The Radian Measure of Angles

In Section **??**, we review the concept of (oriented) angles and degree measure. While degrees are the unit of choice for many applications of trigonometry, we introduce here the concept of the **radian measure** of an angle. As we will see, this concept naturally ties angles to real numbers. While the concept may seem foreign at first, we assure the reader that the utility of radian measure in modeling real-world phenomena is well worth the effort. We begin our development with a definition from Geometry.

Definition 1. The real number π is defined to be the ratio of a circle's circumference to its diameter. In symbols, given a circle of circumference C and diameter d,

$$\pi = \frac{C}{d}$$

While Definition **??** is quite possibly the 'standard' definition of π, the authors would be remiss if we didn't mention that buried in this definition is actually a theorem. As the reader is probably aware, the number π is a mathematical constant - that is, it doesn't matter *which* circle is selected, the ratio of its circumference to its diameter will have the same value as any other circle. While this is indeed true, it is far from obvious and leads to a counterintuitive scenario which is explored in the Exercises. Since the diameter of a circle is twice its radius, we can quickly rearrange the equation in Definition **??** to get a formula more useful for our purposes, namely: $2\pi = \frac{C}{r}$. Hence, for any circle, the ratio of its circumference to its radius is 2π.

Suppose we take a *portion* of a circle and compare some arc measuring s units in length along the circumference to the radius, r. Let θ be the **central angle** subtended by this arc, that is, an angle whose vertex is the center of the circle and whose determining rays pass through the endpoints of the arc.

Using the GeoGebra interactive below, click and drag the point P to change the length of the arc, s (and hence the angle, θ.) If we keep θ the same and adjust the radius of the circle, r, notice that the ratio $\frac{s}{r}$ remains constant.[1]

GeoGebra link: `https://www.geogebra.org/m/nkfhevs7`

The same proportionality (similarity) arguments used to show π is a constant shows the ratio $\frac{s}{r}$ is also constant among all circles. It is this ratio, $\frac{s}{r}$, which defines the **radian measure** of an angle.

To get a better feel for radian measure, consider the GeoGebra interactive below. Note there are two sliders: one to adjust the radian measure of the central angle, θ, and one to adjust the radius, r. The length of the subtended arc, s, is marked off into increments of r. Notice, for instance, that for an angle θ with radian measure 2, the length of s is exactly double the radius, r. That is, s sweeps out two 'radius lengths' worth of circumference. Likewise, if $\theta = 3$ radians, every subtended arc sweeps out three radius lengths.

GeoGebra link: `https://www.geogebra.org/m/brnjyatt`

In general, the radian measure of an angle θ tells us how many 'radius lengths' we need to sweep out along the circle to subtend the angle θ.

Since one revolution sweeps out the circumference $2\pi r$, one revolution has radian measure $\frac{2\pi r}{r} = 2\pi$. From this we can find the radian measure of other central angles using proportions, just like we did with degrees. For instance, half of a revolution has radian measure $\frac{1}{2}(2\pi) = \pi$, a quarter revolution has radian measure $\frac{1}{4}(2\pi) = \frac{\pi}{2}$, and so forth. Note that, by definition, the radian measure of an angle is a length divided by another length so that these measurements are actually dimensionless and are considered 'pure' numbers. For this reason, we do not use any symbols to denote radian measure, but we use the word 'radians' to denote

[1] Also note that as we stretch s to cover one circumference (hence one complete revolution for θ) the ratio approaches 2π.

these dimensionless units as needed. For instance, we say one revolution measures '2π radians,' half of a revolution measures 'π radians,' and so forth.

As with degree measure, the distinction between the angle itself and its measure is often blurred in practice, so when we write '$\theta = \frac{\pi}{2}$', we mean θ is an angle which measures $\frac{\pi}{2}$ radians.[2] We extend radian measure to oriented angles, just as we did with degrees beforehand, so that a positive measure indicates counter-clockwise rotation and a negative measure indicates clockwise rotation.[3] Much like before, two positive angles α and β are supplementary if $\alpha + \beta = \pi$ and complementary if $\alpha + \beta = \frac{\pi}{2}$. Finally, we leave it to the reader to show that when using radian measure, two angles α and β are coterminal if and only if $\beta = \alpha + 2\pi k$ for some integer k.

Example 1. Graph each of the (oriented) angles below in standard position and classify them according to where their terminal side lies. Find three coterminal angles, at least one of which is positive and one of which is negative.

1. $\alpha = \frac{\pi}{6}$
2. $\beta = -\frac{4\pi}{3}$
3. $\gamma = \frac{9\pi}{4}$
4. $\phi = -\frac{5\pi}{2}$

Explanation. 1. The angle $\alpha = \frac{\pi}{6}$ is positive, so we draw an angle with its initial side on the positive x-axis and rotate counter-clockwise $\frac{(\pi/6)}{2\pi} = \frac{1}{12}$ of a revolution. Thus α is a Quadrant I angle. Coterminal angles θ are of the form $\theta = \alpha + 2\pi \cdot k$, for some integer k. To make the arithmetic a bit easier, we note that $2\pi = \frac{12\pi}{6}$, thus when $k = 1$, we get $\theta = \frac{\pi}{6} + \frac{12\pi}{6} = \frac{13\pi}{6}$. Substituting $k = -1$ gives $\theta = \frac{\pi}{6} - \frac{12\pi}{6} = -\frac{11\pi}{6}$ and when we let $k = 2$, we get $\theta = \frac{\pi}{6} + \frac{24\pi}{6} = \frac{25\pi}{6}$.

Using the interactive below, we can visualize sweeping out α and identifying coterminal angles. The circle is divided into 12 equal sectors, each corresponding to a $\frac{\pi}{6}$'s worth of radian measure. Starting with the slider at $n = 0$, we move the slider to $n = (+)1$ which sweeps out $\alpha = (+1)\frac{\pi}{6} = \frac{\pi}{6}$ radians from the positive x-axis.

Now that we've found the terminal side for α, we can look for a coterminal angle by moving the slider more to the right (increasing n and continuing to rotate counter-clockwise) until we reach the same terminal side as θ. Our first such instance occurs when $n = 13$. Hence, $\theta = 13\frac{\pi}{6} = \frac{13\pi}{6}$, is a coterminal angle. Notice that we moved $n = 12$ additional $\frac{\pi}{6}$'s, or $12\frac{\pi}{6} = \frac{12\pi}{6} = 2\pi$ radians, counter-clockwise to reach this coterminal angle.

Continuing to move the slider to the right sweeps out additional $\frac{\pi}{6}$ sectors in the counter-clockwise direction until we find a second (positive) coterminal angle when $n = 25$. Hence, $\theta = \frac{25\pi}{6}$ is a second positive coterminal angle. Notice that here, we moved $n = 24$ additional $\frac{\pi}{6}$'s, that is $24\frac{\pi}{6} = \frac{24\pi}{6} = 4\pi$ radians, counter-clockwise to reach this coterminal angle.

[2] The authors are well aware that we are now identifying radians with real numbers. We will justify this shortly.
[3] This, in turn, endows the subtended arcs with an orientation as well. We address this in short order.

Next, we reset the slider to $n = 1$, so we're back to $\alpha = \frac{\pi}{6}$, and begin moving the slider to the *left*. This corresponds to rotating *clockwise* in increments of $\frac{\pi}{6}$. Eventually we find another coterminal angle when $n = -11$. This value of n corresponds to the coterminal angle $\theta = -\frac{11\pi}{6}$. Notice that in this case, we moved $n = 12$ $\frac{\pi}{6}$'s, in other words $12\frac{\pi}{6} = \frac{12\pi}{6} = 2\pi$ radians, *clockwise* to reach this coterminal angle.

GeoGebra link: `https://www.geogebra.org/m/v42gefvn`

2. Since $\beta = -\frac{4\pi}{3}$ is negative, we start at the positive x-axis and rotate clockwise $\frac{(4\pi/3)}{2\pi} = \frac{2}{3}$ of a revolution. We find β to be a Quadrant II angle. To find coterminal angles, we proceed as before using $2\pi = \frac{6\pi}{3}$, and compute $\theta = -\frac{4\pi}{3} + \frac{6\pi}{3} \cdot k$ for integer values of k. We obtain $\frac{2\pi}{3}$, $-\frac{10\pi}{3}$ and $\frac{8\pi}{3}$ as coterminal angles.

   As above, the GeoGebra interactive below allows us to view the above process geometrically.

   GeoGebra link: `https://www.geogebra.org/m/psvy7avh`

3. Since $\gamma = \frac{9\pi}{4}$ is positive, we rotate counter-clockwise from the positive x-axis. One full revolution accounts for $2\pi = \frac{8\pi}{4}$ of the radian measure with $\frac{\pi}{4}$ or $\frac{1}{8}$ of a revolution remaining. We have γ as a Quadrant I angle. All angles coterminal with γ are of the form $\theta = \frac{9\pi}{4} + \frac{8\pi}{4} \cdot k$, where k is an integer. Working through the arithmetic, we find: $\frac{\pi}{4}$, $-\frac{7\pi}{4}$ and $\frac{17\pi}{4}$.

   We can confirm our answers using the GeoGebra interactive below.

   GeoGebra link: `https://www.geogebra.org/m/qc3vqpvj`

4. To graph $\phi = -\frac{5\pi}{2}$, we begin our rotation clockwise from the positive x-axis. As $2\pi = \frac{4\pi}{2}$, after one full revolution clockwise, we have $\frac{\pi}{2}$ or $\frac{1}{4}$ of a revolution remaining. Since the terminal side of ϕ lies on the negative y-axis, ϕ is a quadrantal angle. To find coterminal angles, we compute $\theta = -\frac{5\pi}{2} + \frac{4\pi}{2} \cdot k$ for a few integers k and obtain $-\frac{\pi}{2}$, $\frac{3\pi}{2}$ and $\frac{7\pi}{2}$.

   We can visualize and confirm our answers using the interactive below.

   GeoGebra link: `https://www.geogebra.org/m/hexty8b3`

It is worth mentioning that we could have plotted the angles in Example **??** by first converting them to degree measure and following the procedure set forth in Example **??**. While converting back and forth from degrees and radians is certainly a good skill to have, it is best that you learn to 'think in radians' as well as you can 'think in degrees'. The authors would, however, be derelict in our duties if we ignored the basic conversion between these systems altogether.

Since one revolution counter-clockwise measures 360° and the same angle measures 2π radians, we can use the proportion $\frac{2\pi \text{ radians}}{360^\circ}$, or its reduced equivalent, $\frac{\pi \text{ radians}}{180^\circ}$, as the conversion factor between the two systems. For example, to convert 60° to radians we find $60^\circ \left(\frac{\pi \text{ radians}}{180^\circ}\right) = \frac{\pi}{3}$ radians, or simply $\frac{\pi}{3}$. To convert from radian measure back to degrees, we multiply by the ratio $\dfrac{180^\circ}{\pi \text{ radian}}$. For example, $-\dfrac{5\pi}{6}$ radians is equal to $\left(-\dfrac{5\pi}{6} \text{ radians}\right)\left(\dfrac{180^\circ}{\pi \text{ radians}}\right) = -150^\circ$.[4] Hence, an angle which measures 1 in radian measure is equal to $\dfrac{180^\circ}{\pi} \approx 57.2958^\circ$. To summarize:

Equation 0.1. Degree - Radian Conversion:

- To convert degree measure to radian measure, multiply by $\dfrac{\pi \text{ radians}}{180^\circ}$
- To convert radian measure to degree measure, multiply by $\dfrac{180^\circ}{\pi \text{ radians}}$

In light of Example **??** and Equation **??**, the reader may well wonder what the allure of radian measure is. The numbers involved are, admittedly, much more complicated than degree measure. The answer lies in how easily angles in radian measure can be identified with real numbers.

Consider the Unit Circle, $x^2 + y^2 = 1$, as drawn below, the angle θ in standard position and the corresponding arc measuring s units in length. By definition, and the fact that the Unit Circle has radius 1, the radian measure of θ is $\dfrac{s}{r} = \dfrac{s}{1} = s$ so that, once again blurring the distinction between an angle and its measure, we have $\theta = s$.

In order to identify real numbers with oriented angles, we essentially 'wrap' the real number line around the Unit Circle and associating to each real number t an *oriented* arc on the Unit Circle with initial point $(1, 0)$.

Viewing the vertical line $x = 1$ as another real number line demarcated like the y-axis, given a real number $t > 0$, we 'wrap' the (vertical) interval $[0, t]$ around the Unit Circle in a counter-clockwise fashion.

This process is best seen in the GeoGebra interactive below on the left. By adjusting the slider upwards from 0, we see as we move up the vertical line $x = 1$ from $(1, 0)$ to the point $(1, t)$, an arc of length t units is traced out counter-clockwise along the Unit Circle. Since this arc has a length of t units, the corresponding angle has radian measure equal to t.

GeoGebra link: `https://www.geogebra.org/m/aapdqza4`

If $t < 0$, we wrap the interval $[t, 0]$ *clockwise* around the Unit Circle. This is seen above using the interactive on the right. Adjusting the slider and moving downwards, we once again trace out an interval on the line $x = 1$ from $(1, 0)$ to $(1, t)$. Here, the arc wraps *clockwise* around the Unit Circle. Since we have defined clockwise rotation as having negative radian measure, the angle determined by this arc has radian measure equal to t in this case as well.

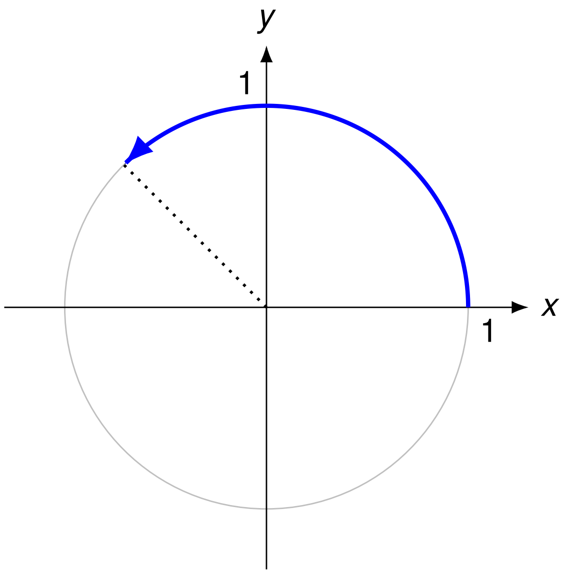
Finally, note that if $t = 0$, we are at the point $(1, 0)$ on the x-axis which corresponds to an angle with radian measure 0. In this way, we identify each real number t with the corresponding angle with radian measure t.

Example 2. Sketch the oriented arc on the Unit Circle corresponding to each of the following real numbers.

1. $t = \dfrac{3\pi}{4}$
2. $t = -2\pi$
3. $t = -2$
4. $t = 117$

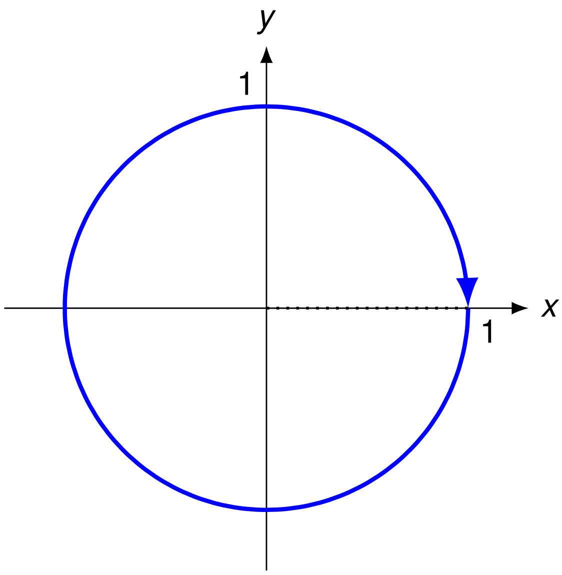
[4] Note that the negative sign indicates clockwise rotation in both systems, and so it is carried along accordingly.

Explanation. 1. The arc associated with $t = \frac{3\pi}{4}$ is the arc on the Unit Circle which subtends the angle $\frac{3\pi}{4}$ in radian measure. Since $\frac{3\pi}{4}$ is $\frac{3}{8}$ of a revolution, we have an arc which begins at the point $(1, 0)$ proceeds counter-clockwise up to midway through Quadrant II.



$$t = \frac{3\pi}{4}$$

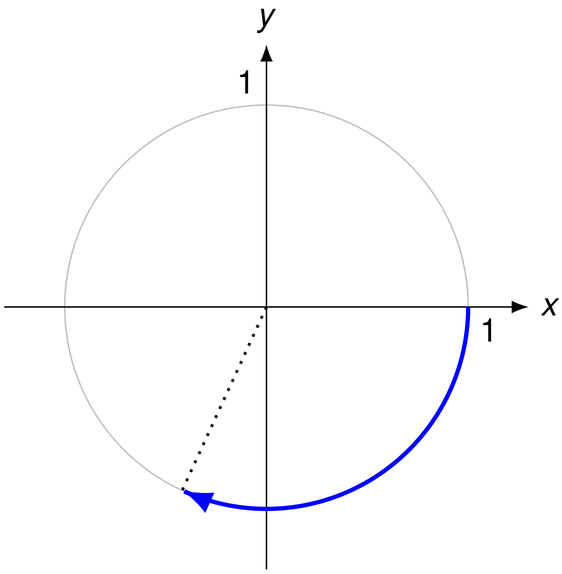
2. Since one revolution is 2π radians, and $t = -2\pi$ is negative, we graph the arc which begins at $(1, 0)$ and proceeds *clockwise* for one full revolution.



$$t = -2\pi$$

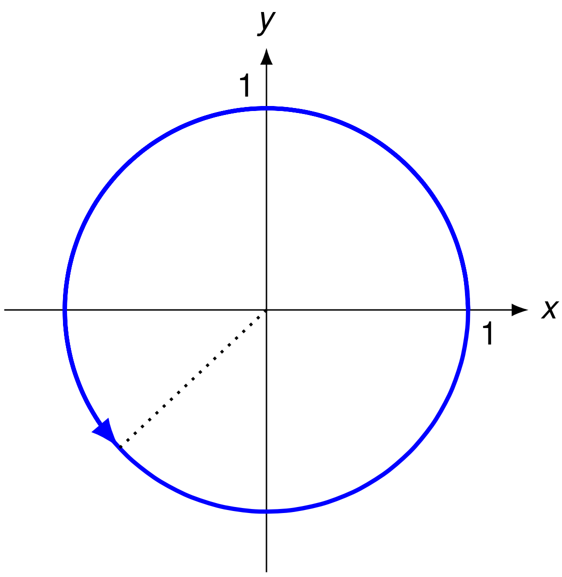
3. Like $t = -2\pi$, $t = -2$ is negative, so we begin our arc at $(1, 0)$ and proceed clockwise around the unit circle. Since $\pi \approx 3.14$ and $\frac{\pi}{2} \approx 1.57$, we find that rotating 2 radians clockwise from the point $(1, 0)$

lands us in Quadrant III. To more accurately place the endpoint, we proceed as we did in Example **??**, successively halving the angle measure until we find $\frac{5\pi}{8} \approx 1.96$ which tells us our arc extends just a bit beyond the quarter mark into Quadrant III.



$t = -2$

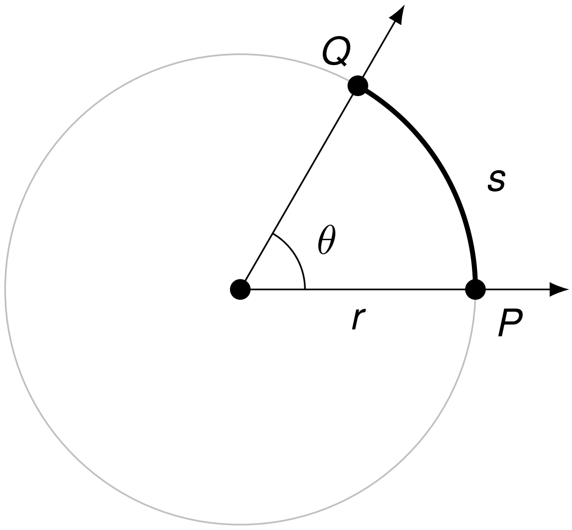
4. Since 117 is positive, the arc corresponding to $t = 117$ begins at $(1, 0)$ and proceeds counter-clockwise. As 117 is much greater than 2π, we wrap around the Unit Circle several times before finally reaching our endpoint. We approximate $\frac{117}{2\pi}$ as 18.62 which tells us we complete 18 revolutions counter-clockwise with 0.62, or just shy of $\frac{5}{8}$ of a revolution to spare. In other words, the terminal side of the angle which measures 117 radians in standard position is just short of being midway through Quadrant III.



$t = 117$

1 Applications of Radian Measure: Circular Motion

Now that we have paired angles with real numbers via radian measure, a whole world of applications awaits us. Our first excursion into this realm comes by way of circular motion. Suppose an object is moving as pictured below along a circular path of radius r from the point P to the point Q in an amount of time t.



Here s represents a *displacement* so that $s > 0$ means the object is traveling in a counter-clockwise direction and $s < 0$ indicates movement in a clockwise direction. Note that with this convention the formula we used to define radian measure, namely $\theta = \frac{s}{r}$, still holds since a negative value of s incurred from a clockwise displacement matches the negative we assign to θ for a clockwise rotation. In Physics, the **average velocity** of the object, denoted $\overline{v}$ and read as 'v-bar', is defined as the average rate of change of the position of the object with respect to time.[5] As a result, we have $\overline{v} = \frac{\text{displacement}}{\text{time}} = \frac{s}{t}$. The quantity $\overline{v}$ has units of $\frac{\text{length}}{\text{time}}$ and conveys two ideas: the direction in which the object is moving and how fast the position of the object is changing. The contribution of direction in the quantity $\overline{v}$ is either to make it positive (in the case of counter-clockwise motion) or negative (in the case of clockwise motion), so that the quantity $|\overline{v}|$ quantifies how fast the object is moving - it is the **speed** of the object. Measuring θ in radians we have $\theta = \frac{s}{r}$ thus $s = r\theta$ and

$$\overline{v} = \frac{s}{t} = \frac{r\theta}{t} = r \cdot \frac{\theta}{t}$$

The quantity $\frac{\theta}{t}$ is called the **average angular velocity** of the object. It is denoted by $\overline{\omega}$ and is read 'omega-bar'. The quantity $\overline{\omega}$ is the average rate of change of the angle θ with respect to time and thus has units $\frac{\text{radians}}{\text{time}}$. If $\overline{\omega}$ is constant throughout the duration of the motion, then it can be shown[6] that the average velocities involved, namely $\overline{v}$ and $\overline{\omega}$, are the same as their instantaneous counterparts, v and ω, respectively. In this case, v is simply called the 'velocity' of the object and ω is called the 'angular velocity.'[7]

If the path of the object were 'uncurled' from a circle to form a line segment, then the velocity of the object on that line segment would be the same as the velocity on the circle. For this reason, the quantity v is often called the *linear* velocity of the object in order to distinguish it from the *angular* velocity, ω. Putting together the ideas of the previous paragraph, we get the following.

Equation 0.2. Velocity for Circular Motion: For an object moving on a circular path of radius r with constant angular velocity ω, the (linear) velocity of the object is given by $v = r\omega$.

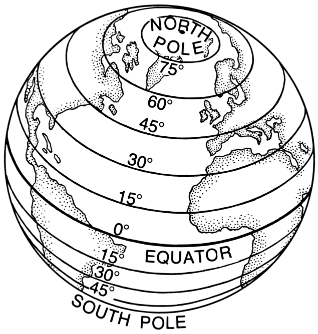
[5] See Definition **??** in Section **??** for a review of this concept.

[6] You guessed it, using Calculus . . .

[7] See Example **??** in Section **??** for more of a discussion on instantaneous velocity.

We need to talk about units here. The units of v are $\frac{\text{length}}{\text{time}}$, the units of r are length only, and the units of ω are $\frac{\text{radians}}{\text{time}}$. Thus the left hand side of the equation $v = r\omega$ has units $\frac{\text{length}}{\text{time}}$, whereas the right hand side has units $\text{length} \cdot \frac{\text{radians}}{\text{time}} = \frac{\text{length} \cdot \text{radians}}{\text{time}}$. The supposed contradiction in units is resolved by remembering that radians are a dimensionless quantity and angles in radian measure are identified with real numbers so that the units $\frac{\text{length} \cdot \text{radians}}{\text{time}}$ reduce to the units $\frac{\text{length}}{\text{time}}$. We are long overdue for an example.

Example 3. Assuming that the surface of the Earth is a sphere, any point on the Earth can be thought of as an object traveling on a circle (this is the **parallel of latitude** of the point) as seen in the figure below.[8] Since it takes the Earth (approximately) 24 hours to rotate, the object takes 24 hours to complete one revolution along this circle. Lakeland Community College is at 41.628° north latitude, and it can be shown[9] that the radius of the earth at this Latitude is approximately 2960 miles. Find the linear velocity, in miles per hour, of Lakeland Community College as the world turns.



Explanation. To use the formula $v = r\omega$, we first need to compute the angular velocity ω. The earth makes one revolution in 24 hours, and one revolution is 2π radians, so $\omega = \frac{2\pi \text{ radians}}{24 \text{ hours}} = \frac{\pi}{12 \text{ hours}}$. Note that once again, we are identifying angles in radian measure as real numbers so we can drop the 'radian' units as they are dimensionless. Also note that for simplicity's sake, we assume that we are viewing the rotation of the earth as counter-clockwise so $\omega > 0$. Hence, the linear velocity is

$$v = 2960 \text{ miles} \cdot \frac{\pi}{12 \text{ hours}} \approx 775 \, \frac{\text{miles}}{\text{hour}}$$

It is worth noting that the quantity $\frac{1 \text{ revolution}}{24 \text{ hours}}$ in Example **??** is called the **ordinary frequency** of the motion and is usually denoted by the variable f. The ordinary frequency is a measure of how often an object makes a complete cycle of the motion. The fact that $\omega = 2\pi f$ suggests that ω is also a frequency. Indeed, it is called the **angular frequency** of the motion. On a related note, the quantity $T = \frac{1}{f}$ is called the **period** of the motion and is the amount of time it takes for the object to complete one cycle of the motion. In the scenario of Example **??**, the period of the motion is 24 hours, or one day.

The concepts of frequency and period help frame the equation $v = r\omega$ in a new light. That is, if ω is fixed, points which are farther from the center of rotation need to travel faster to maintain the same angular frequency since they have farther to travel to make one revolution in one period's time. The distance of the object to the center of rotation is the radius of the circle, r, and is the 'magnification factor' which relates ω and v. We will have more to say about frequencies and periods in Section **??**. While we have exhaustively discussed velocities associated with circular motion, we have yet to discuss a more natural question: if an object is moving on a circular path of radius r with a fixed angular velocity (frequency) ω, what is the position of the object at time t? The answer to this question is the very heart of Trigonometry and is answered in the next section.

[8] Diagram credit: Pearson Scott Foresman [Public domain], via Wikimedia Commons.

[9] We will discuss how we arrived at this approximation in Example **??**.