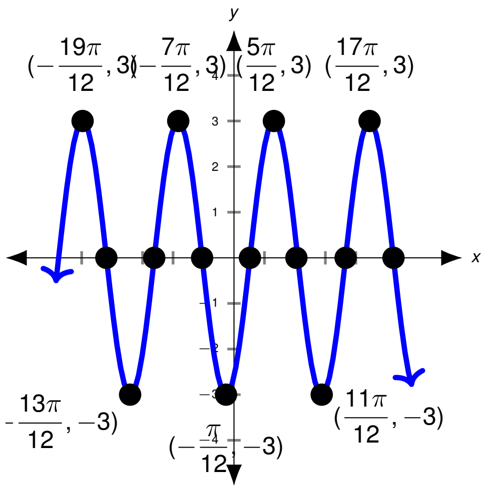


Problem 2.2

$S(t) = -3\sin(t) + 3$, $C(t) = -3\cos\left(t - \frac{\pi}{2}\right) + 3$

Use the Desmos graph below to check that both of your formulas agree.

Desmos link: https://www.desmos.com/calculator/6etbg1ifrh

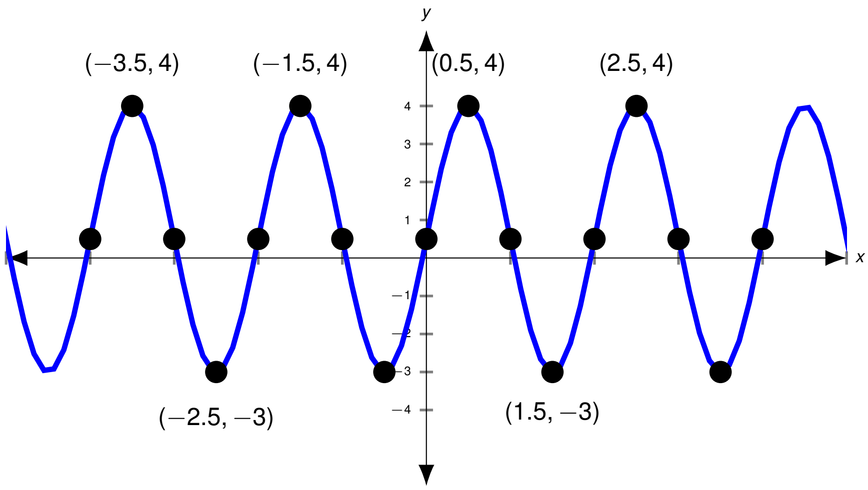


Problem 2.3

$S(t) = 3\sin\left(2t - \frac{\pi}{3}\right)$, $C(t) = 3\cos\left(2t - \frac{5\pi}{6}\right)$

Use the Desmos graph below to check that both of your formulas agree.

Desmos link: `https://www.desmos.com/calculator/6etbg1ifrh`



Problem 2.4

$S(t) = \frac{7}{2}\sin(\pi t) + \frac{1}{2}$, $C(t) = \frac{7}{2}\cos\left(\pi t\frac{\pi}{2}\right) + \frac{1}{2}$

Use the Desmos graph below to check that both of your formulas agree.

Desmos link: `https://www.desmos.com/calculator/6etbg1ifrh`

Problem 3 *Use the graph of $S(t) = 4\sin(t)$ to graph each of the following functions. State the period of each.*

1. $f(t) = |4\sin(t)|$

   Period: π

   Desmos link: `https://www.desmos.com/calculator/kjwmsggfeu`

2. $g(t) = \sqrt{4\sin(t)}$

   Period: 2π.

EXERCISES FOR GRAPHS OF SINE AND COSINE

Desmos link: https://www.desmos.com/calculator/kjwmsggfeu

Question 4 *In Exercises **??** - **??**, use a graphing utility to help you and your classmates discuss the given questions.*

Problem 4.1 *Graph* $f(t) = \cos(3t) + \sin(t)$*. Is this function periodic? If so, what is the period?*

$f(t) = \cos(3t) + \sin(t)$ *has period* 2π*.*

Problem 4.2 *Graph* $f(t) = t\sin(t)$ *along with* $y = \pm t$*. What do you notice?*

The graph of $f(t) = t\sin(t)$ *is bounded by the lines* $y = \pm t$*;* f *has a variable amplitude of* t*.*

Problem 4.3 *Graph* $f(t) = \dfrac{\sin(t)}{t}$ *along with* $y = \pm\dfrac{1}{t}$*. What do you notice?*

The graph of $f(t) = \dfrac{\sin(t)}{t}$ *is bounded by the graphs of* $y = \pm\dfrac{1}{t}$*;* f *has a variable amplitude of* $\dfrac{1}{t}$*.*

1. *What appears to be* $\lim\limits_{t\to\infty} f(t)$*? Interpret your answer graphically.*

   $\lim\limits_{t\to\infty} f(t) = 0$*. We have a horizontal asymptote* $y = 0$*.*

2. *Use the Squeeze Theorem, Theorem **??** from Section **??** to prove your claim.*

   HINT: *Since* $-1 \le \sin(t) \le 1$*, for* $t > 0$*,* $-\dfrac{1}{t} \le \dfrac{\sin(t)}{t} \le \dfrac{1}{t} \dots$

   Since $-\dfrac{1}{t} \le \dfrac{\sin(t)}{t} \le \dfrac{1}{t}$*,* $\lim\limits_{t\to\infty}\left(-\dfrac{1}{t}\right) = \lim\limits_{t\to\infty}\dfrac{1}{t} = 0$*,* $\lim\limits_{t\to\infty}\dfrac{\sin(t)}{t} = 0$ *by the Squeeze Theorem.*

Problem 4.4 *Graph* $f(t) = \cos\left(\dfrac{1}{t}\right)$*.*

1. *Investigate* $\lim\limits_{t\to 0} f(t)$*. Does* $\lim\limits_{t\to 0} f(t)$ *exist? Why or why not?*

   $\lim\limits_{t\to 0} f(t)$ *does not exist. We have infinitely many oscillations as* $t \to 0$*.*

2. *Determine* $\lim\limits_{t\to\infty} f(t)$ *and interpret your answer graphically.*

   $\lim\limits_{t\to\infty} f(t) = 1$ *since as* $t \to \infty$*,* $\dfrac{1}{t} \to 0$*. Since cosine is continuous,* $\cos\left(\dfrac{1}{t}\right) \to \cos(0) = 1$*. We have a horizontal asymptote* $y = 1$*.*

Problem 4.5 *Graph* $f(t) = e^{-0.1t}(\cos(2t) + \sin(2t))$ *along with* $y = \pm 2\,e^{-0.1t}$*. What do you notice?*

The graph of $f(t) = e^{-0.1t}(\cos(2t) + \sin(2t))$ *lies between*[4] *the graphs of* $y = \pm 2\,e^{-0.1t}$*.*

1. *What appears to be* $\lim\limits_{t\to\infty} f(t)$*? Interpret your answer graphically.*

   $\lim\limits_{t\to\infty} f(t) = 0$*. We have a horizontal asymptote* $y = 0$*.*

2. *Use the Squeeze Theorem, Theorem **??** from Section **??** to prove your claim.*

   HINT: *Since* $-1 \le \cos(2t) \le 1$ *and* $-1 \le \sin(2t) \le 1$*,* $-2 \le \cos(2t) + \sin(2t) \le 2$*. Hence,* $-2e^{-0.1t} \le e^{-0.1t}(\cos(2t) + \sin(2t)) \le 2e^{-0.1t} \dots$

   Since $-2e^{-0.1t} \le e^{-0.1t}(\cos(2t) + \sin(2t)) \le 2e^{-0.1t}$*,* $\lim\limits_{t\to\infty}\left(-2e^{-0.1t}\right) = \lim\limits_{t\to\infty} 2e^{-0.1t} = 0$ *, the Squeeze Theorem gives* $\lim\limits_{t\to\infty} e^{-0.1t}(\cos(2t) + \sin(2t)) = 0$*.*

[4] We'll be able to show in Section **??** that the graph of f more perfectly lies between the graphs of $y = \pm\sqrt{2}\,e^{-0.1t} \dots$

EXERCISES FOR GRAPHS OF SINE AND COSINE

Problem 5 *Show every constant function f is periodic by explaining why $f(x+117) = f(x)$ for all real numbers x. Then show that f has no period by showing that you cannot find a* smallest *number p such that $f(x+p) = f(x)$ for all real numbers x.*

Said differently, show that $f(x+p) = f(x)$ for all real numbers x for ALL values of $p > 0$, so no smallest value exists to satisfy the definition of 'period'.

Problem 6 *The sounds we hear are made up of mechanical waves. The note 'A' above the note 'middle C' is a sound wave with ordinary frequency $f = 440$ Hertz $= 440 \frac{\text{cycles}}{\text{second}}$. Find a sinusoid which models this note, assuming that the amplitude is 1 and the phase shift is 0.*

$S(t) = \sin(880\pi t)$

Problem 7 *The voltage V in an alternating current source has amplitude $220\sqrt{2}$ and ordinary frequency $f = 60$ Hertz. Find a sinusoid which models this voltage. Assume that the phase is 0.*

$V(t) = 220\sqrt{2}\sin(120\pi t)$

Problem 8 *The London Eye is a popular tourist attraction in London, England and is one of the largest Ferris Wheels in the world. It has a diameter of 135 meters and makes one revolution (counter-clockwise) every 30 minutes. It is constructed so that the lowest part of the Eye reaches ground level, enabling passengers to simply walk on to, and off of, the ride. Find a sinsuoid which models the height h of the passenger above the ground in meters t minutes after they board the Eye at ground level.*

$h(t) = 67.5\sin\left(\frac{\pi}{15}t - \frac{\pi}{2}\right) + 67.5$

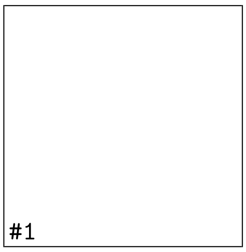
Problem 9 *On page **??** in Section **??**, we found the x-coordinate of counter-clockwise motion on a circle of radius r with angular frequency ω to be $x = r\cos(\omega t)$, where $t = 0$ corresponds to the point $(r, 0)$. Suppose we are in the situation of Exercise **??** above. Find a sinsusoid which models the horizontal displacement x of the passenger from the center of the Eye in meters t minutes after they board the Eye. Here we take $x(t) > 0$ to mean the passenger is to the right of the center, while $x(t) < 0$ means the passenger is to the left of the center.*

$x(t) = 67.5\cos\left(\frac{\pi}{15}t - \frac{\pi}{2}\right) = 67.5\sin\left(\frac{\pi}{15}t\right)$

Problem 10 *In Exercise **??** in Section **??**, we introduced the yo-yo trick 'Around the World' in which a yo-yo is thrown so it sweeps out a vertical circle. As in that exercise, suppose the yo-yo string is 28 inches and it completes one revolution in 3 seconds. If the closest the yo-yo ever gets to the ground is 2 inches, find a sinsuoid which models the height h of the yo-yo above the ground in inches t seconds after it leaves its lowest point.*

$h(t) = 28\sin\left(\frac{2\pi}{3}t - \frac{\pi}{2}\right) + 30$

Problem 11 *Consider the pendulum below. Ignoring air resistance, the angular displacement of the pendulum from the vertical position, θ, can be modeled as a sinusoid.*[5]



[5] Provided θ is kept 'small.' Carl remembers the 'Rule of Thumb' as being 20° or less. Check with your friendly neighborhood physicist to make sure.

The amplitude of the sinusoid is the same as the initial angular displacement, θ_0, of the pendulum and the period of the motion is given by

$$T = 2\pi\sqrt{\frac{\ell}{g}}$$

where ℓ is the length of the pendulum and g is the acceleration due to gravity.

1. *Find a sinusoid which gives the angular displacement θ as a function of time, t. Arrange things so $\theta(0) = \theta_0$.*
   $\theta(t) = \theta_0 \sin\left(\sqrt{\frac{g}{l}}\, t + \frac{\pi}{2}\right)$
2. *In Exercise **??** section **??**, you found the length of the pendulum needed in Jeff's antique Seth-Thomas clock to ensure the period of the pendulum is $\frac{1}{2}$ of a second. Assuming the initial displacement of the pendulum is 15°, find a sinusoid which models the displacement of the pendulum θ as a function of time, t, in seconds.*
   $\theta(t) = \frac{\pi}{12} \sin\left(4\pi t + \frac{\pi}{2}\right)$

Problem 12 *The table below lists the average temperature of Lake Erie as measured in Cleveland, Ohio on the first of the month for each month during the years 1971 – 2000.*[6] *For example, $t = 3$ represents the average of the temperatures recorded for Lake Erie on every March 1 for the years 1971 – 2000.*

Month Number, t	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*
Temperature (° F), T	*36*	*33*	*34*	*38*	*47*	*57*	*67*	*74*	*73*	*67*	*56*	*46*

1. *Using the techniques discussed in Example **??**, fit a sinusoid to these data.*
2. *Graph your model along with the data set to judge the reasonableness of the fit.*
3. *Use the model from **??** to predict the average temperature recorded for Lake Erie on April 15^{th} and September 15^{th} during the years 1971–2000.*[7]
4. *Compare your results to those obtained using a graphing utility.*

Problem 13 *The fraction of the moon illuminated at midnight Eastern Standard Time on the t^{th} day of June, 2009 is given in the table below.*[8]

Day of June, t	*3*	*6*	*9*	*12*	*15*	*18*	*21*	*24*	*27*	*30*
Fraction Illuminated, F	*0.81*	*0.98*	*0.98*	*0.83*	*0.57*	*0.27*	*0.04*	*0.03*	*0.26*	*0.58*

1. *Using the techniques discussed in Example **??**, fit a sinusoid to these data.*[9]
2. *Graph your model along with the data set to judge the reasonableness of the fit.*

[6] See this website: http://www.erh.noaa.gov/cle/climate/cle/normals/laketempcle.html.
[7] The computed average is 41°F for April 15^{th} and 71°F for September 15^{th}.
[8] See this website: http://www.usno.navy.mil/USNO/astronomical-applications/data-services/frac-moon-ill.
[9] You may want to plot the data before you find the phase shift.

3. *Use the model from* **??** *to predict the fraction of the moon illuminated on June 1, 2009.* [10]

4. *Compare your results to those obtained using a graphing utility.*

Problem 14 *Use a graphing utility to graph* $y = 8.36\sin(-294.81t - 26.53) + 12.06$. *(This is the regression model produced by desmos in Example* **??***.) Zoom in, as needed, until you start to see the wave-like nature of the graph. Use Theorem* **??** *to determine a window which produces exactly one complete cycle of this sinusoid and check your answer graphically.*

Problem 15 *Use Theorem* **??** *to prove Theorem* **??**.

Problem 16 *With the help of your classmates, research Amplitude Modulation and Frequency Modulation.*

Problem 17 *What other things in the world might be roughly sinusoidal? Look to see what models you can find for them and share your results with your class.*

[10] The listed fraction is 0.62.