Graphs of Sine and Cosine

On page **??**, we discussed how to interpret the sine and cosine of real numbers. To review, we identify a real number t with an oriented angle θ measuring t radians[1] and define $\sin(t) = \sin(\theta)$ and $\cos(t) = \cos(\theta)$. Since every real number can be identified with one and only one angle θ this way, the domains of the functions $f(t) = \sin(t)$ and $g(t) = \cos(t)$ are all real numbers, $(-\infty, \infty)$.

When it comes to range, recall that the sine and cosine of angles are coordinates of points on the Unit Circle and hence, each fall between -1 and 1 inclusive. Since the real number line,[2] when wrapped around the Unit Circle completely covers the circle, we can be assured that every point on the Unit Circle corresponds to at least one real number. Putting these two facts together, we conclude the range of $f(t) = \sin(t)$ and $g(t) = \cos(t)$ are both $[-1, 1]$. We summarize these two important facts below.

Theorem 1. Domain and Range of the Cosine and Sine Functions:

- The function $f(t) = \sin(t)$
  - has domain $(-\infty, \infty)$
  - has range $[-1, 1]$
- The function $g(t) = \cos(t)$
  - has domain $(-\infty, \infty)$
  - has range $[-1, 1]$

Our aim in this section is to become familiar with the graphs of $f(t) = \sin(t)$ and $g(t) = \cos(t)$. To that end, we begin by making a table and plotting points. We'll start by graphing $f(t) = \sin(t)$ by making a table of values and plotting the corresponding points. We'll keep the independent variable 't' for now and use the default 'y' as our dependent variable.[3]

t	$\sin(t)$	$(t, \sin(t))$
0	0	$(0, 0)$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$
$\frac{\pi}{2}$	1	$\left(\frac{\pi}{2}, 1\right)$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\left(\frac{3\pi}{4}, \frac{\sqrt{2}}{2}\right)$
π	0	$(\pi, 0)$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\left(\frac{5\pi}{4}, -\frac{\sqrt{2}}{2}\right)$
$\frac{3\pi}{2}$	-1	$\left(\frac{3\pi}{2}, -1\right)$
$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\left(\frac{7\pi}{4}, -\frac{\sqrt{2}}{2}\right)$
2π	0	$(2\pi, 0)$

We use desmos to help us construct the graph below. Note the scale of the horizontal and vertical axis is far from 1:1, but you are invited to take advantage of desmos's 'Zoom Square' feature to see a more accurate graph.

Desmos link: `https://www.desmos.com/calculator/m6sshcvnod`

If we plot additional points, we soon find that the graph repeats itself. This shouldn't come as too much of a surprise considering Theorem **??**. In fact, in light of that theorem, we expect the function to repeat itself every 2π units. Below is a more accurately scaled graph highlighting the portion we had already graphed above. The graph is often described as having a 'wavelike' nature and is sometimes called a **sine wave** or, more technically, a **sinusoid**.

[1] which, you'll recall essentially 'wraps the real number line around the Unit Circle

[2] in particular the interval $[0, 2\pi)$

[3] Keep in mind that we're using 'y' here to denote the *output* from the sine function. It is a coincidence that the y-values on the graph of $y = \sin(t)$ correspond to the y-values on the Unit Circle.

Desmos link: https://www.desmos.com/calculator/i8zvkblwhp

Note that by copying the highlighted portion of the graph and pasting it end-to-end, we obtain the entire graph of $f(t) = \sin(t)$. We give this 'repeating' property a name.

Definition 1. Periodic Functions: A function f is said to be **periodic** if there is a real number c so that $f(t + c) = f(t)$ for all real numbers t in the domain of f. The smallest positive number p for which $f(t + p) = f(t)$ for all real numbers t in the domain of f, if it exists, is called the **period** of f.

We have already seen a family of periodic functions in Section **??**: the constant functions. However, despite being periodic a constant function has no period. (We'll leave that odd gem as an exercise for you.)

Returning to $f(t) = \sin(t)$, we see that by Definition **??**, f is periodic since $\sin(t + 2\pi) = \sin(t)$. To determine the period of f, we need to find the smallest real number p so that $f(t + p) = f(t)$ for all real numbers t or, said differently, the smallest positive real number p such that $\sin(t + p) = \sin(t)$ for all real numbers t.

We know that $\sin(t + 2\pi) = \sin(t)$ for all real numbers t but the question remains if any smaller real number will do the trick. Suppose $p > 0$ and $\sin(t + p) = \sin(t)$ for all real numbers t. Then, in particular, $\sin(0 + p) = \sin(0) = 0$ so that $\sin(p) = 0$. From this we know p is a multiple of π. Since $\sin\left(\frac{\pi}{2}\right) \neq \sin\left(\frac{\pi}{2} + \pi\right)$, we know $p \neq \pi$. Hence, $p = 2\pi$ so the period of $f(t) = \sin(t)$ is 2π.

Having period 2π essentially means that we can completely understand everything about the function $f(t) = \sin(t)$ by studying *one* interval of length 2π, say $[0, 2\pi]$.[4] For this reason, when graphing sine (and cosine) functions, we typically restrict our attention to graphing these functions over the course of one period to produce one **cycle** of the graph.

Not surprisingly, the graph of $g(t) = \cos(t)$ exhibits similar behavior as $f(t) = \sin(t)$ as seen below.[6]

We start with a table of values.

t	$\cos(t)$	$(t, \cos(t))$
0	1	$(0, 1)$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$
$\frac{\pi}{2}$	0	$\left(\frac{\pi}{2}, 0\right)$
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\left(\frac{3\pi}{4}, -\frac{\sqrt{2}}{2}\right)$
π	-1	$(\pi, -1)$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\left(\frac{5\pi}{4}, -\frac{\sqrt{2}}{2}\right)$
$\frac{3\pi}{2}$	0	$\left(\frac{3\pi}{2}, 0\right)$
$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\left(\frac{7\pi}{4}, \frac{\sqrt{2}}{2}\right)$
2π	1	$(2\pi, 1)$

Next, we use desmos to help us graph one cycle.

Desmos link: https://www.desmos.com/calculator/0xabgsnluo

[4] Technically, we should study the interval $[0, 2\pi)$,[5] since whatever happens at $t = 2\pi$ is the same as what happens at $t = 0$. As we will see shortly, $t = 2\pi$ gives us an extra 'check' when we go to graph these functions.

[5] In some advanced texts, the interval of choice is $[-\pi, \pi)$.

[6] Here note that the dependent variable 'y' represents the outputs from $g(t) = \cos(t)$ which are x-coordinates on the Unit Circle.

Like $f(t) = \sin(t)$, $g(t) = \cos(t)$ is a wavelike curve with period 2π. Moreover, the graphs of the sine and cosine functions have the same shape - differing only in what appears to be a horizontal shift. As we'll prove in Section **??**, $\sin\left(t + \frac{\pi}{2}\right) = \cos(t)$, which means we can obtain the graph of $y = \cos(t)$ by shifting the graph of $y = \sin(t)$ to the left $\frac{\pi}{2}$ units.[7]

Desmos link: https://www.desmos.com/calculator/4xh96xwznz

While arguably the most important property shared by $f(t) = \sin(t)$ and $g(t) = \cos(t)$ is their periodic 'wavelike' nature,[8] their graphs suggest these functions are both continuous and smooth. Recall from Section **??** that, like polynomial functions, the graphs of the sine and cosine functions have no jumps, gaps, holes in the graph, vertical asymptotes, corners or cusps.

Note the graphs of both $f(t) = \sin(t)$ and $g(t) = \cos(t)$ meander as $t \to -\infty$ or as $\to \infty$. Said differently, none of the limits $\lim\limits_{t\to-\infty} \sin(t)$, $\lim\limits_{t\to\infty} \sin(t)$, $\lim\limits_{t\to-\infty} \cos(t)$, or $\lim\limits_{t\to\infty} \cos(t)$ exist. Even though these functions are 'trapped' (or **bounded**) between -1 and 1, neither graph has any horizontal asymptotes.

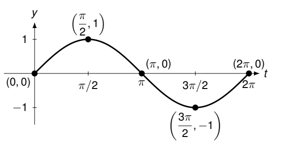
Lastly, the graphs of $f(t) = \sin(t)$ and $g(t) = \cos(t)$ suggest each enjoy one of the symmetries introduced in Section **??**. The graph of $y = \sin(t)$ appears to be symmetric about the origin while the graph of $y = \cos(t)$ appears to be symmetric about the y-axis. Indeed, as we'll prove in Section **??**, $f(t) = \sin(t)$ is, in fact, an odd function: [9] that is, $\sin(-t) = -\sin(t)$ and $g(t) = \cos(t)$ is an even function, so $\cos(-t) = \cos(t)$.

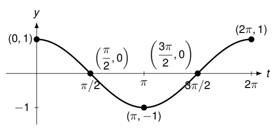
We summarize all of these properties in the following result.

Theorem 2. Properties of the Cosine and Sine Functions

- The function $f(t) = \sin(t)$
  - has domain $(-\infty, \infty)$
  - has range $[-1, 1]$
  - is continuous and smooth
  - is odd
  - has period 2π
- The function $g(t) = \cos(t)$
  - has domain $(-\infty, \infty)$
  - has range $[-1, 1]$
  - is continuous and smooth
  - is even
  - has period 2π
- Conversion formulas: $\sin\left(t + \frac{\pi}{2}\right) = \cos(t)$ and $\cos\left(t - \frac{\pi}{2}\right) = \sin(t)$

Now that we know the basic shapes of the graphs of $y = \sin(t)$ and $y = \cos(t)$, we can use the results of Section **??** to graph more complicated functions using transformations. The fact that both of these functions are periodic means we only have to know what happens over the course of one period of the function in order to determine what happens to all points on the graph. To that end, we graph the '**fundamental cycle**' - the portion of each graph generated over the interval $[0, 2\pi]$ - for each sine and cosine:





In working through Section **??** , it was very helpful to track 'key points' through the transformations. The 'key points' we've indicated on the graphs above correspond to the quadrantal angles and generate the zeros and the extrema of functions. Since the quadrantal angles divide the interval $[0, 2\pi]$ into four equal pieces, we shall refer to these angles henceforth as the 'quarter marks.'

[7] Hence, we can obtain the graph of $y = \sin(t)$ by shifting the graph of $y = \cos(t)$ to the *right* $\frac{\pi}{2}$ units: $\cos\left(t - \frac{\pi}{2}\right) = \sin(t)$.

[8] this is the reason they are so useful in the Sciences and Engineering

[9] The reader may wish to review Definitions **??** and **??** as needed.

It is worth noting that because the transformations discussed in Section **??** are linear,[10] the *relative spacing* of the points before and after the transformations remains the same.[11] In particular, wherever the interval $[0, 2\pi]$ is mapped, the quarter marks of the new interval correspond to the quarter marks of $[0, 2\pi]$. (Can you see why?) We will exploit this fact in the following example.

Example 1. Graph one cycle of the following functions. State the period of each.

1. $f(t) = 3\sin(2t)$
2. $g(t) = 2\cos\left(t + \frac{\pi}{2}\right) + 1$

Explanation. 1. One way to proceed is to use Theorem **??** and follow the procedure outlined there. Starting with the fundamental cycle of $y = \sin(t)$, we divide each t-coordinate by 2 and multiply each y-coordinate by 3 to obtain one cycle of $y = 3\sin(2t)$.

Using the desmos interactive below, we can select each transformation, in sequence, to see the progression from the graph of $y = \sin(t)$ to $y = f(t) = 3\sin(2t)$.

Desmos link: `https://www.desmos.com/calculator/0y2nwkw9yo`

Since one cycle of $y = f(t)$ is completed over the interval $[0, \pi]$, the period of f is π.

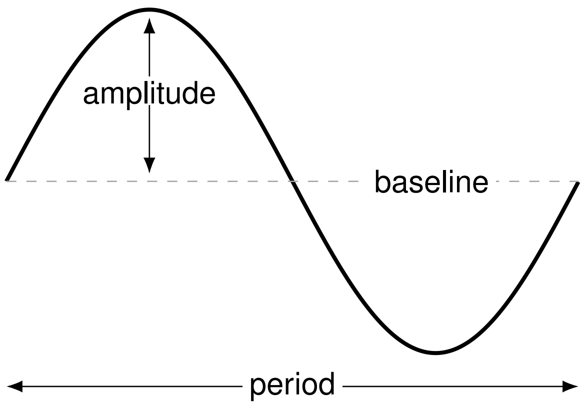
2. Starting with the fundamental cycle of $y = \cos(t)$ and using Theorem **??**, we subtract $\frac{\pi}{2}$ from each of the t-coordinates, then multiply each y-coordinate by 2, and add 1 to each y-coordinate.

Using the desmos interactive below, we can select each transformation, in sequence, to see the progression from the graph of $y = \cos(t)$ to $y = g(t) = 2\cos\left(t + \frac{\pi}{2}\right) + 1$.

Desmos link: `https://www.desmos.com/calculator/pwzvj3bqjl`

We find one cycle of $y = g(t)$ is completed over the interval $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$, the period is $\frac{3\pi}{2} - \left(-\frac{\pi}{2}\right) = 2\pi$.

As previously mentioned, the curves graphed in Example **??** are examples of sinusoids. A sinusoid is the result of taking the graph of $y = \sin(t)$ or $y = \cos(t)$ and performing any of the transformations mentioned in Section **??**. We graph one cycle of a generic sinusoid below. Sinusoids can be characterized by four properties: period, phase shift, vertical shift (or 'baseline'), and amplitude.



[10] See the remarks at the beginning of Section **??**.

[11] If we use a linear function $f(t) = mt + b$ to transform the inputs, t, then $\Delta[f(t)] = m\Delta t$. That is, the change *after* the the transformation, $m\Delta t$, is just a multiple of the change *before* the transformation, Δt.

We have already discussed the period of a sinusoid. If we think of t as measuring time, the period is how long it takes for the sinusoid to complete one cycle and is usually represented by the letter T. The standard period of both $\sin(t)$ and $\cos(t)$ is 2π, but horizontal scalings will change this.

In Example **??**, for instance, the function $f(t) = 3\sin(2t)$ has period π instead of 2π because the graph is horizontally compressed by a factor of 2 as compared to the graph of $y = \sin(t)$. However, the period of $g(t) = 2\cos\left(t + \frac{\pi}{2}\right) + 1$ is the same as the period of $\cos(t)$, 2π, since there are no horizontal scalings.

The **phase shift** of the sinusoid is the horizontal shift. Again, thinking of t as time, the phase shift of a sinusoid can be thought of as when the sinusoid 'starts' as compared to $t = 0$. Assuming there are no reflections across the y-axis, we can determine the phase shift of a sinusoid by finding where the value $t = 0$ on the graph of $y = \sin(t)$ or $y = \cos(t)$ is mapped to under the transformations.

For $f(t) = 3\sin(2t)$, the phase shift is '0' since the value $t = 0$ on the graph of $y = \sin(t)$ remains stationary under the transformations. Loosely speaking, this means both $y = \sin(t)$ and $y = 3\sin(2t)$ 'start' at the same time. The phase shift of $g(t) = 2\cos\left(t + \frac{\pi}{2}\right) + 1$ is $-\frac{\pi}{2}$ or '$\frac{\pi}{2}$ to the *left*' since the value $t = 0$ on the graph of $y = \cos(t)$ is mapped to $t = -\frac{\pi}{2}$ on the graph of $y = 2\cos\left(t + \frac{\pi}{2}\right) + 1$. Again, loosely speaking, this means $y = 2\cos\left(t + \frac{\pi}{2}\right) + 1$ starts $\frac{\pi}{2}$ time units *earlier* than $y = \cos(t)$.

The vertical shift of a sinusoid is exactly the same as the vertical shifts in Section **??** and determines the new 'baseline' of the sinusoid. Thanks to symmetry, the vertical shift can always be found by averaging the maximum and minimum values of the sinusoid. For $f(t) = 3\sin(2t)$, the vertical shift is 0 whereas the vertical shift of $g(t) = 2\cos\left(t + \frac{\pi}{2}\right) + 1$ is 1 or '1 up.'

The **amplitude** of the sinusoid is a measure of how 'tall' the wave is, as indicated in the figure below. Said differently, the amplitude measures how much the curve gets displaced from its 'baseline. ' The amplitude of the standard cosine and sine functions is 1, but vertical scalings can alter this.

In Example **??**, the amplitude of $f(t) = 3\sin(2t)$ is 3, owing to the vertical stretch by a factor of 3 as compared with the graph of $y = \sin(t)$. In the case of $g(t) = 2\cos\left(t + \frac{\pi}{2}\right) + 1$, the amplitude is 2 due to its vertical stretch as compared with the graph of $y = \cos(t)$. Note that the '+1' here does *not* affect the amplitude of the curve; it merely changes the 'baseline' from $y = 0$ to $y = 1$.

The following theorem shows how these four fundamental quantities relate to the parameters which describe a generic sinusoid. The proof follows from Theorem **??** and is left to the reader in Exercise **??**.

Theorem 3. For $\omega > 0$, the graphs of

$$S(t) = A\sin(\omega t + \phi) + B \quad \text{and} \quad C(t) = A\cos(\omega t + \phi) + B$$

- have period $T = \frac{2\pi}{\omega}$
- have amplitude $|A|$
- have phase shift $-\frac{\phi}{\omega}$
- have vertical shift or 'baseline' B

The parameter ω mentioned above is called the **angular frequency**, or more simply, the **frequency** of the sinusoid and is the number of cycles the sinusoid completes over an interval of length 2π. That is, ω measures how 'frequently' the sinusoid repeats over an interval of length 2π. As we'll see in the next example, we can always ensure $\omega > 0$ using the even and odd properties of the cosine and sine functions, respectively. If t represents time, ω as represents how fast the sinusoid is being generated in terms of *radians* per unit time. In essence, it is the *angular speed* of the curve.

A quantity closely related to the angular frequency of the sinusoid is the **ordinary frequency** of the sinusoid, usually denoted f. The ordinary frequency of a sinusoid measures the number of cycles the sinusoid completes over an interval of length 1. Since the period, T represents the length of the interval required for a sinusoid to make one complete cycle, we have $f = \frac{1}{T}$. Once again, if t represents time, the ordinary frequency measures how fast the sinusoid is being generated in terms of *complete cycles* per unit time.[12]

Note that since $T = \frac{2\pi}{\omega}$, $f = \frac{1}{T} = \frac{\omega}{2\pi}$. Rewriting, we get $\omega = 2\pi f$. To understand this equation in terms of units, recall 1 complete cycle (revolution) around the Unit Circle counts for 2π radians. Hence, to get from f, measured in cycles per unit time, to ω, measured in radians per unit time, we need to multiply by 2π.

If the concepts of period and frequency seem familiar, they should. In Section **??**, we discussed these very same ideas in the context of Example **??** and revisited them again in Section **??** in Equation **??**. and Example **??**. On the one hand, the notions presented here are more general, since they are not tied directly to circular motion. On the other hand, the stipulation in this section that $\omega > 0$ means we are restricting our attention to angular *speeds* instead of the more general angular *velocities*.[13]

Last, but not least, the quantity ϕ mentioned in Theorem **??** is called the **phase** or **phase angle** of the sinusoid. The phase of a sinusoid is the angle in the argument which corresponds to $t = 0$, and is important in describing waves in fields such as physics and electronics. When *graphing* sinusoids, however, we focus our attention on the horizontal shift induced by ϕ, $-\frac{\phi}{\omega}$.

We put Theorem **??** to good use in the next example.

Example 2. Use Theorem **??** to determine the frequency, period, phase shift, amplitude, and vertical shift of each of the following functions and use this information to graph one cycle of each function.

1. $f(t) = 3\cos\left(\frac{\pi t - \pi}{2}\right) + 1$
2. $g(t) = \frac{1}{2}\sin(\pi - 2t) + \frac{3}{2}$

Explanation. 1. To use Theorem **??**, we first need to rewrite $f(t)$ in the form prescribed by Theorem **??**. To that end, we rewrite: $f(t) = 3\cos\left(\frac{\pi t - \pi}{2}\right) + 1 = 3\cos\left(\frac{\pi}{2}t + \left(-\frac{\pi}{2}\right)\right) + 1$.

From this, we identify $A = 3$, $\omega = \frac{\pi}{2}$, $\phi = -\frac{\pi}{2}$ and $B = 1$. According to Theorem **??**, the frequency is $\omega = \frac{\pi}{2}$, the period is $T = \frac{2\pi}{\omega} = \frac{2\pi}{\pi/2} = 4$, the phase shift is $-\frac{\phi}{\omega} = -\frac{-\pi/2}{\pi/2} = 1$ (indicating a shift to the *right* 1 unit), the amplitude is $|A| = |3| = 3$, and the vertical shift is $B = 1$ (indicating a shift *up* 1 unit.)

To graph $y = f(t)$, we know one cycle begins at $t = 1$ (the phase shift.) Since the period is 4, we know the cycle ends 4 units later at $t = 1 + 4 = 5$. If we divide the interval $[1, 5]$ into four equal pieces, each piece has length $\frac{4}{4} = 1$. Hence, we to get our quarter marks, we start with $t = 1$ and add 1 unit until we reach the endpoint, $t = 5$. Our new quarter marks are: $t = 1$, $t = 2$, $t = 3$, $t = 4$, and $t = 5$.

We now substitute these new quarter marks into $f(t)$ to obtain the corresponding y-values on the graph.[14]

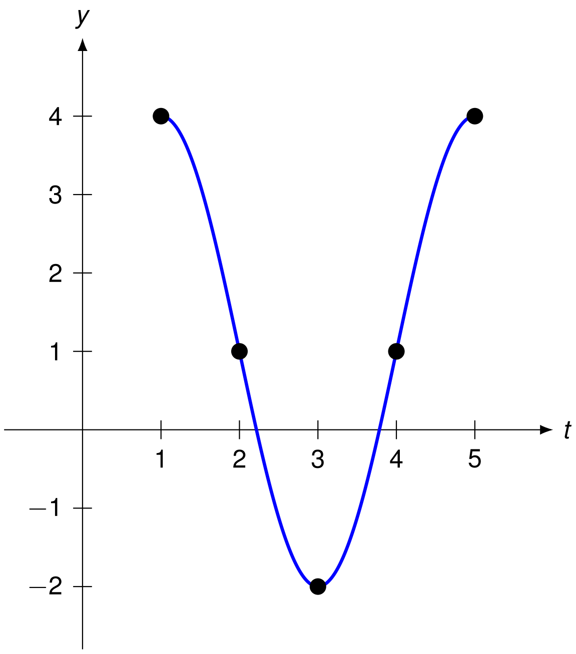
[12] If t is time measured in seconds, then one cycle per second is 1 **Hertz**.

[13] Recall that velocity is speed with a direction. In **??**, $\omega > 0$ indicated counter-clockwise motion while $\omega < 0$ indicated clockwise motion.

[14] Note when we substitute the quarter marks into $f(t)$, the argument of the cosine function simplifies to the quadrantal angles. That is, when we substitute $t = 1$, the argument of cosine simplifies to 0; when we substitute $t = 2$, the argument simplifies $\frac{\pi}{2}$ and so on. This provides a quick check of our calculations.

t	$f(t)$	$(t, f(t))$
1	4	$(1, 4)$
2	1	$(2, 1)$
3	-2	$(3, -2)$
4	1	$(4, 1)$
5	4	$(5, 4)$

We connect the dots in a 'wavelike' fashion to produce the graph below.



Note that we can (partially) spot-check our answer by noting the average of the maximum and minimum is $\frac{4+(-2)}{2} = 1$ (our vertical shift) and the amplitude, $4 - 1 = 1 - (-2) = 3$ is indeed 3.

Thought not asked for, this example provides a nice opportunity to interpret the ordinary frequency: $f = \frac{1}{T} = \frac{1}{4}$. Hence, $\frac{1}{4}$ of the sinusoid is traced out over an interval that is 1 unit long.

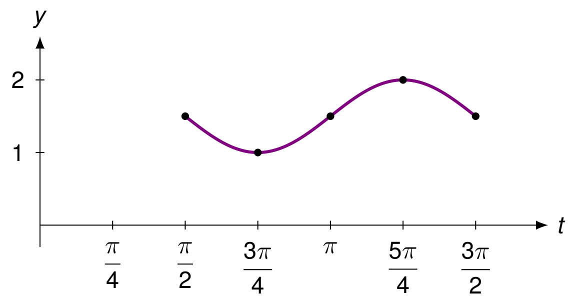
2. Turning our attention now to the function g, we first note that the coefficient of t is negative. In order to use Theorem **??**, we need that coefficient to be positive. Hence, we first use the odd property of the sine function to rewrite $\sin(\pi - 2t)$ so that instead of a coefficient of -2, t has a coefficient of 2. We get $\sin(\pi - 2t) = \sin(-2t + \pi) = \sin(-(2t - \pi)) = -\sin(2t - \pi)$. Hence, $g(t) = -\frac{1}{2}\sin(2t + (-\pi)) + \frac{3}{2}$.

   We identify $A = -\frac{1}{2}$, $\omega = 2$, $\phi = -\pi$ and $B = \frac{3}{2}$. The frequency is $\omega = 2$, the period is $T = \frac{2\pi}{2} = \pi$, the phase shift is $-\frac{-\pi}{2} = \frac{\pi}{2}$ (indicating a shift *right* $\frac{\pi}{2}$ units), the amplitude is $\left|-\frac{1}{2}\right| = \frac{1}{2}$, and, finally, the vertical shift is *up* $\frac{3}{2}$.

   Proceeding as before, we know one cycle of g starts at $t = \frac{\pi}{2}$ and ends at $t = \frac{\pi}{2} + \pi = \frac{3\pi}{2}$. Dividing the interval $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ into four equal pieces gives pieces of length $\frac{\pi}{4}$ units. Hence, to obtain our new quarter marks, we start at $t = \frac{\pi}{2}$ and add $\frac{\pi}{4}$ until we reach $t = \frac{3\pi}{2}$. Our new quarter marks are: $t = \frac{\pi}{2}$, $t = \frac{3\pi}{4}$, $t = \pi$, $t = \frac{5\pi}{4}$, $t = \frac{3\pi}{2}$. Substituting these values into g gives us the table below.

t	$g(t)$	$(t, g(t))$
$\frac{\pi}{2}$	$\frac{3}{2}$	$\left(\frac{\pi}{2}, \frac{3}{2}\right)$
$\frac{3\pi}{4}$	1	$\left(\frac{3\pi}{4}, 1\right)$
π	$\frac{3}{2}$	$\left(\pi, \frac{3}{2}\right)$
$\frac{5\pi}{4}$	2	$\left(\frac{5\pi}{4}, 2\right)$
$\frac{3\pi}{2}$	$\frac{3}{2}$	$\left(\frac{3\pi}{2}, \frac{3}{2}\right)$

From this table, we produce the graph below.



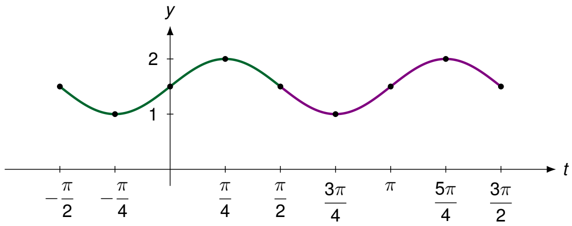
Again, we can quickly check the vertical shift by averaging the maximum and minimum values: $\frac{2+1}{2} = \frac{3}{2}$ and verify the amplitude: $2 - \frac{3}{2} = \frac{3}{2} - 1 = \frac{1}{2}$.

Note that in this section, we have discussed *two* ways to graph sinusoids: using Theorem **??** from Section **??** and using Theorem **??**. Both methods will produce one cycle of the resulting sinusoid, but each method may produce a *different* cycle of the same sinusoid.

For example, if we graphed the function $g(t) = \frac{1}{2}\sin(\pi - 2t) + \frac{3}{2}$ from Example **??** using Theorem **??**, we obtain the following:

t	$g(t)$	$(t, g(t))$
$\frac{\pi}{2}$	$\frac{3}{2}$	$\left(\frac{\pi}{2}, \frac{3}{2}\right)$
$\frac{\pi}{4}$	2	$\left(\frac{\pi}{4}, 2\right)$
0	$\frac{3}{2}$	$\left(0, \frac{3}{2}\right)$
$-\frac{\pi}{4}$	1	$\left(-\frac{\pi}{4}, 1\right)$
$-\frac{\pi}{2}$	$\frac{3}{2}$	$\left(-\frac{\pi}{2}, \frac{3}{2}\right)$

Comparing this result (in green) with the one obtained in Example **??** (in purple) side by side, we see that one cycle ends right where the other starts. The cause of this discrepancy goes back to using the odd property of sine.

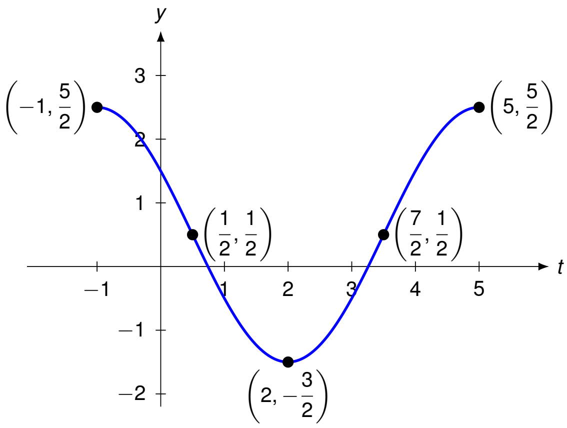


Essentially, the odd property of the sine function converts a reflection across the y-axis into a reflection across the t-axis. (Can you see why?) For this reason, whenever the coefficient of t is negative, Theorems **??** and **??** will produce different results.

In the Exercises, we assume the problems are worked using Theorem **??**. If you choose to use Theorems **??** instead, your answer may look different than what is provided even though both your answer and the textbook's answer represent *one* cycle of the *same* function.

In the next example, we use Theorem **??** to determine the formula of a sinusoid given the graph of one cycle. Note that in some disciplines, sinusoids are written in terms of sines whereas in others, cosines functions are preferred. To cover all bases, we ask for both.

Example 3. Below is the graph of one complete cycle of a sinusoid $y = f(t)$.



1. Write $f(t)$ in the form $C(t) = A\cos(\omega t + \phi) + B$, for $\omega > 0$.
2. Write $f(t)$ in the form $S(t) = A\sin(\omega t + \phi) + B$, for $\omega > 0$.

Explanation.

1. Since one cycle is graphed over the interval $[-1, 5]$, its period is $T = 5 - (-1) = 6$. According to Theorem **??**, $6 = T = \frac{2\pi}{\omega}$, so that $\omega = \frac{\pi}{3}$. Next, we see that the phase shift is -1, so we have $-\frac{\phi}{\omega} = -1$, or $\phi = \omega = \frac{\pi}{3}$.

   To find the baseline, we average the maximum and minimum values: $B = \frac{1}{2}\left[\frac{5}{2} + \left(-\frac{3}{2}\right)\right] = \frac{1}{2}(1) = \frac{1}{2}$.

   To find the amplitude, we subtract the maximum value from the baseline: $A = \frac{5}{2} - \frac{1}{2} = 2$.

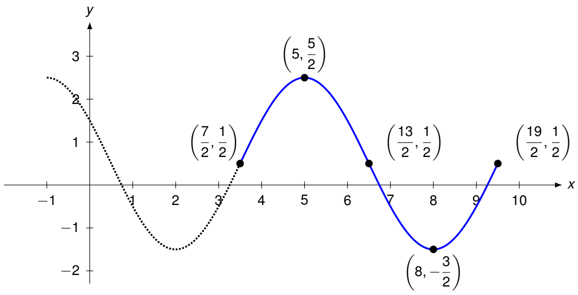
   Putting this altogether, we obtain our final answer is $f(t) = 2\cos\left(\frac{\pi}{3}t + \frac{\pi}{3}\right) + \frac{1}{2}$.

2. Since we have written $f(t)$ in terms of cosines, we can use the conversion from sine to cosine as listed in Theorem **??**. Since $\cos(t) = \sin\left(t + \frac{\pi}{2}\right)$, $\cos\left(\frac{\pi}{3}t + \frac{\pi}{3}\right) = \sin\left(\left[\frac{\pi}{3}t + \frac{\pi}{3}\right] + \frac{\pi}{2}\right)$, so $\cos\left(\frac{\pi}{3}t + \frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}t + \frac{5\pi}{6}\right)$. Our final answer is $f(t) = 2\sin\left(\frac{\pi}{3}t + \frac{5\pi}{6}\right) + \frac{1}{2}$.

However, for the sake of completeness, we provide another solution strategy which enables us to write $f(t)$ in terms of sines without starting with our answer from part 1.

Note that we obtain the period, amplitude, and vertical shift as before: $\omega = \frac{\pi}{3}$, $A = 2$ and $B = \frac{1}{2}$. The trickier part is finding the phase shift.

To that end, we imagine extending the graph of the given sinusoid as in the figure below so that we can identify a cycle beginning at $\left(\frac{7}{2}, \frac{1}{2}\right)$. Taking the phase shift to be $\frac{7}{2}$, we get $-\frac{\phi}{\omega} = \frac{7}{2}$, or $\phi = -\frac{7}{2}\omega = -\frac{7}{2}\left(\frac{\pi}{3}\right) = -\frac{7\pi}{6}$. Hence, our answer is $f(t) = 2\sin\left(\frac{\pi}{3}t - \frac{7\pi}{6}\right) + \frac{1}{2}$.



Note that each of the answers given in Example **??** is one choice out of many possible answers. For example, when fitting a sine function to the data, we could have chosen to start at $\left(\frac{1}{2}, \frac{1}{2}\right)$ taking $A = -2$. In this case, the phase shift is $\frac{1}{2}$ so $\phi = -\frac{\pi}{6}$ for an answer of $f(t) = -2\sin\left(\frac{\pi}{3}t - \frac{\pi}{6}\right) + \frac{1}{2}$. The ultimate check of any solution is to graph the answer and check it matches the given data.

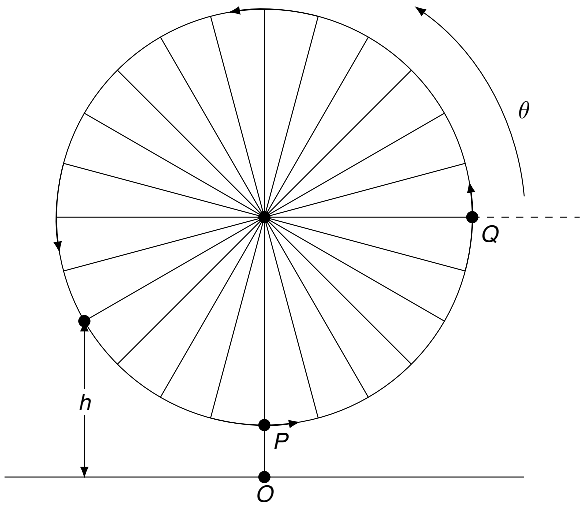
1 Applications of Sinusoids

In the same way exponential functions can be used to model a wide variety of phenomena in nature,[15] the sine and cosine functions can be used to model their fair share of natural behaviors. Our first foray into sinusoidal motion revisits circular motion - in particular Equation **??** .

Example 4. Recall from Exercise **??** in Section **??** that The Giant Wheel at Cedar Point is a circle with diameter 128 feet which sits on an 8 foot tall platform making its overall height 136 feet. It completes two revolutions in 2 minutes and 7 seconds. Assuming that the riders are at the edge of the circle, find a sinusoid which describes the height of the passengers above the ground t seconds after they pass the point on the wheel closest to the ground.

Explanation. We sketch the problem situation below and assume a counter-clockwise rotation.[16]

[15] See Section **??**.
[16] Otherwise, we could just observe the motion of the wheel from the other side.



We know from the Equation **??** in Section **??** that the y-coordinate for counter-clockwise motion on a circle of radius r centered at the origin with constant angular velocity (frequency) ω is given by $y = r\sin(\omega t)$. Here, $t = 0$ corresponds to the point $(r, 0)$ so that θ, the angle measuring the amount of rotation, is in standard position.

In our case, the diameter of the wheel is 128 feet, so the radius is $r = 64$ feet. Since the wheel completes two revolutions in 2 minutes and 7 seconds (which is 127 seconds) the period $T = \frac{1}{2}(127) = \frac{127}{2}$ seconds. Hence, the angular frequency is $\omega = \frac{2\pi}{T} = \frac{4\pi}{127}$ radians per second.

Putting these two pieces of information together, we have that $y = 64\sin\left(\frac{4\pi}{127}t\right)$ describes the y-coordinate on the Giant Wheel after t seconds, assuming it is centered at $(0, 0)$ with $t = 0$ corresponding to point Q.

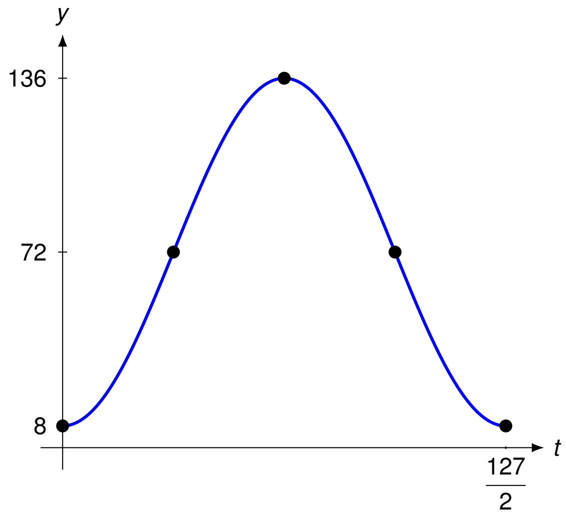
In order to find an expression for h, we take the point O in the figure as the origin. Since the base of the Giant Wheel ride is 8 feet above the ground and the Giant Wheel itself has a radius of 64 feet, its center is 72 feet above the ground. To account for this vertical shift upward,[17] we add 72 to our formula for y to obtain the new formula $h = y + 72 = 64\sin\left(\frac{4\pi}{127}t\right) + 72$.

Next, we need to adjust things so that $t = 0$ corresponds to the point P instead of the point Q. This is where the phase comes into play. Geometrically, we need to shift the angle θ in the figure back $\frac{\pi}{2}$ radians.

From the discussion motivating Equation **??**, we know $\theta = \omega t = \frac{4\pi}{127}t$, so we (temporarily) write the height in terms of θ as $h = 64\sin(\theta)+72$. Subtracting $\frac{\pi}{2}$ from θ gives $h(t) = 64\sin\left(\theta - \frac{\pi}{2}\right)+72 = 64\sin\left(\frac{4\pi}{127}t - \frac{\pi}{2}\right)+72$.

We can check the reasonableness of our answer by graphing $y = h(t)$ over the interval $\left[0, \frac{127}{2}\right]$ and visualizing the path of a person on the Big Wheel ride over the course of one rotation.

[17] We are readjusting our 'baseline' from $y = 0$ to $y = 72$.



A few remarks about Example **??** are in order. First, note that the amplitude of 64 in our answer corresponds to the radius of the Giant Wheel. This means that passengers on the Giant Wheel never stray more than 64 feet vertically from the center of the Wheel, which makes sense. Second, the phase shift of our answer works out to be $\frac{\pi/2}{4\pi/127} = \frac{127}{8} = 15.875$. This represents the 'time delay' (in seconds) we introduce by starting the motion at the point P as opposed to the point Q. Said differently, passengers which 'start' at P take 15.875 seconds to 'catch up' to the point Q.

Our next example revisits the daylight data first introduced in Section **??**, Exercise **??**.

Example 5. According to the U.S. Naval Observatory website, the number of hours H of daylight that Fairbanks, Alaska received on the 21st day of the nth month of 2009 is given below. Here $t = 1$ represents January 21, 2009, $t = 2$ represents February 21, 2009, and so on.

Month Number	1	2	3	4	5	6	7	8	9	10	11	12
Hours of Daylight	5.8	9.3	12.4	15.9	19.4	21.8	19.4	15.6	12.4	9.1	5.6	3.3

1. Find a sinusoid which models these data and graph your answer along with the data.
2. Compare your answer to part **??** to one obtained using the regression feature of a graphing utility.

Explanation. 1. To get a feel for the data, we plot it below on the left. At first glance, the data appear to be more like '$\wedge$'-shaped instead of sinusoidal, harkening back to Section **??**. However, from experience, the hours of daylight is a cyclical process, which is why we attempt to fit this data to a sine function.

Please note that when it comes down to it, fitting a sinusoid to data manually is not an exact science. We do our best to find the constants A, ω, ϕ and B so that the function $H(t) = A\sin(\omega t + \phi) + B$ closely matches the data. In this example, we first go after the vertical shift B to determine the baseline.

In a typical sinusoid, the value of B is the average of the maximum and minimum values. So here we take $B = \frac{3.3 + 21.8}{2} = 12.55$.

Next is the amplitude A which is the displacement from the baseline to the maximum (and minimum) values. We find $A = 21.8 - 12.55 = 12.55 - 3.3 = 9.25$. At this stage, our sinusoid has the form: $H(t) = 9.25\sin(\omega t + \phi) + 12.55$.

We proceed to find the angular frequency, ω. Since the data collected is over the span of a year (12 months), we take the period $T = 12$ months.[18] This means $\omega = \frac{2\pi}{T} = \frac{2\pi}{12} = \frac{\pi}{6}$.

[18] Even though the data collected lies in the interval $[1, 12]$, which has a length of 11, we need to think of the data point at $t = 1$ as a

The last quantity to find is the phase ϕ. Unlike the previous example, it is easier in this case to find the phase shift $-\dfrac{\phi}{\omega}$. Since we picked $A > 0$, the phase shift corresponds to the first value of t with $H(t) = 12.55$ (the baseline value).[19]

Here, we choose $t = 3$, since its corresponding H value of 12.4 is closer to 12.55 than the next value, 15.9, which corresponds to $t = 4$. Hence, $-\dfrac{\phi}{\omega} = 3$, so $\phi = -3\omega = -3\left(\dfrac{\pi}{6}\right) = -\dfrac{\pi}{2}$. We have $H(t) = 9.25\sin\left(\dfrac{\pi}{6}t - \dfrac{\pi}{2}\right) + 12.55$. Below is a graph of our data with the curve $y = H(t)$ courtesy of desmos.

Desmos link: `https://www.desmos.com/calculator/ofjrvq5g0w`

2. Using desmos to find a regression model produces $A \approx 8.36$ and $B \approx 12.06$, which closely match our answers from number **??**. However, desmos calculates $\omega \approx -294.81$ and $\phi \approx -26.53$, which are far from our values $\omega = \dfrac{\pi}{6}$ and $\phi = -\dfrac{\pi}{2}$.

   Indeed, in Exercise **??**, we invite the reader to graph $y = 8.36\sin(-294.81t - 26.53) + 12.06$ using desmos. At first glance, appears to be a solid gray rectangle. After zooming in, however, we see this rectangle is made up of literally hundreds of oscillations. Though desmos boasts this regression as having an R^2 value of 0.9886, meaning mathematically, this is a very good fit to the data, this curve doesn't model the real-world phenomenon of daylight hours in any sort of reasonable manner.

   Since we know the period of the sinusoid is 12 months, we set $\omega = \dfrac{\pi}{6}$ and ask desmos to run a regression for the remaining three parameters. Desmos returns the values $A \approx -8.13$, $B \approx 12.5$, and $\phi \approx -4.70$, with an R^2 value of 0.987, so its regression curve is a much more reasonable $H(t) = -8.13\sin\left(\dfrac{\pi}{6}t - 4.70\right) + 12.5$. Even though, at first glance, this curve appears much different from our solution to number **??**, the we graph both functions below, along with our data, and see they are very similar.[20] (Our solution to number **??** is the dashed red curve.)

Desmos link: `https://www.desmos.com/calculator/zje3uhkmbc`

The scenario described in Example **??** is a typical example of where the circular functions are useful outside the context of angles or circular motion. Indeed, sine and cosine functions are used extensively to model a wide range of periodic phenomena including signal analysis, wave physics, and even quantum mechanics. We close this section discussing limits involving sine and cosine.

2 Limits involving Sine and Cosine

We've already stated as fact (but not proven) that $f(t) = \sin(t)$ and $g(t) = \cos(t)$ are continuous. Per Definition **??**, this means that $\lim\limits_{t\to a}\sin(t) = \sin(a)$ and $lim_{t\to a}\cos(t) = \cos(a)$ for all real numbers, a. This means, for instance, we can compute $\lim\limits_{t\to\pi}\cos(t) = \cos(\pi) = -1$. Per the discussion following Definition **??**, so long as we avoid the usual domain pitfalls, we can also compute:

$$\lim_{t\to\pi}\frac{2t\cos(t) - \sin(3t)}{\sqrt{\cos(2t)}} = \frac{2\pi\cos(\pi) - \sin(3\pi)}{\sqrt{\cos(2\pi)}} = \frac{-2\pi - 0}{\sqrt{1}} = -2\pi$$

In the next example we investigate a few (similar looking) limits of sinusoids.

Example 6. Investigate the following limits numerically, graphically, and analytically.

representative sample of the amount of daylight for every day in January. That is, it represents $H(t)$ over the interval $[0, 1]$. Similarly, $t = 2$ is a sample of $H(t)$ over $[1, 2]$, and so forth.

[19] See the figure on page **??**.

[20] In Section **??**, we can show how similar these two functions are analytically. See Exercise **??**.

1. $\lim\limits_{t \to 0} \sin\left(\frac{\pi}{t}\right)$
2. $\lim\limits_{t \to 0} \sin\left(10^{23}\, t\right)$
3. $\lim\limits_{t \to 0} \frac{\sin(t)}{t}$

Explanation.

1. To analyze $\lim\limits_{t \to 0} \sin\left(\frac{\pi}{t}\right)$ numerically, we construct a table of values 'near' $t = 0$ for $F(t) = \sin\left(\frac{\pi}{t}\right)$. Choosing values such as $t = \pm 0.001$, $t = \pm 0.01$, etc., we get a list of '0's. This result shouldn't be too surprising since in each of these cases, $\frac{\pi}{t}$ results in an integer multiple of π. However, the graph near $t = 0$ (literally) paints a different picture. The graph appears to wildly oscillate near $t = 0$, suggesting that $\lim\limits_{t \to 0} \sin\left(\frac{\pi}{t}\right)$ does not exist.

   Desmos link: `https://www.desmos.com/calculator/ag6olhwkkb`

   Indeed, as $t \to 0^{-}$, $\frac{\pi}{t} \to -\infty$ and as $t \to 0^{+}$, $\frac{\pi}{t} \to \infty$. In other words, all of the infinite oscillations experienced by the standard sine function, $f(t) = \sin(t)$ as $t \to -\infty$ and as $t \to \infty$ are being brought back to near $t = 0$ in $F(t) = \sin\left(\frac{\pi}{t}\right)$. Hence, $\lim\limits_{t \to 0} \sin\left(\frac{\pi}{t}\right)$ does not exist.

2. We proceed as above to investigate $\lim\limits_{t \to 0} \sin\left(10^{23}\, t\right)$. A table and a graph of $F(t) = \sin\left(10^{23}\, t\right)$ 'near' $t = 0$ produces numerical and graphical noise. Neither the table nor the graph suggest $\lim\limits_{t \to 0} F(t)$ exists. However, we know $F(t) = \sin\left(10^{23}\, t\right)$ is a sinusoid with frequency $\omega = 10^{23}$. Hence, resetting the window on the desmos interactive below to $-\frac{\pi}{10^{23}} \leq x \leq \frac{\pi}{10^{23}}$ and $-1 \leq y \leq 1$, we see one cycle of F which suggests the limit is, in fact, 0.

   Desmos link: `https://www.desmos.com/calculator/jue9qbhzmb`

   Indeed, we know $F(t) = \sin\left(10^{23}\, t\right)$ is continuous, so $\lim\limits_{t \to 0} \sin\left(10^{23}\, t\right) = \sin\left(10^{23}(0)\right) = \sin(0) = 0$.

3. Next we investigate $F(t) = \frac{\sin(t)}{t}$ near $t = 0$. The table and graph both suggest $\lim\limits_{t \to 0} \frac{\sin(t)}{t} = 1$.

   Desmos link: `https://www.desmos.com/calculator/ohffbwnwsf`

   Attempting to analyze this limit analytically, we see that $\lim\limits_{t \to 0} \frac{\sin(t)}{t}$ produces the indeterminate form '$\frac{0}{0}$'. At this stage, we do not have the tools to either prove or disprove our conjecture that $\lim\limits_{t \to 0} \frac{\sin(t)}{t} = 1$, but we will soon, [21] so stay tuned!

[21] See Exercises **??** through **??** in Section **??**.