Exercises for The Algebra of Difference Quotients

Question 1 *In Exercises - , find and simplify the difference quotient* $\frac{f(x+h)-f(x)}{h}$ *for the given function.*

Problem 1.1 $f(x) = 2x - 5$

2

Problem 1.2 $f(x) = -3x + 5$

-3

Problem 1.3 $f(x) = 6$

0

Problem 1.4 $f(x) = 3x^2 - x$

$6x + 3h - 1$

Problem 1.5 $f(x) = -x^2 + 2x - 1$

$-2x - h + 2$

Problem 1.6 $f(x) = 4x^2$

$8x + 4h$

Problem 1.7 $f(x) = x - x^2$

$-2x - h + 1$

Problem 1.8 $f(x) = x^3 + 1$

$3x^2 + 3xh + h^2$

Problem 1.9 $f(x) = mx + b$ where $m \neq 0$

m

Problem 1.10 $f(x) = ax^2 + bx + c$ where $a \neq 0$

$2ax + ah + b$

Problem 1.11 $f(x) = \frac{2}{x}$

$$\frac{-2}{x(x+h)}$$

Problem 1.12 $f(x) = \frac{3}{1-x}$

$$\frac{3}{(1-x-h)(1-x)}$$

Problem 1.13 $f(x) = \frac{1}{x^2}$

$$\frac{-(2x+h)}{x^2(x+h)^2}$$

Problem 1.14 $f(x) = \dfrac{2}{x+5}$

$\dfrac{-2}{(x+5)(x+h+5)}$

Problem 1.15 $f(x) = \dfrac{1}{4x-3}$

$\dfrac{-4}{(4x-3)(4x+4h-3)}$

Problem 1.16 $f(x) = \dfrac{3x}{x+1}$

$\dfrac{3}{(x+1)(x+h+1)}$

Problem 1.17 $f(x) = \dfrac{x}{x-9}$

$\dfrac{-9}{(x-9)(x+h-9)}$

Problem 1.18 $f(x) = \dfrac{x^2}{2x+1}$

$\dfrac{2x^2+2xh+2x+h}{(2x+1)(2x+2h+1)}$

Problem 1.19 $f(x) = \sqrt{x-9}$

$\dfrac{1}{\sqrt{x+h-9}+\sqrt{x-9}}$

Problem 1.20 $f(x) = \sqrt{2x+1}$

$\dfrac{2}{\sqrt{2x+2h+1}+\sqrt{2x+1}}$

Problem 1.21 $f(x) = \sqrt{-4x+5}$

$\dfrac{-4}{\sqrt{-4x-4h+5}+\sqrt{-4x+5}}$

Problem 1.22 $f(x) = \sqrt{4-x}$

$\dfrac{-1}{\sqrt{4-x-h}+\sqrt{4-x}}$

Problem 1.23 $f(x) = \sqrt{ax+b}$*, where* $a \neq 0$.

$\dfrac{a}{\sqrt{ax+ah+b}+\sqrt{ax+b}}$

Problem 1.24 $f(x) = x\sqrt{x}$

$\dfrac{3x^2+3xh+h^2}{(x+h)^{3/2}+x^{3/2}}$

Problem 1.25 $f(x) = \sqrt[3]{x}$

Hint: $(a - b)\left(a^2 + ab + b^2\right) = a^3 - b^3$

$$\frac{1}{(x+h)^{2/3} + (x+h)^{1/3}x^{1/3} + x^{2/3}}$$

Question 2 *In Exercises - , $C(x)$ denotes the cost to produce x items and $p(x)$ denotes the price-demand function in the given economic scenario. In each Exercise, do the following:*

- *Find and interpret $C(0)$.*
- *Find and interpret $\overline{C}(10)$.*
- *Find and interpret $p(5)$*
- *Find and simplify $R(x)$.*
- *Find and simplify $P(x)$.*
- *Solve $P(x) = 0$ and interpret.*

Problem 2.1 *The cost, in dollars, to produce x "I'd rather be a Sasquatch" T-Shirts is $C(x) = 2x + 26$, $x \geq 0$ and the price-demand function, in dollars per shirt, is $p(x) = 30 - 2x$, $0 \leq x \leq 15$.*

- *Find and interpret $C(0)$.*

  $C(0) = 26$*, so the fixed costs are* \$26.
- *Find and interpret $\overline{C}(10)$.*

  $\overline{C}(10) = 4.6$*, so when 10 shirts are produced, the cost per shirt is* \$4.60.
- *Find and interpret $p(5)$*

  $p(5) = 20$*, so to sell* 5 *shirts, set the price at* \$20 *per shirt.*
- *Find and simplify $R(x)$.*

  $R(x) = -2x^2 + 30x$, $0 \leq x \leq 15$
- *Find and simplify $P(x)$.*

  $P(x) = -2x^2 + 28x - 26$, $0 \leq x \leq 15$
- *Solve $P(x) = 0$ and interpret.*

  $P(x) = 0$ *when* $x = 1$ *and* $x = 13$. *These are the 'break even' points, so selling* 1 *shirt or* 13 *shirts will guarantee the revenue earned exactly recoups the cost of production.*

Problem 2.2 *The cost, in dollars, to produce x bottles of* 100% *All-Natural Certified Free-Trade Organic Sasquatch Tonic is $C(x) = 10x + 100$, $x \geq 0$ and the price-demand function, in dollars per bottle, is $p(x) = 35 - x$, $0 \leq x \leq 35$.*

- *Find and interpret $C(0)$.*

  $C(0) = 100$*, so the fixed costs are* \$100.
- *Find and interpret $\overline{C}(10)$.*

  $\overline{C}(10) = 20$*, so when 10 bottles of tonic are produced, the cost per bottle is* \$20.
- *Find and interpret $p(5)$*

  $p(5) = 30$*, so to sell* 5 *bottles of tonic, set the price at* \$30 *per bottle.*

- *Find and simplify $R(x)$.*

  $R(x) = -x^2 + 35x$, $0 \leq x \leq 35$

- *Find and simplify $P(x)$.*

  $P(x) = -x^2 + 25x - 100$, $0 \leq x \leq 35$

- *Solve $P(x) = 0$ and interpret.*

  $P(x) = 0$ *when* $x = 5$ *and* $x = 20$. *These are the 'break even' points, so selling* 5 *bottles of tonic or* 20 *bottles of tonic will guarantee the revenue earned exactly recoups the cost of production.*

Problem 2.3 *The cost, in cents, to produce x cups of Mountain Thunder Lemonade at Junior's Lemonade Stand is* $C(x) = 18x + 240$, $x \geq 0$ *and the price-demand function, in cents per cup, is* $p(x) = 90 - 3x$, $0 \leq x \leq 30$.

- *Find and interpret $C(0)$.*

  $C(0) = 240$, *so the fixed costs are* 240¢ *or* \$2.40.

- *Find and interpret $\overline{C}(10)$.*

  $\overline{C}(10) = 42$, *so when 10 cups of lemonade are made, the cost per cup is* 42¢.

- *Find and interpret $p(5)$*

  $p(5) = 75$, *so to sell* 5 *cups of lemonade, set the price at* 75¢ *per cup.*

- *Find and simplify $R(x)$.*

  $R(x) = -3x^2 + 90x$, $0 \leq x \leq 30$

- *Find and simplify $P(x)$.*

  $P(x) = -3x^2 + 72x - 240$, $0 \leq x \leq 30$

- *Solve $P(x) = 0$ and interpret.*

  $P(x) = 0$ *when* $x = 4$ *and* $x = 20$. *These are the 'break even' points, so selling* 4 *cups of lemonade or* 20 *cups of lemonade will guarantee the revenue earned exactly recoups the cost of production.*

Problem 2.4 *The daily cost, in dollars, to produce x Sasquatch Berry Pies* $C(x) = 3x + 36$, $x \geq 0$ *and the price-demand function, in dollars per pie, is* $p(x) = 12 - 0.5x$, $0 \leq x \leq 24$.

- *Find and interpret $C(0)$.*

  $C(0) = 36$, *so the daily fixed costs are* \$36.

- *Find and interpret $\overline{C}(10)$.*

  $\overline{C}(10) = 6.6$, *so when 10 pies are made, the cost per pie is* \$6.60.

- *Find and interpret $p(5)$*

  $p(5) = 9.5$, *so to sell* 5 *pies a day, set the price at* \$9.50 *per pie.*

- *Find and simplify $R(x)$.* $R(x) = -0.5x^2 + 12x$, $0 \leq x \leq 24$

- *Find and simplify $P(x)$.*

  $P(x) = -0.5x^2 + 9x - 36$, $0 \leq x \leq 24$

- *Solve $P(x) = 0$ and interpret.*

  $P(x) = 0$ *when* $x = 6$ *and* $x = 12$. *These are the 'break even' points, so selling* 6 *pies or* 12 *pies a day will guarantee the revenue earned exactly recoups the cost of production.*

Problem 2.5 *The monthly cost, in hundreds of dollars, to produce x custom built electric scooters is* $C(x) = 20x + 1000$, $x \geq 0$ *and the price-demand function, in hundreds of dollars per scooter, is* $p(x) = 140 - 2x$, $0 \leq x \leq 70$.

- *Find and interpret* $C(0)$.

  $C(0) = 1000$, *so the monthly fixed costs are* 1000 *hundred dollars, or* \$100,000.

- *Find and interpret* $\overline{C}(10)$.

  $\overline{C}(10) = 120$, *so when 10 scooters are made, the cost per scooter is* 120 *hundred dollars, or* \$12,000.

- *Find and interpret* $p(5)$

  $p(5) = 130$, *so to sell* 5 *scooters a month, set the price at* 130 *hundred dollars, or* \$13,000 *per scooter.*

- *Find and simplify* $R(x)$.

  $R(x) = -2x^2 + 140x$, $0 \leq x \leq 70$

- *Find and simplify* $P(x)$.

  $P(x) = -2x^2 + 120x - 1000$, $0 \leq x \leq 70$

- *Solve* $P(x) = 0$ *and interpret.*

  $P(x) = 0$ *when* $x = 10$ *and* $x = 50$. *These are the 'break even' points, so selling* 10 *scooters or* 50 *scooters a month will guarantee the revenue earned exactly recoups the cost of production.*

Problem 2.6 *Let us return to Example **??** where* $C(x) = .03x^3 - 4.5x^2 + 225x + 250$ *denotes the cost, in dollars, of producing x PortaBoy game systems. Recall the* ***average cost***[1] *is defined as* $\overline{C}(x) = \dfrac{C(x)}{x}$, $x > 0$, *is the cost per item.*

1. *Find and interpret* $\overline{C}(75)$.
2. *Define the* ***marginal cost*** $MC(x) = C(x + 1) - C(x)$. *Find and interpret* $MC(75)$.
3. *How do your answers to parts **??** and **??** compare?*
4. *Graph* $y = \overline{C}(x)$ *with help from a graphing utility. What is happening graphically near* $x = 75$*?*
5. *Use Theorem **??** to show that, in general,* $ARoC[\overline{C}(x)] = 0$ *when* $MC(x) = \overline{C}(x)$.

   HINT: *Note that, by definition,* $MC(x) = C(x + 1) - C(x) = \Delta[C(x)]$ *when* $\Delta x = 1$.

   Hence, $ARoC[C(x)] = \dfrac{\Delta[C(x)]}{\Delta x} = \dfrac{\Delta[C(x)]}{1} = \Delta[C(x)] = MC(x)$ *in this case ...*

[1] First mentioned in Definition **??** in Section **??**.