Exercises for Inverse Functions

Question 1 *In Exercises **??** - **??**, verify the given pairs of functions are inverses algebraically and graphically.*

Problem 1.1 $f(x) = 2x + 7$ *and* $g(x) = \dfrac{x-7}{2}$

Problem 1.2 $f(x) = \dfrac{5-3x}{4}$ *and* $g(x) = -\dfrac{4}{3}x + \dfrac{5}{3}$.

Problem 1.3 $f(t) = \dfrac{5}{t-1}$ *and* $g(t) = \dfrac{t+5}{t}$

Problem 1.4 $f(t) = \dfrac{t}{t-1}$ *and* $g(t) = f(t) = \dfrac{t}{t-1}$

Problem 1.5 $f(x) = \sqrt{4-x}$ *and* $g(x) = -x^2 + 4$, $x \geq 0$

Problem 1.6 $f(x) = 1 - \sqrt{x+1}$ *and* $g(x) = x^2 - 2x$, $x \leq 1$.

Problem 1.7 $f(t) = (t-1)^3 + 5$ *and* $g(t) = \sqrt[3]{t-5} + 1$

Problem 1.8 $f(t) = -\sqrt[4]{t-2}$ *and* $g(t) = t^4 + 2$, $t \leq 0$.

Question 2 ***Problem 2.1*** $f(x) = 6x - 2$

$f^{-1}(x) = \dfrac{x+2}{6}$

Problem 2.2 $f(x) = 42 - x$

$f^{-1}(x) = 42 - x$

Problem 2.3 $g(t) = \dfrac{t-2}{3} + 4$

$g^{-1}(x) = 3x - 10$

Problem 2.4 $g(t) = 1 - \dfrac{4+3t}{5}$

$g^{-1}(t) = -\dfrac{5}{3}t + \dfrac{1}{3}$

Problem 2.5 $f(x) = \sqrt{3x-1} + 5$

$f^{-1}(x) = \dfrac{(x-5)^2 + 1}{3}, \quad x \geq 5$

Problem 2.6 $f(x) = 2 - \sqrt{x-5}$

$f^{-1}(x) = (2-x)^2 + 5, \quad x \leq 2$

Problem 2.7 $g(t) = 3\sqrt{t-1} - 4$

$g^{-1}(x) = \left(\dfrac{x+4}{3}\right)^2 + 1, \quad x \geq -4$

Problem 2.8 $g(t) = 1 - 2\sqrt{2t+5}$

$g^{-1}(x) = \dfrac{(1-x)^2}{8} - \dfrac{5}{2}, \quad x \leq 1$

Problem 2.9 $f(x) = \sqrt[5]{3x-1}$

$f^{-1}(x) = \dfrac{x^5+1}{3}$

Problem 2.10 $f(x) = 3 - \sqrt[3]{x-2}$

$f^{-1}(x) = (3-x)^3 + 2$

Problem 2.11 $g(t) = t^2 - 10t,\ t \geq 5$

$g^{-1}(x) = 5 + \sqrt{x+25}, \quad x \geq -25$

Problem 2.12 $g(t) = 3(t+4)^2 - 5,\ t \leq -4$

$g^{-1}(x) = -4 - \sqrt{\dfrac{x+5}{3}}, \quad x \geq -5$

Problem 2.13 $f(x) = x^2 - 6x + 5,\ x \leq 3$

$f^{-1}(x) = 3 - \sqrt{x+4}, \quad x \geq -4$

Problem 2.14 $f(x) = 4x^2 + 4x + 1,\ x < -1$

$f^{-1}(x) = -\dfrac{\sqrt{x}+1}{2},\ x > 1$

Problem 2.15 $g(t) = \dfrac{3}{4-t}$

$g^{-1}(t) = \dfrac{4t-3}{t}$

Problem 2.16 $g(t) = \dfrac{t}{1-3t}$

$g^{-1}(t) = \dfrac{t}{3t+1}$

Problem 2.17 $f(x) = \dfrac{2x-1}{3x+4}$

$f^{-1}(x) = \dfrac{4x+1}{2-3x}$

Problem 2.18 $f(x) = \dfrac{4x+2}{3x-6}$

$f^{-1}(x) = \dfrac{6x+2}{3x-4}$

Problem 2.19 $g(t) = \dfrac{-3t-2}{t+3}$

$g^{-1}(t) = \dfrac{-3t-2}{t+3}$

Problem 2.20 $g(t) = \dfrac{t-2}{2t-1}$

$g^{-1}(t) = \dfrac{t-2}{2t-1}$

Problem 3 *Explain why each set of ordered pairs below represents a one-to-one function and find the inverse.*

1. $F = \{(0,0), (1,1), (2,-1), (3,2), (4,-2), (5,3), (6,-3)\}$

   None of the first coordinates of the ordered pairs in F are repeated, so F is a function and none of the second coordinates of the ordered pairs of F are repeated, so F is one-to-one.

   $F^{-1} = \{(0,0), (1,1), (-1,2), (2,3), (-2,4), (3,5), (-3,6)\}$

2. $G = \{(0,0), (1,1), (2,-1), (3,2), (4,-2), (5,3), (6,-3), ...\}$

   NOTE: The difference between F and G is the '....'

   Because of the '...' it is helpful to determine a formula for the matching. For the even numbers n, $n = 0, 2, 4, ...$*, the ordered pair* $\left(n, -\frac{n}{2}\right)$ *is in G. For the odd numbers* $n = 1, 3, 5, ...$*, the ordered pair* $\left(n, \frac{n+1}{2}\right)$ *is in G. Hence, given any input to G, n, whether it be even or odd, there is only one output from G, either* $-\frac{n}{2}$ *or* $\frac{n+1}{2}$*, both of which are functions of n. To show G is one to one, we note that if the output from G is* 0 *or less, then it must be of the form* $-\frac{n}{2}$ *for an even number n. Moreover, if* $-\frac{n}{2} = -\frac{m}{2}$*, then* $n = m$*. In the case we are looking at outputs from G which are greater than* 0*, then it must be of the form* $\frac{n+1}{2}$ *for an odd number n. In this, too, if* $\frac{n+1}{2} = \frac{m+1}{2}$*, then* $n = m$*. Hence, in any case, if the outputs from G are the same, then the inputs to G had to be the same so G is one-to-one and* $G^{-1} = \{(0,0), (1,1), (-1,2), (2,3), (-2,4), (3,5), (-3,6), ...\}$

3. $P = \{(2t^5, 3t-1) \mid t \text{ is a real number.}\}$

   To show P is a function we note that if we have the same inputs to P, say $2t^5 = 2u^5$*, then* $t = u$*. Hence the corresponding outputs,* $2t-1$ *and* $3u-1$*, are equal, too. To show P is one-to-one, we note that if we have the same outputs from P,* $3t-1 = 3u-1$*, then* $t = u$*. Hence, the corresponding inputs* $2t^5$ *and* $2u^5$ *are equal, too. Hence P is one-to-one and* $P^{-1} = \{(3t-1, 2t^5) \mid t \text{ is a real number.}\}$

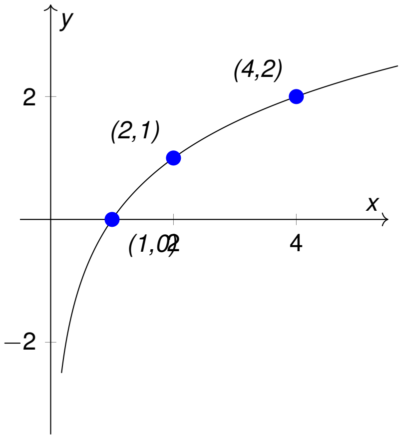
4. $Q = \{(n, n^2) \mid n \text{ is a natural number.}\}$[1]

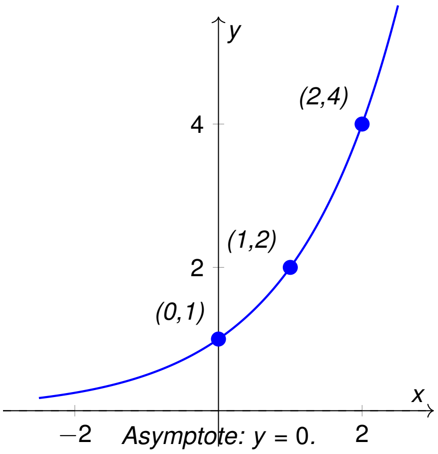
   To show Q is a function, we note that if we have the same inputs to Q, say $n = m$*, then the outputs from Q, namely* n^2 *and* m^2 *are equal. To show Q is one-to-one, we note that if we get the same output from Q, namely* $n^2 = m^2$*, then* $n = \pm m$*. However since n and m are* natural *numbers, both n and m are positive so* $n = m$*. Hence Q is one-to-one and* $Q^{-1} = \{(n^2, n) \mid n \text{ is a natural number.}\}$.

Problem 4 $y = f(x)$

$y = f^{-1}(x)$*. Asymptote:* $x = 0$.

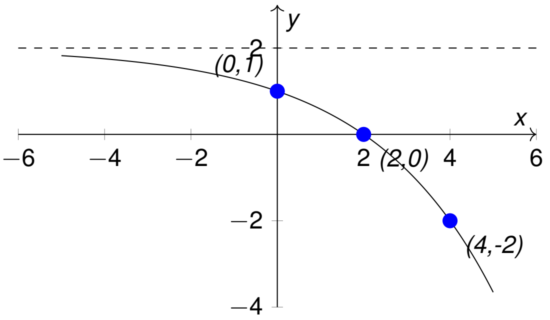
[1] Recall this means $n = 0, 1, 2, ...$

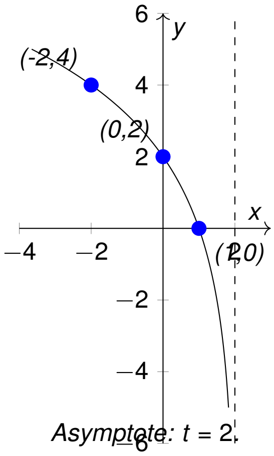




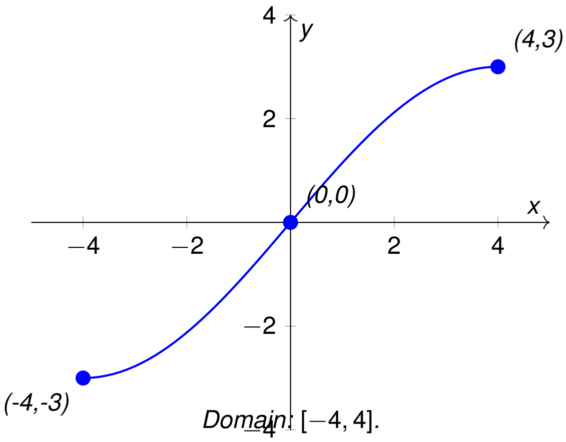
Problem 5 $y = g(t)$

$y = g^{-1}(t)$. *Asymptote:* $y = 2$.





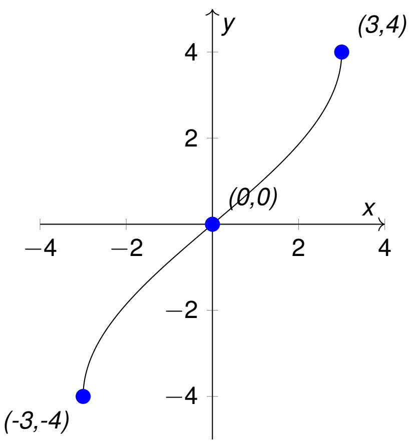
Problem 6 $y = S(t)$

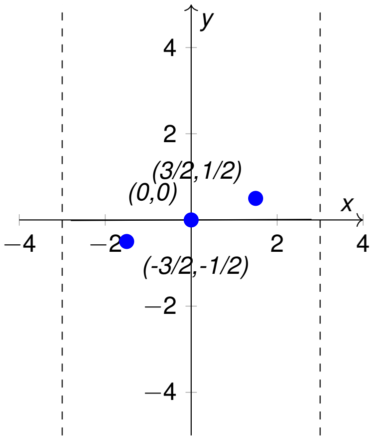


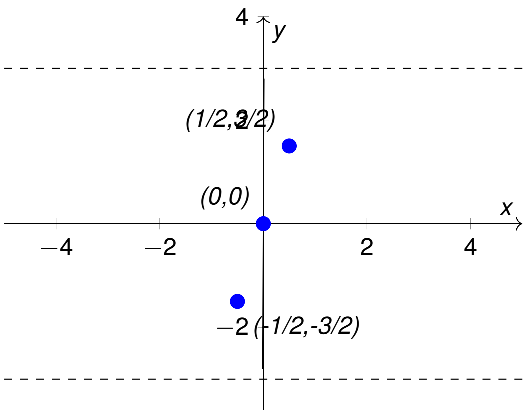
Problem 7 $y = R(s)$

$y = R^{-1}(s)$. *Asymptotes:* $s = \pm 3$.

$y = S^{-1}(t)$. *Domain* $[-3, 3]$.







Asymptotes: $y = \pm 3$.

Problem 8 *The price of a dOpi media player, in dollars per dOpi, is given as a function of the weekly sales* x *according to the formula* $p(x) = 450 - 15x$ *for* $0 \leq x \leq 30$.

1. *Find* $p^{-1}(x)$ *and state its domain.*

   $p^{-1}(x) = \dfrac{450 - x}{15}$. *The domain of* p^{-1} *is the range of* p *which is* $[0, 450]$

2. *Find and interpret* $p^{-1}(105)$.

   $p^{-1}(105) = 23$. *This means that if the price is set to* \$105 *then* 23 *dOpis will be sold.*

3. *The profit (in dollars) made from producing and selling* x *dOpis per week is given by the formula* $P(x) = -15x^2 + 350x - 2000$, *for* $0 \leq x \leq 30$. *Find* $\left(P \circ p^{-1}\right)(x)$ *and determine what price per dOpi would yield the maximum profit. What is the maximum profit? How many dOpis need to be produced and sold to achieve the maximum profit?*

   $\left(P \circ p^{-1}\right)(x) = -\dfrac{1}{15}x^2 + \dfrac{110}{3}x - 5000$, $0 \leq x \leq 450$.

   The graph of $y = \left(P \circ p^{-1}\right)(x)$ *is a parabola opening downwards with vertex* $\left(275, \dfrac{125}{3}\right) \approx (275, 41.67)$. *This means that the maximum profit is a whopping* \$41.67 *when the price per dOpi is set to* \$275. *At this price, we can produce and sell* $p^{-1}(275) = 11.\overline{6}$ *dOpis. Since we cannot sell part of a system, we need to adjust the price to sell either* 11 *dOpis or* 12 *dOpis. We find* $p(11) = 285$ *and* $p(12) = 270$, *which means we set the price per dOpi at either* \$285 *or* \$270, *respectively. The profits at these prices are* $\left(P \circ p^{-1}\right)(285) = 35$ *and* $\left(P \circ p^{-1}\right)(270) = 40$, *so it looks as if the maximum profit is* \$40 *and it is made by producing and selling* 12 *dOpis a week at a price of* \$270 *per dOpi.*

Problem 9 *Show that the Fahrenheit to Celsius conversion function found in Exercise* **??** *in Section* **??** *is invertible and that its inverse is the Celsius to Fahrenheit conversion function.*

Problem 10 *Analytically show that the function* $f(x) = x^3 + 3x + 1$ *is one-to-one. Use Theorem* **??** *to help you compute* $f^{-1}(1)$, $f^{-1}(5)$, *and* $f^{-1}(-3)$. *What happens when you attempt to find a formula for* $f^{-1}(x)$*?*

Problem 11 *Let* $f(x) = \dfrac{2x}{x^2 - 1}$.

1. *Graph* $y = f(x)$ *using the techniques in Section* **??**. *Check your answer using a graphing utility.*
2. *Verify that* f *is one-to-one on the interval* $(-1, 1)$.
3. *Use the procedure outlined on Page* **??** *to find the formula for* $f^{-1}(x)$ *for* $-1 < x < 1$.
4. *Since* $f(0) = 0$, *it should be the case that* $f^{-1}(0) = 0$. *What goes wrong when you attempt to substitute* $x = 0$ *into* $f^{-1}(x)$*? Discuss with your classmates how this problem arose and possible remedies.*

Problem 12 *With the help of your classmates, explain why a function which is either strictly increasing or strictly decreasing on its entire domain would have to be one-to-one, hence invertible.*

Problem 13 *If* f *is odd and invertible, prove that* f^{-1} *is also odd.*

Problem 14 *Let* f *and* g *be invertible functions. With the help of your classmates show that* $(f \circ g)$ *is one-to-one, hence invertible, and that* $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$.

Question 15 *With help from your classmates, find the inverses of the functions in Exercises* **??** - **??**.

Problem 15.1 $f(x) = ax + b,\ a \neq 0$

Problem 15.2 $f(x) = a\sqrt{x - h} + k,\ a \neq 0, x \geq h$

Problem 15.3 $f(x) = ax^2 + bx + c$ *where* $a \neq 0,\ x \geq -\dfrac{b}{2a}$.

Problem 15.4 $f(x) = \dfrac{ax + b}{cx + d}$, *(See Exercise* **??** *below.)*

Problem 16 *What conditions must you place on the values of* a, b, c *and* d *in Exercise* **??** *in order to guarantee that the function is invertible?*

Problem 17 *The function given in number* **??** *is an example of a function which is its own inverse.*

1. *Algebraically verify every function of the form:* $f(x) = \dfrac{ax + b}{cx - a}$ *is its own inverse. What assumptions do you need to make about the values of a, b, and c?*
2. *Under what conditions is* $f(x) = mx + b$, $m \neq 0$ *its own inverse? Prove your answer.*
   If $b = 0$, *then* $m = \pm 1$. *If* $b \neq 0$, *then* $m = -1$ *and* b *can be any real number.*