Theorems **??**, **??**, **??**, **??**, **??** and **??** all describe ways in which the graph of a function can changed, or 'transformed' to obtain the graph of a related function. The results and proofs of each of these theorems are virtually identical, and with the language of function composition, we can see better why.

Consider, for instance, Theorem **??**, in which we describe how to transform the graph of $f(x) = x^r$ to $F(x) = a(bx - h)^r + k$. We may think of F as being build up from f by composing f with linear functions. Specifically, if we let $i(x) = bx - h$, then $(f \circ i)(x) = f(i(x)) = f(bx - h) = (bx - h)^r$. If, additionally, we let and $j(x) = ax + k$, then $(j \circ (f \circ i))(x) = j((f \circ i)(x)) = j((bx - h)^r) = a(bx - h)^r + k = F(x)$. Hence, we can view $F = j \circ f \circ i$.

In this section, our goal is to generalize the aforementioned theorems to the graphs of *all* functions. Along the way, you'll see some very familiar arguments, but, additionally, we hope this section affords the reader an opportunity to not only see *how* these transformations work they way they do, but *why*.

Our motivational example for the results in this section is the graph of $y = f(x)$ below. While we could formulate an expression for $f(x)$ as a piecewise-defined function consisting of linear and constant parts, we wish to focus more on the geometry here. That being said, we do record some of the function values - the 'key points' if you will - to track through each transformation.

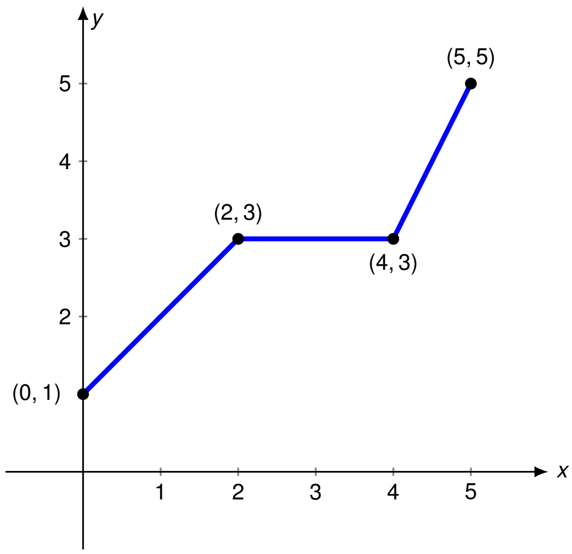

Figure 1: $y = f(x)$

Alt text: Graph of a piecewise linear function passing through the points (0, 1), (2, 3), (4, 3), and (5, 5). The graph rises from (0, 1) to (2, 3), remains horizontal from (2, 3) to (4, 3), then rises again to (5, 5).

x	$(x, f(x))$	$f(x)$
0	$(0, 1)$	1
2	$(2, 3)$	3
4	$(4, 3)$	3
5	$(5, 5)$	5

1 Vertical and Horizontal Shifts

Suppose we wished to graph $g(x) = f(x) + 2$. From a procedural point of view, we start with an input x to the function f and we obtain the output $f(x)$. The function g takes the output $f(x)$ and adds 2 to it. Using the sample values for f from the table above we can create a table of values for g below, hence generating points on the graph of g.

x	$(x, f(x))$	$f(x)$	$g(x) = f(x) + 2$	$(x, g(x))$
0	$(0, 1)$	1	$1 + 2 = 3$	$(0, 3)$
2	$(2, 3)$	3	$3 + 2 = 5$	$(2, 5)$
4	$(4, 3)$	3	$3 + 2 = 5$	$(4, 5)$
5	$(5, 5)$	5	$5 + 2 = 7$	$(5, 7)$

In general, if (a, b) is on the graph of $y = f(x)$, then $f(a) = b$. Hence, $g(a) = f(a) + 2 = b + 2$, so the point $(a, b + 2)$ is on the graph of g. In other words, to obtain the graph of g, we add 2 to the y-coordinate of each point on the graph of f.

Geometrically, adding 2 to the y-coordinate of a point moves the point 2 units above its previous location. Adding 2 to every y-coordinate on a graph *en masse* is moves or 'shifts' the entire graph of f up 2 units. Notice that the graph retains the same basic shape as before, it is just 2 units above its original location. In other words, we connect the four 'key points' we moved in the same manner in which they were connected before.

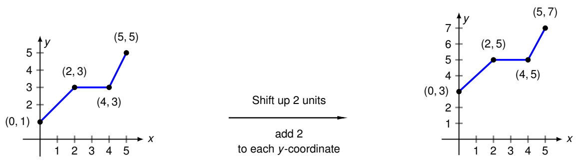

Figure 2: Graphs of $y = f(x)$ (left) and $y = g(x) = f(x) + 2$ (right).

Alt text: Two graphs illustrate a vertical shift. The graph on the left passes through the points $(0, 1)$, $(2, 3)$, $(4, 3)$, and $(5, 5)$. The graph on the right passes through the points $(0, 3)$, $(2, 5)$, $(4, 5)$, and $(5, 7)$. An arrow indicates that the graph is shifted up 2 units by adding 2 to each y-coordinate.

You'll note that the domain of f and the domain of g are the same, namely $[0, 5]$, but that the range of f is $[1, 5]$ while the range of g is $[3, 7]$. In general, shifting a function vertically like this will leave the domain unchanged, but could very well affect the range.

You can easily imagine what would happen if we wanted to graph the function $j(x) = f(x) - 2$. Instead of adding 2 to each of the y-coordinates on the graph of f, we'd be subtracting 2. Geometrically, we would be moving the graph down 2 units. We leave it to the reader to verify that the domain of j is the same as f, but the range of j is $[-1, 3]$. In general, we have:

Theorem 1. Vertical Shifts. Suppose f is a function and k is a real number.

To graph $F(x) = f(x) + k$, add k to each of the y-coordinates of the points on the graph of $y = f(x)$.

NOTE: This results in a vertical shift up k units if $k > 0$ or down k units if $k < 0$.

To prove Theorem **??**, we first note that f and F have the same domain (why?) Let c be an element in the domain of F and, hence, the domain of f. The fact that f and F are *functions* guarantees there is *exactly one* point on each of their graphs corresponding to $x = c$. On $y = f(x)$, this point is $(c, f(c))$; on $y = F(x)$, this point is $(c, F(c)) = (c, f(c) + k)$. This sets up a nice correspondence between the two graphs and shows that each of the points on the graph of F can be obtained to by adding k to each of the y-coordinates of the corresponding point on the graph of f. This proves Theorem **??**. In the language of 'inputs' and 'outputs', Theorem **??** says adding to the *output* of a function causes the graph to shift *vertically*.

Keeping with the graph of $y = f(x)$ above, suppose we wanted to graph $g(x) = f(x + 2)$. In other words, we are looking to see what happens when we add 2 to the input of the function. Let's try to generate a table of values of g based on those we know for f. We quickly find that we run into some difficulties. For instance, when we substitute $x = 4$ into the formula $g(x) = f(x+2)$, we are asked to find $f(4+2) = f(6)$ which doesn't exist because the domain of f is only $[0, 5]$. The same thing happens when we attempt to find $g(5)$.

x	$(x, f(x))$	$f(x)$	$g(x) = f(x+2)$	$(x, g(x))$
0	$(0, 1)$	1	$g(0) = f(0+2) = f(2) = 3$	$(0, 3)$
2	$(2, 3)$	3	$g(2) = f(2+2) = f(4) = 3$	$(2, 3)$
4	$(4, 3)$	3	$g(4) = f(4+2) = f(6) = ?$	
5	$(5, 5)$	5	$g(5) = f(5+2) = f(7) = ?$	

What we need here is a new strategy. We know, for instance, $f(0) = 1$. To determine the corresponding point on the graph of g, we need to figure out what value of x we must substitute into $g(x) = f(x+2)$ so that the quantity $x + 2$, works out to be 0. Solving $x + 2 = 0$ gives $x = -2$, and $g(-2) = f((-2) + 2) = f(0) = 1$ so $(-2, 1)$ on the graph of g. To use the fact $f(2) = 3$, we set $x + 2 = 2$ to get $x = 0$. Substituting gives $g(0) = f(0+2) = f(2) = 3$. Continuing in this fashion, we produce the table below.

x	$x+2$	$g(x) = f(x+2)$	$(x, g(x))$
-2	0	$g(-2) = f(-2+2) = f(0) = 1$	$(-2, 1)$
0	2	$g(0) = f(0+2) = f(2) = 3$	$(0, 3)$
2	4	$g(2) = f(2+2) = f(4) = 3$	$(2, 3)$
3	5	$g(3) = f(3+2) = f(5) = 5$	$(3, 5)$

In summary, the points $(0, 1)$, $(2, 3)$, $(4, 3)$ and $(5, 5)$ on the graph of $y = f(x)$ give rise to the points $(-2, 1)$, $(0, 3)$, $(2, 3)$ and $(3, 5)$ on the graph of $y = g(x)$, respectively. In general, if (a, b) is on the graph of $y = f(x)$, then $f(a) = b$. Solving $x + 2 = a$ gives $x = a - 2$ so that $g(a-2) = f((a-2)+2) = f(a) = b$. As such, $(a - 2, b)$ is on the graph of $y = g(x)$. The point $(a - 2, b)$ is exactly 2 units to the *left* of the point (a, b) so the graph of $y = g(x)$ is obtained by shifting the graph $y = f(x)$ to the left 2 units, as pictured below.

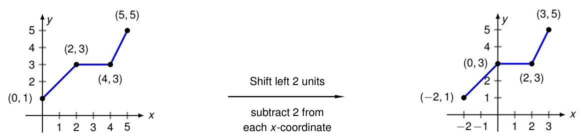

Figure 3: Graphs of $y = f(x)$ (left) and $y = g(x) = f(x+2)$ (right).

Alt text: Two graphs illustrate a horizontal shift. The graph on the left passes through the points $(0, 1)$, $(2, 3)$, $(4, 3)$, and $(5, 5)$. The graph on the right passes through $(-2, 1)$, $(0, 3)$, $(2, 3)$, and $(3, 5)$. An arrow indicates that the graph is shifted left 2 units by subtracting 2 from each x-coordinate.

Note that while the ranges of f and g are the same, the domain of g is $[-2, 3]$ whereas the domain of f is $[0, 5]$. In general, when we shift the graph horizontally, the range will remain the same, but the domain could change. If we set out to graph $j(x) = f(x-2)$, we would find ourselves *adding* 2 to all of the x values of the points on the graph of $y = f(x)$ to effect a shift to the *right* 2 units. Generalizing these notions produces the following result.

Theorem 2. Horizontal Shifts. Suppose f is a function and h is a real number.

To graph $F(x) = f(x - h)$, add h to each of the x-coordinates of the points on the graph of $y = f(x)$.

NOTE: This results in a horizontal shift right h units if $h > 0$ or left h units if $h < 0$.

To prove Theorem **??**, we first note the domains of f and F may be different. If c is in the domain of f, then the only number we know for sure is in the domain of F is $c + h$, since $F(c+h) = f((c+h)-h) = f(c)$. This sets up a nice correspondence between the domain of f and the domain of F which spills over to a correspondence between their graphs, The point $(c, f(c))$ is the one and only point on the graph of $y = f(x)$ corresponding to $x = c$ just as the point $(c + h, F(c + h)) = (c + h, f(c))$ is the one and only point on the graph of $y = F(x)$ corresponding to $x = c + h$. This correspondence shows we may obtain the graph of F by adding h to each x-coordinate of each point on the graph of f, which establishes the theorem. In words, Theorem **??** says that subtracting from the *input* to a function amounts to shifting the graph *horizontally*.

Theorems **??** and **??** present a theme which will run common throughout the section: changes to the *outputs* from a function result in some kind of *vertical change*; changes to the *inputs* to a function result in some kind of *horizontal* change. We demonstrate Theorems **??** and **??** in the example below.

Example 1. Use Theorems **??** and **??** to answer the questions below. Check your answers using a graphing utility where appropriate.

1. Suppose $(-1, 3)$ is on the graph of $y = f(x)$. Find a point on the graph of:

   (i) $y = f(x) + 5$ (ii) $y = f(x + 5)$ (iii) $y = f(x - 7) + 4$

2. Let $f(t) = |t| - 2t$.

   (i) Find a formula for a function $g(t)$ whose graph is the same as the graph of f but shifted to the right 4 units.

   (ii) Find a formula for a function $h(t)$ whose graph is the same as the graph of f but shifted down 2 units.

3. Predict how the graph of $F(x) = \dfrac{(x-2)^{\frac{2}{3}}}{x}$ relates to the graph of $f(x) = \dfrac{x^{\frac{2}{3}}}{x+2}$.

4. Below is the graph of $y = f(x)$.

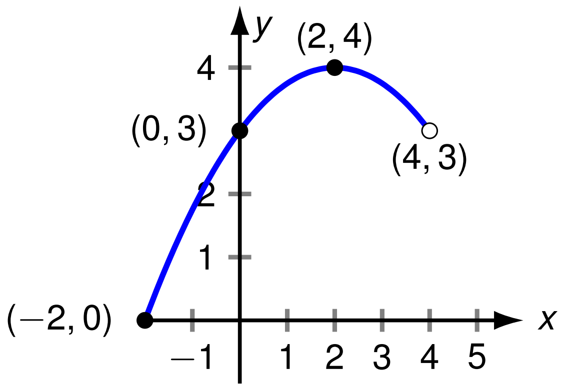

Figure 4: Graph of $y = f(x)$.

Alt text: The graph is a downward-opening parabola segment with a closed endpoint at $(-2, 0)$, passing through $(0, 3)$ and reaching a maximum at $(2, 4)$. The graph ends with an open endpoint at $(4, 3)$.

   Use it to sketch the graph of

   (i) $F(x) = f(x - 2)$ (ii) $F(x) = f(x) + 1$ (iii) $F(x) = f(x + 1) - 2$

5. Below on the left is the graph of $y = f(x)$ and below on the right is the graph of $y = g(x)$. Write $g(x)$ in terms of $f(x)$ and vice-versa.

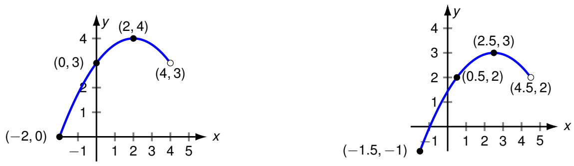

Figure 5: Graphs of $y = f(x)$ (left) and $y = g(x)$ (right).

Alt text: The graph on the left is a downward-opening parabola segment with a closed endpoint at $(-2, 0)$, passing through $(0, 3)$ and reaching a maximum at $(2, 4)$. The graph ends with an open endpoint at $(4, 3)$. The graph on the right is a downward-opening parabola segment with a closed endpoint at $(-1.5, -1)$, passing through $(0.5, 2)$ and reaching a maximum at $(2.5, 3)$. The graph ends with an open endpoint at $(4.5, 2)$.

Solution.

1. (i) To apply Theorem **??**, we identify $f(x) + 5 = f(x) + k$ so $k = 5$. Hence, we add 5 to the y-coordinate of $(-1, 3)$ and get $(-1, 3 + 5) = (-1, 8)$. To check our answer note since $(-1, 3)$ is on the graph of f this means $f(-1) = 3$. Substituting $x = -1$ into the formula $y = f(x) + 5$, we get $y = f(-1) + 5 = 3 + 5 = 8$. Hence, $(-1, 8)$ is on the graph of $f(x) + 5$.

   (ii) We note that $f(x+5)$ can be written as $f(x-(-5)) = f(x-h)$ so we apply Theorem **??** with $h = -5$. Adding -5 to (subtracting 5 from) the x-coordinate of $(-1, 3)$ gives $(-1 + (-5), 3) = (-6, 3)$. To check our answer, since $(-1, 3)$ is on the graph of f, $f(-1) = 3$. Substituting $x = -6$ into $y = f(x+5)$ gives $y = f(-6 + 5) = f(-1) = 3$, proving $(-6, 3)$ is on the graph of $y = f(x + 5)$.

   (iii) Note that the expression $f(x - 7) + 4$ differs from $f(x)$ in *two* ways indicating two different transformations. In situations like this, its best if we handle each transformation in turn, starting with the graph of $y = f(x)$ and 'building up' to the graph of $y = f(x - 7) + 4$.

   We choose to work from the 'inside' (argument) out and use Theorem **??** to first get a point on the graph of $y = f(x - 7) = f(x - h)$. Identifying $h = 7$, we add 7 to the x-coordinate of $(-1, 3)$ to get $(-1 + 7, 3) = (6, 3)$. Hence, $(6, 3)$ is a point on the graph of $y = f(x - 7)$.

   Next, we apply Theorem **??** to graph $y = f(x - 7) + 4$ starting with $y = f(x - 7)$. Viewing $f(x - 7) + 4 = f(x - 7) + k$, we identify $k = 4$ and add 4 to the y-coordinate of $(6, 3)$ to get $(6, 3 + 4) = (6, 7)$. To check, we note that if we substitute $x = 6$ into $y = f(x - 7) + 4$, we get $y = f(6 - 7) + 4 = f(-1) + 4 = 3 + 4 = 7$.

2. Here the independent variable is t instead of x which doesn't affect the geometry in any way since our convention is the independent variable is used to label the horizontal axis and the dependent variable is used to label the vertical axis.

   (i) Per Theorem **??**, the graph of $g(t) = f(t - 4) = |t - 4| - 2(t - 4) = |t - 4| - 2t + 8$ should be the graph of $f(t) = |t| - 2t$ shifted to the right 4 units.

   (ii) Per Theorem **??**, the graph of $h(t) = f(t) + (-2) = |t| - 2t + (-2) = |t| - 2t - 2$ should be the graph of $f(t) = |t| - 2t$ shifted down 2 units.

   Using the Desmos interactive below, we can check our answers graphically by adjusting the sliders for two parameters 'a' and 'b.' For the function $g(t) = f(t - a)$, adjusting the slider for a from $a = 0$ to $a = 4$ shows the graph of f gradually shifting horizontally to to end up 4 units to the right of its starting position. Likewise, adjusting the slider for the parameter b, we see the graph of the function $h(t) = f(t) - b$ gradually shifts the graph of f vertically until it stops 2 units below its starting position.

   Desmos link: `https://www.desmos.com/calculator/xrmcgegn3j`

3. Comparing *formulas*, it appears as if $F(x) = f(x - 2)$. We check:

$$f(x-2) = \frac{(x-2)^{\frac{2}{3}}}{(x-2)+2} = \frac{(x-2)^{\frac{2}{3}}}{x} = F(x),$$

   so, per Theorem **??**, the graph of $y = F(x)$ should be the graph of $y = f(x)$ but shifted to the right 2 units. Once again, we can use a slider in the Desmos interactive below to visualize this horizontal shift by adjusting the parameter 'a' from 0 to 2 to graph the transformed function $F(x) = f(x - a)$.

   Desmos link: `https://www.desmos.com/calculator/yc4sxelkkp`

4. (i) We recognize $F(x) = f(x - 2) = f(x - h)$. With $h = 2$, Theorem **??** tells us to add 2 to each of the x-coordinates of the points on the graph of f, moving the graph of f to the *right* two units.

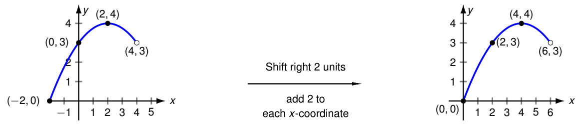

Figure 6: Graphs of $y = f(x)$ (left) and $y = F(x) = f(x-2)$ (right).

Alt text: The graph on the left is a downward-opening parabola segment with closed endpoint $(-2, 0)$, vertex $(2, 4)$, and open endpoint $(4, 3)$. The graph on the right has the same shape and is obtained by shifting the first graph 0.5 unit to the right and 1 unit downward. The corresponding points are $(-1.5, -1)$, $(2.5, 3)$, and $(4.5, 2)$.

We can check our answer by showing each ordered pair (x, y) listed on our final graph satisfies the equation $y = f(x-2)$. Starting with $(0, 0)$, we substitute $x = 0$ into $y = f(x-2)$ and get $y = f(0-2) = f(-2)$. Since $(-2, 0)$ is on the graph of f, we know $f(-2) = 0$. Hence, $y = f(0-2) = f(-2) = 0$, showing the point $(0, 0)$ is on the graph of $y = f(x-2)$. We invite the reader to check the remaining points.

(ii) We have $F(x) = f(x) + 1 = f(x) + k$ where $k = 1$, so Theorem **??** tells us to move the graph of f *up* 1 unit by adding 1 to each of the y-coordinates of the points on the graph of f.

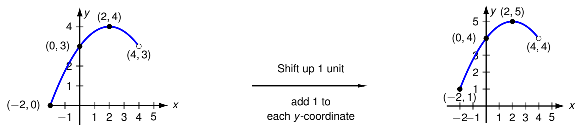

Figure 7: Graphs of $y = f(x)$ (left) and $y = F(x) = f(x) + 1$ (right).

Alt text: Two graphs illustrate a vertical shift. The graph on the left is a downward-opening parabola segment with closed endpoint $(-2, 0)$, vertex $(2, 4)$, and open endpoint $(4, 3)$. The graph on the right has the same shape and is obtained by shifting the first graph upward 1 unit. The corresponding points are $(-2, 1)$, $(2, 5)$, and $(4, 4)$.

To check our answer, we proceed as above. Starting with the point $(-2, 1)$, we substitute $x = -2$ into $y = f(-2) + 1$ to get $y = f(-2) + 1$. Since $(-2, 0)$ is on the graph of f, we know $f(-2) = 0$. Hence, $y = f(-2) + 1 = 0 + 1 = 1$. This proves $(-2, 1)$ is on the graph of $y = f(x) + 1$. We encourage the reader to check the remaining points in kind.

(iii) We are asked to graph $F(x) = f(x+1) - 2$. As above, when we have more than one modification to do, we work from the inside out and build up to $F(x) = f(x+1) - 2$ from $f(x)$ in stages. First, we apply Theorem **??** to graph $y = f(x+1)$ from $y = f(x)$. Rewriting $f(x+1) = f(x - (-1))$, we identify $h = -1$, so we add -1 to (subtract 1 from) each of the x-coordinates on the graph of f, shifting it to the *left* 1 unit.

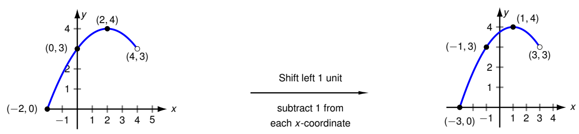

Figure 8: Graphs of $y = f(x)$ (left) and $y = f(x+1)$ (right).

Alt text: Two graphs illustrate a horizontal shift. The graph on the left is a downward-opening parabola segment with closed endpoint $(-2, 0)$, vertex $(2, 4)$, and open endpoint $(4, 3)$. The graph on the right has the same shape and is obtained by shifting the first graph left 1 unit. The corresponding points are $(-3, 0)$, $(1, 4)$, and $(3, 3)$.

Next, we apply Theorem **??** to graph $y = f(x+1) - 2$ starting with the graph of $y = f(x+1)$. Writing $f(x+1) - 2 = f(x+1) + (-2) = f(x+1) + k$, we identify $k = -2$ so Theorem **??** instructs us to add -2 to (subtract 2 from) each of the y-coordinates on the graph of $y = f(x+1)$, thereby shifting the graph *down* two units.

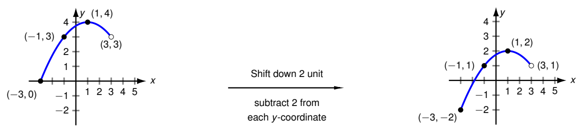

Figure 9: Graphs of $y = f(x+1)$ (left) and $y = f(x+1) - 2$ (right).

Alt text: Two graphs illustrate a vertical shift. The graph on the left is a downward-opening parabola segment with closed endpoint $(-3, 0)$, vertex $(1, 4)$, and open endpoint $(3, 3)$. The graph on the right has the same shape and is obtained by shifting the first graph downward 2 units. The corresponding points are $(-3, -2)$, $(1, 2)$, and $(3, 1)$.

To check, we start with the point $(-3, -2)$. We find when we substitute $x = -3$ into the equation $y = f(x+1) - 2$ we get $y = f(-3+1) - 2 = f(-2) - 2$. Since $(-2, 0)$ is on the graph of f, we know $f(-2) = 0$, so $y = f(-3+1) - 2 = f(-2) - 2 = 0 - 2 = -2$. This proves $(-3, -2)$ is on the graph of $y = f(x+1) - 2$. We leave the checks of the remaining points to the reader.

5. To write $g(x)$ in terms of $f(x)$, we note that based on points which are labeled, it appears as if the graph of g can be obtained from the graph of f by shifting the graph of f to the right 0.5 units and down 1 unit.

   Per Theorems **??** and **??**, $g(x)$ must take the form $g(x) = f(x - h) + k$. Since the horizontal shift is to the *right* 0.5 units, $h = 0.5$ and since the vertical shift is *down* 1 unit, $k = -1$. Hence, we get $g(x) = f(x - 0.5) - 1$.

   We can check our answer by working through both transformations, in sequence, as in the previous example. To write $f(x)$ in terms of $g(x)$, we need to reverse the process - that is, we need to shift the graph of g *left* one half of a unit and *up* one unit. Theorems **??** and **??** suggest the formula $f(x) = g(x + 0.5) + 1$. We leave it to the reader to check. ■

2 Reflections about the Coordinate Axes

We now turn our attention to reflections. We know from Section **??** that to reflect a point (x, y) across the x-axis, we replace y with $-y$. If (x, y) is on the graph of f, then $y = f(x)$, so replacing y with $-y$ is the same as replacing $f(x)$ with $-f(x)$. Hence, the graph of $y = -f(x)$ is the graph of f reflected across the x-axis. Similarly, the graph of $y = f(-x)$ is the graph of $y = f(x)$ reflected across the y-axis.[1]

Theorem 3. Reflections. Suppose f is a function.

To graph $F(x) = -f(x)$, multiply each of the y-coordinates of the points on the graph of $y = f(x)$ by -1.

NOTE: This results in a reflection across the x-axis.

To graph $F(x) = f(-x)$, multiply each of the x-coordinates of the points on the graph of $y = f(x)$ by -1.

NOTE: This results in a reflection across the y-axis.

The proof of Theorem **??** follows in much the same way as the proofs of Theorems **??** and **??**. If c is an element of the domain of f and $F(x) = -f(x)$, then the point $(c, f(c))$ corresponds to the point $(c, F(c)) = (c, -f(c))$. Comparing the corresponding points $(c, f(c))$ and $(c, -f(c))$, we see they only difference is the y-coordinates are the exact opposite - indicating they are mirror-images across the x-axis. Similarly, if c is an element in the domain of f, then c corresponds to the element $-c$ in the domain of $F(x) = f(-x)$ since $F(-c) = f(-(-c)) = f(c)$. Hence, the corresponding points here are $(c, f(c))$ and $(-c, F(-c)) = (-c, f(c))$. Comparing $(c, f(c))$ with $(-c, f(c))$, we see they are reflections about the y-axis.

Using the language of inputs and outputs, Theorem **??** says that multiplying the *outputs* from a function by -1 reflects its graph across the *horizontal* axis, while multiplying the *inputs* to a function by -1 reflects the graph across the *vertical* axis.

Applying Theorem **??** to the graph of $y = f(x)$ given at the beginning of the section, we can graph $y = -f(x)$ by reflecting the graph of f about the x-axis.

[1] The expressions $-f(x)$ and $f(-x)$ should look familiar - they are the quantities we used in Section **??** to determine if a function was even, odd or neither. We explore impact of symmetry on reflections in Exercise **??**.

x	$(x, f(x))$	$f(x)$	$g(x) = -f(x)$	$(x, g(x))$
0	$(0, 1)$	1	-1	$(0, -1)$
2	$(2, 3)$	3	-3	$(2, -3)$
4	$(4, 3)$	3	-3	$(4, -3)$
5	$(5, 5)$	5	-5	$(5, -5)$

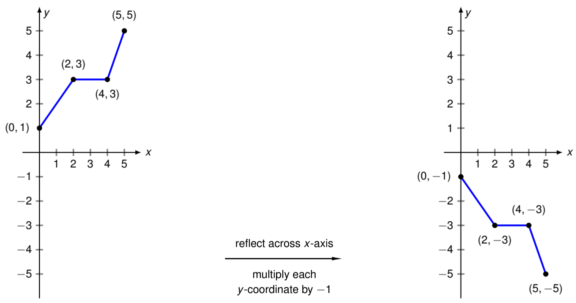

Figure 10: Graphs of $y = f(x)$ (left) and $y = -f(x)$ (right).

Alt text: Two graphs illustrate a reflection across the x-axis. The graph on the left is a piecewise linear function passing through the points $(0, 1)$, $(2, 3)$, $(4, 3)$, and $(5, 5)$. The graph on the right is obtained by reflecting the first graph across the x-axis. The corresponding points are $(0, -1)$, $(2, -3)$, $(4, -3)$, and $(5, -5)$.

By reflecting the graph of f across the y-axis, we obtain the graph of $y = f(-x)$.

x	$-x$	$g(x) = f(-x)$	$(x, g(x))$
0	0	$g(0) = f(-(-0)) = f(0) = 1$	$(0, 1)$
-2	2	$g(-2) = f(-(-2)) = f(2) = 3$	$(-2, 3)$
-4	4	$g(-4) = f(-(-4)) = f(4) = 3$	$(-4, 3)$
-5	5	$g(-5) = f(-(-5)) = f(5) = 5$	$(-5, 5)$

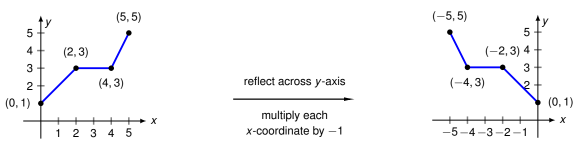

Figure 11: Graphs of $y = f(x)$ (left) and $y = f(-x)$ (right).

Alt text: Two graphs illustrate a reflection across the y-axis. The graph on the left is a piecewise linear function passing through the points $(0, 1)$, $(2, 3)$, $(4, 3)$, and $(5, 5)$. The graph on the right is obtained by reflecting the first graph across the y-axis. The corresponding points are $(0, 1)$, $(-2, 3)$, $(-4, 3)$, and $(-5, 5)$.

Example 2. Use Theorems **??**, **??** and **??** to answer the questions below. Check your answers using a graphing utility where appropriate.

1. Suppose $(2, -5)$ is on the graph of $y = f(x)$. Find a point on the graph of:

   (i) $y = f(-x)$ (ii) $y = -f(x + 2)$ (iii) $y = f(8 - x)$

2. Find a formula for a function $H(s)$ whose graph is the same as $t = h(s) = s^3 - s^2$ but is reflected across the t-axis.

3. Predict how the graph of $G(t) = \dfrac{t+4}{t-3}$ relates to the graph of $g(t) = \dfrac{t+4}{3-t}$.

4. Below is the graph of $y = f(x)$. Use it to sketch the graph of

   (i) $F(x) = f(-x) + 1$

   (ii) $F(x) = 1 - f(2 - x)$

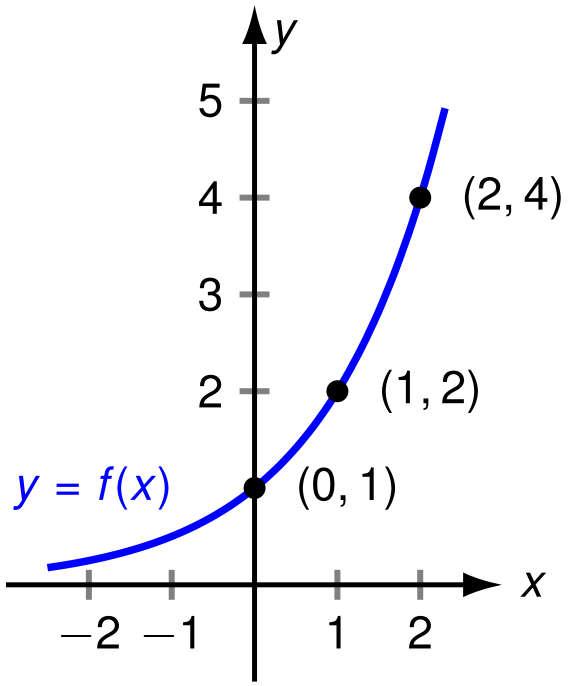

Figure 12: Graph of $y = f(x)$

Alt text: Graph of an increasing function. The curve rises gradually at first and then becomes steeper as it moves from left to right. The points $(0, 1)$, $(1, 2)$, and $(2, 4)$ are marked and labeled on the graph.

5. Below on the right is the graph of $y = g(x)$. Write $g(x)$ in terms of $f(x)$ and vice-versa.

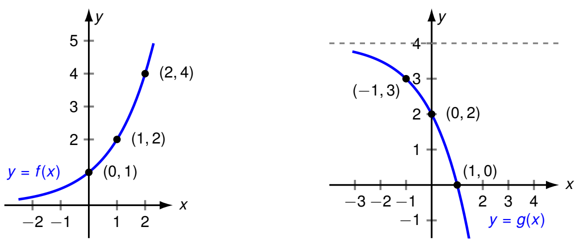

Figure 13: Graphs of $y = f(x)$ and $y = g(x)$

Alt text: The graph on the left shows a function passing through the points $(0, 1)$, $(1, 2)$, and $(2, 4)$. The curve increases from left to right, rising gradually at first and then becoming steeper.

The graph on the right shows a decreasing function passing through the points $(-1, 3)$, $(0, 2)$, and $(1, 0)$. The curve decreases from left to right, becoming steeper as it approaches the point $(1, 0)$. A dashed horizontal line at $y = 4$ is shown above the graph.

Note that the x-axis, $y = 0$, is a horizontal asymptote to the graph of $y = f(x)$ and the line $y = 4$ is a horizontal asymptote to the graph of $y = g(x)$.

Solution.

1. (i) To find a point on the graph of $y = f(-x)$, Theorem **??** tells us to multiply the x-coordinate of the point on the graph of $y = f(x)$ by -1: $((-1)2, -5) = (-2, -5)$.

   To check, since $(2, -5)$ is on the graph of f, we know $f(2) = -5$. Hence, when we substitute $x = -2$ into $y = f(-x)$, we get $y = f(-(-2)) = f(2) = -5$, proving $(-2, -5)$ is on the graph of $y = f(-x)$.

   (ii) To find a point on the graph of $y = -f(x + 2)$, we first note we have two transformations at work here, so we work our way from the inside out and build $f(x)$ to $-f(x + 2)$.

   First, we find a point on the graph of $y = f(x+2)$. Writing $f(x+2) = f(x-(-2))$, we apply Theorem **??** with $h = -2$ and add -2 to (or subtract 2 from) the x-coordinate of the point we know is on $y = f(x)$: $(2 - 2, -5) = (0, -5)$.

   Next we apply Theorem **??** to the graph of $y = f(x+2)$ to get a point on the graph of $y = -f(x+2)$ by multiplying the y-coordinate of $(0, -5)$ by -1: $(0, (-1)(-5)) = (0, 5)$.

   To check, recall $f(2) = -5$ so that when we substitute $x = 0$ into the equation $y = -f(x + 2)$, we get $y = -f(0 + 2) = -f(2) = -(-5) = 5$, as required.

   (iii) Rewriting $f(8-x) = f(-x+8)$ we see we have two transformations at play here: a reflection across the y-axis and a horizontal shift. Since both of these transformations affect the x-coordinates of the graph, the question becomes which transformation to address first. To help us with this decision, we attack the problem algebraically.

   Recall that since $(2, -5)$ is on the graph of f, we know $f(2) = -5$. Hence, to get a point on the graph of $y = f(-x + 8)$, we need to match up the arguments of $f(-x + 8)$ and $f(2)$: $-x + 8 = 2$.

   To solve this equation, we first subtract 8 from both sides to get $-x = -6$. Geometrically, subtracting 8 from the x-coordinate of $(2, -5)$, shifts the point $(2, -5)$ left 8 units to get the point $(-6, -5)$.

   Next, we multiply both sides of the equation $-x = -6$ by -1 to get $x = 6$. Geometrically, multiplying the x-coordinate of $(-6, -5)$ by -1 reflects the point $(-6, -5)$ across the y-axis to $(6, -5)$.

   To check we substitute $x = 6$ into $y = f(-x + 8)$, and obtain $y = f(-6 + 8) = f(2) = -5$.

   Even though we have found our answer, we re-examine this process from a 'build' perspective. We began with a point on the graph of $y = f(x)$ and first shifted the graph to the left 8 units. Per Theorem **??**, this point is on the graph of $y = f(x + 8)$.

   Next we took a point on the graph of $y = f(x + 8)$ and reflected it about the y-axis. Per Theorem **??**, this put the point on the graph of $y = f(-x + 8)$.

   In general, when faced with graphing functions in which there is both a horizontal shift and a reflection about the y-axis, we'll deal with the shift first.

2. In this example, the independent variable is s and the dependent variable is t. We are asked to reflect the graph of h about the t-axis, which in this case is the *vertical* axis. Hence, $H(s) = h(-s) = (-s)^3 - (-s)^2 = -s^3 - s^2$.

   Using the Desmos interactive below, we can use the slider for s and observe that points on the graph of the function $t = H(s)$ are indeed reflections across the (vertical) t-axis of points on the graph of the function $t = h(s)$.

   Desmos link: `https://www.desmos.com/calculator/96sn2hkp5g`

3. Comparing the formulas for $G(t) = \dfrac{t+4}{t-3}$ and $g(t) = \dfrac{t+4}{3-t}$, we have the same numerators, but in the denominator, we have $(t - 3) = -(3 - t)$:

$$G(t) = \frac{t+4}{t-3} = \frac{t+4}{-(3-t)} = -\frac{t+4}{3-t} = -g(t).$$

Hence, the graph of $y = G(t)$ should be the graph of $y = g(t)$ reflected across the t-axis.

Using the Desmos interactive below, adjusting the slider for t traces corresponding points on both $y = g(t)$ and $y = G(t)$, confirming that these two graphs are reflections about the (horizontal) t-axis.

Desmos link: `https://www.desmos.com/calculator/lwjz5zvtva`

4. (i) We have two transformations indicated with the formula $F(x) = f(-x) + 1$: a reflection across the y-axis and a vertical shift. Working from the inside out, we first tackle the reflection. Per Theorem **??**, to obtain the graph of $y = f(-x)$ from $y = f(x)$, we multiply each of the x-coordinates of each of the points on the graph of $y = f(x)$ by (-1).

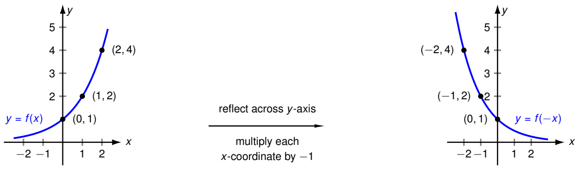

Figure 14: Graphs of $y = f(x)$ and $y = f(-x)$

Alt text: Two graphs illustrate a reflection across the y-axis. The graph on the left shows a function passing through the points $(0, 1)$, $(1, 2)$, and $(2, 4)$. The graph on the right is obtained by reflecting the first graph across the y-axis and passes through the points $(0, 1)$, $(-1, 2)$, and $(-2, 4)$.

Next, we use Theorem **??** to obtain the graph of $y = f(-x) + 1$ from the graph of $y = f(-x)$ by adding 1 to each of the y-coordinates of each of the points on the graph of $y = f(-x)$. This shifts the graph of $y = f(-x)$ up one unit. Note, the horizontal asymptote $y = 0$ is also shifted up 1 unit to $y = 1$.

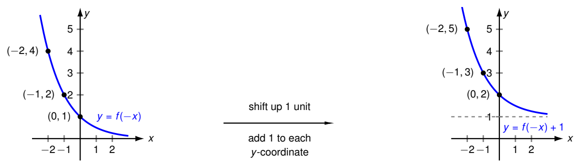

Figure 15: Graphs of $y = f(x)$ and $y = f(x) + 1$

Alt text: A decreasing function passing through the points $(0, 1)$, $(-1, 2)$, and $(-2, 4)$ is translated upward 1 unit. The translated graph passes through the points $(0, 2)$, $(-1, 3)$, and $(-2, 5)$. A dashed horizontal line at $y = 1$ is shown to indicate the new horizontal asymptote.

To check our answer, we begin with the point $(0, 2)$. Substituting $x = 0$ into $y = f(-x) + 1$, we get $y = f(-0) + 1 = f(0) + 1$. Since the point $(0, 1)$ is on the graph of f, we know $f(0) = 1$. Hence, $y = f(0) + 1 = 1 + 1 = 2$, so $(0, 2)$ is, indeed, on the graph of $y = f(-x) + 1$. We leave it to the reader to check the remaining points.

(ii) In order to graph $F(x) = 1 - f(2 - x)$, we first rewrite as $F(x) = -f(-x + 2) + 1$ and note there are *four* modifications to the formula $f(x)$ indicated here.

Working from the inside out, we see we have a reflection about the y-axis indicated as well as a horizontal shift. From our work above, we know we first handle the shift: that is, we apply Theorem **??** to graph $y = f(x + 2) = f(x - (-2))$ by adding -2 to (subtracting 2 from) the x-coordinates of the points on the graph of $y = f(x)$.

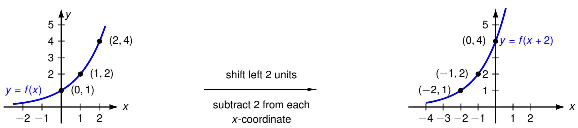

Figure 16: Graphs of $y = f(x)$ and $y = f(x + 2)$

Alt text: An increasing function passing through the points $(0, 1)$, $(1, 2)$, and $(2, 4)$ is translated left 2 units. The translated graph passes through the points $(-2, 1)$, $(-1, 2)$, and $(0, 4)$.

Next, we use Theorem **??** to graph $y = f(-x + 2)$ starting with the graph of $y = f(x + 2)$ by multiplying each of the x-coordinates of the points of the graph of $y = f(x + 2)$ by -1. This reflects the graph of $f(x + 2)$ about the y-axis.

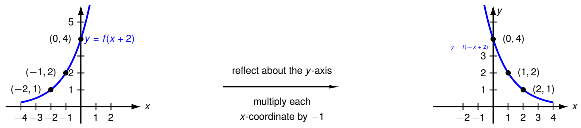

Figure 17: Graphs of $y = f(x + 2)$ and $y = f(-x + 2)$

Alt text: An increasing function passing through the points $(-2, 1)$, $(-1, 2)$, and $(0, 4)$ is reflected across the y-axis. The reflected graph passes through the points $(2, 1)$, $(1, 2)$, and $(0, 4)$.

We have the graph of $y = f(-x + 2)$ and need to build towards the graph of $y = -f(-x + 2) + 1$. The transformations that remain are a reflection about the x-axis and a vertical shift. The question is which to do first.

Once again, we can think algebraically about the problem. We know the point $(0, 1)$ is on the graph of f which means $f(0) = 1$. This point corresponds to the point $(2, 1)$ on the graph of $f(-x + 2)$. Indeed, when we substitute $x = 2$ into $y = f(-x + 2)$, we get $y = f(-2 + 2) = f(0) = 1$.

If we substitute $x = 2$ into the formula $y = -f(-x + 2) + 1$, we get $y = -f(-2 + 2) + 1 = -f(0) + 1 = -1(1) + 1 = 0$. That is, we first multiply the y-coordinate of $(2, 1)$ by -1 then add 1. This suggests we take care of the reflection about the x-axis first, then the vertical shift.

We proceed below to obtain the graph of $y = -f(-x + 2)$ from $y = f(-x + 2)$ by multiplying each of the y-coordinates on the graph of $y = f(-x + 2)$ by -1. Note the horizontal asymptote remains unchanged: $y = (-1)(0) = 0$.

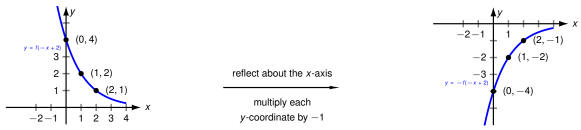

Figure 18: Graphs of $y = f(-x + 2)$ and $y = -f(-x + 2)$

Alt text: A decreasing function passing through the points $(0, 4)$, $(1, 2)$, and $(2, 1)$ is reflected across the x-axis. The reflected graph passes through the points $(0, -4)$, $(1, -2)$, and $(2, -1)$.

Finally, we take care of the vertical shift. Per Theorem **??**, we graph $y = -f(-x + 2) + 1$ by adding 1 to the y-coordinates of each of the points on the graph of $y = -f(-x + 2)$. This moves the graph up one unit, including the horizontal asymptote: $y = 0 + 1 = 1$.

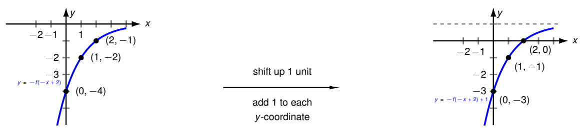

Figure 19: Graphs of $y = -f(-x + 2)$ and $y = -f(-x + 2) + 1$

Alt text: A function passing through the points $(0, -4)$, $(1, -2)$, and $(2, -1)$ is translated upward 1 unit. The translated graph passes through the points $(0, -3)$, $(1, -1)$, and $(2, 0)$. A dashed horizontal line is shown above the graph to indicate the new horizontal asymptote.

To check, we begin with the point $(2, 0)$. Substituting $x = 2$ into $y = 1 - f(2 - x)$, we obtain $y = 1 - f(2 - 2) = 1 - f(0)$. Since $(0, 1)$ is on the graph of f, we know $f(0) = 1$. This means $y = 1 - f(2 - 2) = 1 - f(0) = 1 - 1 = 0$. This proves $(2, 0)$ is on the graph of $y = 1 - f(2 - x)$, and we recommend the reader check the remaining points.

5. With the transformations at our disposal, our task amounts to finding values of h and k and choosing between signs $\pm$ so that $g(x) = \pm f(\pm x - h) + k$.

   Based on the horizontal asymptote, $y = 4$, we choose $k = 4$. Note, however, in the graph of $y = f(x) + 4$, the entire graph is *above* the line $y = 4$. Since the graph of g approaches the asymptote from below, we know $y = -f(\pm x - h) + 4$.

   Hence, two of transformations applied to the graph of f are a reflection across the x-axis followed by a shift up 4 units. This means the point $(0, 1)$ on the graph of f must correspond to the point $(-1, 3)$ on the graph of g, since these are the points closest to the asymptote on each graph.

   Likewise, the points $(1, 2)$ and $(2, 4)$ on the graph of f must correspond to $(0, 2)$ and $(1, 0)$, respectively, on the graph of g. Looking at the x-coordinates only, we have $x = 0$ moves to $x = -1$, $x = 1$ moves to $x = 0$, and $x = 2$ moves to $x = 1$. Hence, the net effect on the x-values is a shift left 1 unit. Hence, we guess the formula for $g(x)$ to be $g(x) = -f(x + 1) + 4$.

   We can readily check by going through the transformations: first, shift left 1 unit; next, reflect across the x-axis; finally, shift up 4. We leave it to the reader to verify that tracking each of the points on the graph of f along with the horizontal asymptote through this sequence of transformations results in the graph of g.

   One way to recover the graph of f from the graph of g is to reverse the process by which we obtained g from f. The challenge here comes from the fact that two different operations were done which affected the y-values: reflection and shifting - and the order in which these are done matters.

   To motivate our methodology, let's consider a more down-to-earth example like putting on socks and then putting on shoes. Unless we're very talented, to reverse this process, we take off the shoes first, then the socks - that is, we undo each step in the reverse order.[2] In the same way, when we think about reversing the steps transforming the graph of f to the graph of g, we need to undo each transformation in the opposite order.

   To review, we obtained the graph of g from the graph of f by first shifting the graph to the left 1 unit, then reflecting the graph about the x-axis, then, finally, shifting the graph up 4 units. Hence, we first undo the vertical shift. Instead of shifting the graph *up* four units, we shift the graph *down* four units. This takes the graph of $y = g(x)$ to $y = g(x) - 4$.

   Next, we have to undo the refection across the x-axis. Thinking at the level of points, to recover the point (a, b) from its reflection across the x-axis, $(a, -b)$, we simply reflect across the x-axis again: $(a, -(-b)) = (a, b)$. Per Theorem **??**, this takes the graph the graph of $y = g(x) - 4$ to the graph of $y = -[g(x) - 4] = -g(x) + 4$.[3]

   Last, to undo moving the graph to the *left* 1 unit, we move the graph of $y = -g(x) + 4$ to the *right* 1 unit. Per Theorem **??**, we accomplish this by graphing $y = -g(x - 1) + 4$. We leave it to the reader to start with the graph of $y = g(x)$ and graph $y = -g(x - 1) + 4$ and show it matches the graph of $y = f(x)$. ■

Some remarks about Example **??** are in order. In number **??** above, to find a point on the graph of $y = f(-x + 8)$, we took the given x-coordinate on our starting graph, 2, and subtracted 8 first then multiplied by -1. If this seems somehow 'backwards' it should.

When *evaluating* the expression $-x + 8$, the order of operations mandates we multiply by -1 first then add 8. Here, however, we weren't *evaluating* an expression - we were *solving* an equation: $-x + 8 = 2$, which meant

[2] We'll have more to say about this sort of thing in Section **??**.

[3] To see this better, let us temporarily write $F(x) = g(x) - 4$. Theorem **??** tells us to reflect the graph of F about the x-axis, graph $y = -F(x) = -[g(x) - 4] = -g(x) + 4$.

we did the exact opposite steps in the opposite order.[4] This exemplifies a larger theme with transformations: when adjusting inputs, the resulting points on the graph are obtained by applying the opposite operations indicated by the formula in the opposite order of operations.

On the other hand, when it came to multiple transformations involving the y-coordinates, we followed the order of operations. As in **??** above, when it came to applying a reflection about the x-axis and a vertical shift, we applied the reflection first, then the shift. This is because instead of *solving* an *equation* to find the new y-coordinates, we were *simplifying* an *expression*. Again, this is an example of a much larger theme: when adjusting outputs, the resulting points on the graph are obtained by applying the stated operations in the usual order.

Last but not least, in number **??**, to find f in terms of g, we reversed the steps used to transform f into g. Another tact is to approach the problem in the same way we approached transforming f into g: namely, starting with the graph of g, determine values h and k and signs $\pm$ so that $f(x) = \pm g(\pm x - h) + k$. We leave this to the reader.

3 Scalings

We now turn our attention to our last class of transformations: **scalings**. A thorough discussion of scalings can get complicated because they are not as straight-forward as the previous transformations. A quick review of what we've covered so far, namely vertical shifts, horizontal shifts and reflections, will show you why those transformations are known as **rigid transformations**.

Simply put, rigid transformations preserve the distances between points on the graph - only their position and orientation in the plane change.[5] If, however, we wanted to make a new graph twice as tall as a given graph, or one-third as wide, we would be affecting the distance between points. These sorts of transformations are hence called **non-rigid**. As always, we motivate the general theory with an example.

Suppose we wish to graph the function $g(x) = 2f(x)$ where $f(x)$ is the function whose graph is given at the beginning of the section. From its graph, we can build a table of values for g as before.

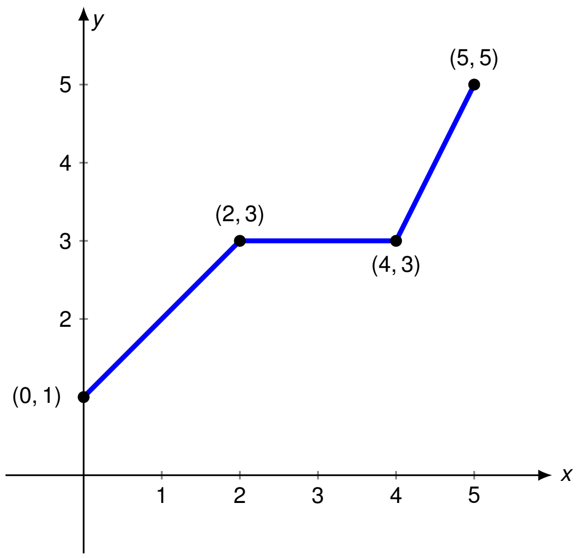

Figure 20: $y = f(x)$

Alt text: Graph of a piecewise linear function passing through the points $(0, 1)$, $(2, 3)$, $(4, 3)$, and $(5, 5)$. The graph rises from $(0, 1)$ to $(2, 3)$, remains horizontal from $(2, 3)$ to $(4, 3)$, then rises again to $(5, 5)$.

[4] Note that dividing by -1 is the same as multiplying by -1, so to keep with the 'opposite steps in opposite order' theme, we could more precisely say we subtracted 8 and *divided* by -1.

[5] Another word that can be used here instead of 'rigid transformation' is 'isometry' - meaning 'same distance.'

x	$(x, f(x))$	$f(x)$	$g(x) = 2f(x)$	$(x, g(x))$
0	$(0, 1)$	1	2	$(0, 2)$
2	$(2, 3)$	3	6	$(2, 6)$
4	$(4, 3)$	3	6	$(4, 6)$
5	$(5, 5)$	5	10	$(5, 10)$

Graphing, we get:

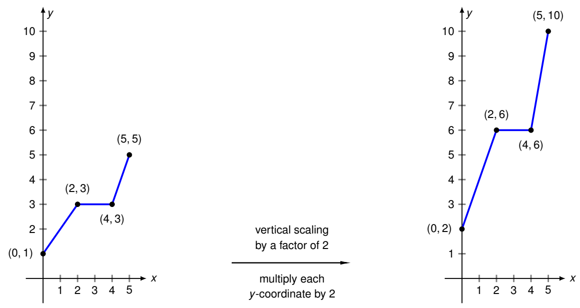

Figure 21: Graphs of $y = f(x)$ and $y = 2f(x)$

Alt text: A piecewise linear function passing through the points $(0, 1)$, $(2, 3)$, $(4, 3)$, and $(5, 5)$ is vertically stretched by a factor of 2. The transformed graph passes through the points $(0, 2)$, $(2, 6)$, $(4, 6)$, and $(5, 10)$.

In general, if (a, b) is on the graph of f, then $f(a) = b$ so that $g(a) = 2f(a) = 2b$ puts $(a, 2b)$ on the graph of g. In other words, to obtain the graph of g, we multiply all of the y-coordinates of the points on the graph of f by 2. Multiplying all of the y-coordinates of all of the points on the graph of f by 2 causes what is known as a 'vertical scaling[6] by a factor of 2.'

If we wish to graph $y = \frac{1}{2}f(x)$, we multiply the all of the y-coordinates of the points on the graph of f by $\frac{1}{2}$. This creates a 'vertical scaling[7] by a factor of $\frac{1}{2}$' as seen below.

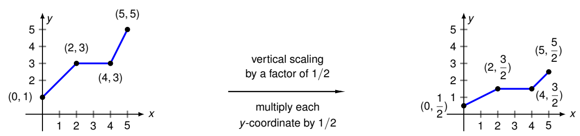

Figure 22: Graphs of $y = f(x)$ and $y = \frac{1}{2}f(x)$

Alt text: A piecewise linear function passing through the points $(0, 1)$, $(2, 3)$, $(4, 3)$, and $(5, 5)$ is vertically compressed by a factor of $\frac{1}{2}$. The transformed graph passes through the points $\left(0, \frac{1}{2}\right)$, $\left(2, \frac{3}{2}\right)$, $\left(4, \frac{3}{2}\right)$, and $\left(5, \frac{5}{2}\right)$.

These results are generalized in the following theorem.

Theorem 4. Vertical Scalings. Suppose f is a function and $a > 0$ is a real number.

To graph $F(x) = af(x)$, multiply each of the y-coordinates of the points on the graph of $y = f(x)$ by a.

[6] Also called a 'vertical stretch,' 'vertical expansion' or 'vertical dilation' by a factor of 2.
[7] Also called 'vertical shrink,' 'vertical compression' or 'vertical contraction' by a factor of 2.

- If $a > 1$, we say the graph of f has undergone a vertical stretch[8] by a factor of a.
- If $0 < a < 1$, we say the graph of f has undergone a vertical shrink[9] by a factor of $\frac{1}{a}$.

The proof of Theorem **??** mimics the proofs of Theorems **??** and **??**. If c is in the domain of f, then $(c, f(c))$ is on the graph of f and the corresponding point on the graph of $F(x) = af(x)$ is $(c, F(c)) = (c, af(c))$. Comparing the points $(c, f(c))$ and $(c, af(c))$ proves the theorem.

A few remarks about Theorem **??** are in order. First, a note about the verbiage. To the authors, the words 'stretch', 'expansion', and 'dilation' all indicate something getting bigger. Hence, 'stretched by a factor of 2' makes sense if we are scaling something by multiplying it by 2. Similarly, we believe words like 'shrink', 'compression' and 'contraction' all indicate something getting smaller, so if we scale something by a factor of $\frac{1}{2}$, we would say it 'shrinks by a factor of 2' - not 'shrinks by a factor of $\frac{1}{2}$'. This is why we have written the descriptions 'stretch by a factor of a' and 'shrink by a factor of $\frac{1}{a}$' in the statement of the theorem.

Second, in terms of inputs and outputs, Theorem **??** says multiplying the *outputs* from a function by positive number a causes the graph to be vertically scaled by a factor of a. It is natural to ask what would happen if we multiply the *inputs* of a function by a positive number. This leads us to our last transformation of the section.

Referring to the graph of f given at the beginning of this section, suppose we want to graph $g(x) = f(2x)$. In other words, we are looking to see what effect multiplying the inputs to f by 2 has on its graph. If we attempt to build a table directly, we quickly run into the same problem we had in our discussion leading up to Theorem **??**, as seen in the table on the left below.

We solve this problem in the same way we solved this problem before. For example, if we want to determine the point on g which corresponds to the point $(2, 3)$ on the graph of f, we set $2x = 2$ so that $x = 1$. Substituting $x = 1$ into $g(x)$, we obtain $g(1) = f(2 \cdot 1) = f(2) = 3$, so that $(1, 3)$ is on the graph of g. Continuing in this fashion, we obtain the table on the lower right.

x	$(x, f(x))$	$f(x)$	$g(x) = f(2x)$	$(x, g(x))$
0	$(0, 1)$	1	$f(2 \cdot 0) = f(0) = 1$	$(0, 1)$
2	$(2, 3)$	3	$f(2 \cdot 2) = f(4) = 3$	$(2, 3)$
4	$(4, 3)$	3	$f(2 \cdot 4) = f(8) = ?$	
5	$(5, 5)$	5	$f(2 \cdot 5) = f(10) = ?$	

x	$2x$	$g(x) = f(2x)$	$(x, g(x))$
0	0	$g(0) = f(2 \cdot 0) = f(0) = 1$	$(0, 0)$
1	2	$g(1) = f(2 \cdot 1) = f(2) = 3$	$(1, 3)$
2	4	$g(2) = f(2 \cdot 2) = f(4) = 3$	$(2, 3)$
$\frac{5}{2}$	5	$g\left(\frac{5}{2}\right) = f\left(2 \cdot \frac{5}{2}\right) = f(5) = 5$	$\left(\frac{5}{2}, 5\right)$

In general, if (a, b) is on the graph of f, then $f(a) = b$. Hence $g\left(\frac{a}{2}\right) = f\left(2 \cdot \frac{a}{2}\right) = f(a) = b$ so that $\left(\frac{a}{2}, b\right)$ is on the graph of g. In other words, to graph g we divide the x-coordinates of the points on the graph of f by 2. This results in a horizontal scaling[10] by a factor of $\frac{1}{2}$.

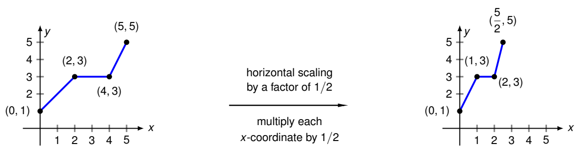

Figure 23: Graphs of $y = f(x)$ and $y = f(2x)$

Alt text: A piecewise linear function passing through the points $(0, 1)$, $(2, 3)$, $(4, 3)$, and $(5, 5)$ is horizontally scaled by a factor of $\frac{1}{2}$. The transformed graph passes through the points $(0, 1)$, $(1, 3)$, $(2, 3)$, and $\left(\frac{5}{2}, 5\right)$.

[8] expansion, dilation
[9] compression, contraction
[10] Also called 'horizontal shrink,' 'horizontal compression' or 'horizontal contraction' by a factor of 2.

If, on the other hand, we wish to graph $y = f\left(\frac{1}{2}x\right)$, we end up multiplying the x-coordinates of the points on the graph of f by 2 which results in a horizontal scaling[11] by a factor of 2, as demonstrated below.

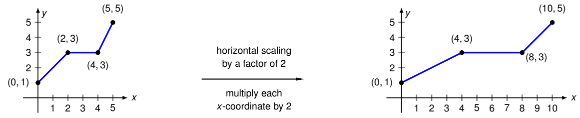

Figure 24: Graphs of $y = f(x)$ and $y = f(\frac{1}{2}x)$

Alt text: A piecewise linear function passing through the points (0, 1), (2, 3), (4, 3), and (5, 5) is horizontally scaled by a factor of 2. The transformed graph passes through the points (0, 1), (4, 3), (8, 3), and (10, 5).

We have the following theorem.

Theorem 5. Horizontal Scalings. Suppose f is a function and $b > 0$ is a real number.

To graph $F(x) = f(bx)$, divide each of the x-coordinates of the points on the graph of $y = f(x)$ by b.

- If $0 < b < 1$, we say the graph of f has undergone a horizontal stretch[12] by a factor of $\frac{1}{b}$.
- If $b > 1$, we say the graph of f has undergone a horizontal shrink[13] by a factor of b.

The proof of Theorem **??** follows closely the spirit of the proof of Theorems **??** and **??**. If c is an element of the domain of f, them the number $\frac{c}{b}$ corresponds to a domain element of $F(x) = f(bx)$ since $F\left(\frac{c}{b}\right) = f\left(b \cdot \frac{c}{b}\right) = f(c)$. Hence, there is a correspondence between the point $(c, f(c))$ on the graph of f and the point $\left(\frac{c}{b}, F\left(\frac{c}{b}\right)\right) = \left(\frac{c}{b}, f(c)\right)$ on the graph of F. We can obtain $\left(\frac{c}{b}, f(c)\right)$ by dividing the x-coordinate of $(c, f(c))$ by b and the result follows.

Theorem **??** tells us that if we multiply the input to a function by b, the resulting graph is scaled horizontally by a factor of $\frac{1}{b}$. The next example explores how vertical and horizontal scalings sometimes interact with each other and with the other transformations introduced in this section.

Example 3. Use Theorems **??**, **??**, **??**, **??** and **??** to answer the questions below. Check your answers using a graphing utility where appropriate.

1. Suppose $(-1, 4)$ is on the graph of $y = f(x)$. Find a point on the graph of:

   (i) $y = 3f(x - 2)$ (ii) $y = f\left(-\frac{1}{2}x\right)$ (iii) $y = f(2x - 3) + 1$

2. Find a formula for a function $G(t)$ whose graph is the same as $y = g(t) = \frac{2t + 1}{t - 1}$ but is vertically stretched by a factor of 4.
3. Predict how the graph of $H(s) = 8s^3 - 12s^2$ relates to the graph of $h(s) = s^3 - 3s^2$.
4. Below is the graph of $y = f(x)$. Use it to sketch the graph of

[11] Also called 'horizontal stretch,' 'horizontal expansion' or 'horizontal dilation' by a factor of 2.
[12] expansion, dilation
[13] compression, contraction

(i) $F(x) = \dfrac{1 - f(x)}{2}$

(ii) $F(x) = f\left(\dfrac{1-x}{2}\right)$

GeoGebra link: `https://www.geogebra.org/m/whqkfda2`

5. Below on the right is the graph of $y = g(x)$. Write $g(x)$ in terms of $f(x)$ and vice-versa.

GeoGebra link: `https://www.geogebra.org/m/zwprjjme`

NOTE: The y-axis, $x = 0$, is a vertical asymptote to the graph of $y = f(x)$ and the line $x = 2$ is a vertical asymptote to the graph of $y = g(x)$.

Solution.

1. (i) As we examine the formula $y = 3f(x-2)$, we note two modifications from $y = f(x)$. Building from the inside out, we start with obtaining a point on the graph of $y = f(x-2)$.

   Per Theorem **??**, this shifts all of the points on the graph of $y = f(x)$ 2 units to the right. Hence, the point $(-1, 4)$ on the graph of $y = f(x)$ moves to the point $(-1+2, 4) = (1, 4)$ on the graph of $y = f(x-2)$.

   To get a point on the graph of $y = 3f(x-2) = af(x-3)$, we apply Theorem **??** with $a = 3$ to the point $(1, 4)$ on the graph of $y = f(x-2)$ to get the point $(1, 3(4)) = (1, 12)$ on the graph of $y = 3f(x-2)$.

   To check, we note that since $(-1, 4)$ is on the graph of $y = f(x)$, we know $f(-1) = 4$. Hence, when we substitute $x = 1$ into the $y = 3f(x-2)$, we get $y = 3f(1-2) = 3f(-1) = 3(4) = 12$.

   (ii) The formula $y = f\left(-\dfrac{1}{2}x\right)$ also indicates two transformations: a horizontal scaling, indicated by $\dfrac{1}{2}$ factor, as well as a reflection across the y-axis. The question before us is which to do first.

   If we return to algebra for inspiration, we know $f(-1) = 4$, so we match up the arguments of $f\left(-\dfrac{1}{2}x\right)$ and $f(-1)$ and get the equation $-\dfrac{1}{2}x = -1$. We solve this equation by multiplying both sides by -2: $x = (-2)(-1) = 2$. That is, we take the original x-value on the graph of $y = f(x)$ and multiply it by -2.

   If we think of $-2 = (-1)(2)$ then multiplying by the '2' in '$(-1)(2)$' produces a horizontal stretch by a factor of 2 while multiplying by the '-1' reflects the point across the y-axis.

   Applying the horizontal stretch first, we use Theorem **??** and start with the point $(-1, 4)$ on the graph of $y = f(x)$ and multiply the x-coordinate by 2 to obtain a point on the graph of $y = f\left(\dfrac{1}{2}x\right)$: $(-1(2), 4) = (-2, 4)$.

   Next, we take care of the reflection about the y-axis using Theorem **??** . Starting with $(-2, 4)$ on the graph of $y = f\left(\dfrac{1}{2}x\right)$, we multiply the x-coordinate by -1 to obtain a point on the graph of $y = f\left(\dfrac{1}{2}(-x)\right) = f\left(-\dfrac{1}{2}x\right)$: $((-1)(-2), 4) = (2, 4)$.

   To check, note when $x = 2$ is substituted into $y = f\left(-\dfrac{1}{2}x\right)$, we get $y = f\left(-\dfrac{1}{2}(2)\right) = f(-1) = 4$.

   Of course, we could have equally written the multiple $-2 = (2)(-1)$ and reversed these steps: doing the reflection first, then the horizontal scaling.

Proceeding this way, we start with the point $(-1, 4)$ on the graph of $y = f(x)$ and reflect across the y-axis to obtain the point $((-1)(-1), 4) = (1, 4)$ on the graph of $y = f(-x)$.

Next, we stretch the graph of $y = f(-x)$ by a factor of 2 by multiplying the x-coordinates of the points on the graph by 2 and obtain $(2(1), 4) = (2, 4)$ on the graph of $y = f\left(-\frac{1}{2}x\right)$.

In general when it comes to reflections and scalings, whether horizontal or, as we'll see soon, vertical, either order will produce the same results.

(iii) The formula $f(2x - 3) + 1$ indicates *three* transformations: a horizontal shift, a horizontal scaling, and a vertical shift. As usual, we appeal to algebra to give us guidance on which horizontal transformation to apply first.

Since we know $f(-1) = 4$, we set $2x - 3 = -1$ and solve. Our first step is to add 3 to both sides: $2x = (-1) + 3 = 2$. Since we are adding 3 to the given x-value -1, this corresponds to a shift to the right 3 units, so the point $(-1, 4)$ is moved to the point $(2, 4)$.

Next, to solve $2x = 2$, we divide this new x-coordinate 2 by 2 and get $x = \frac{2}{2} = 1$ which corresponds to a horizontal compression by a factor of 2. This moves the point $(2, 4)$ to $(1, 4)$.

Hence, the algebra suggests we use Theorem **??** first and follow it up with Theorem **??**. Starting with $(-1, 4)$ on the graph of $y = f(x)$, we shift to the right 3 units to obtain the point $(-1 + 3, 4) = (2, 4)$ on the graph of $y = f(x - 3)$.

Next, we start with the point $(2, 4)$ on the graph of $y = f(x - 3)$ and horizontally shrink the x-axis by a factor of 2 to get the point $\left(\frac{2}{2}, 4\right) = (1, 4)$ on the graph of $y = f(2x - 3)$.

Last, but not least, we take care of the vertical shift using Theorem **??**. Starting with the point $(1, 4)$ on the graph of $y = f(2x - 3)$, we add 1 to the y-coordinate to get the point $(1, 4 + 1) = (1, 5)$ on the graph of $y = f(2x - 3) + 1$.

To check, we substitute $x = 1$ into the formula $y = f(2x - 3) + 1$ and get $y = f(2(1) - 3) + 1 = f(-1) + 1 = 4 + 1 = 5$, as required.

2. To vertically stretch the graph of $y = g(t)$ by 4, we use Theorem **??** with $a = 4$ to get

$$G(t) = 4g(t) = 4\frac{2t + 1}{t - 1} = \frac{4(2t + 1)}{t - 1} = \frac{8t + 4}{t - 1}.$$

Using the Desmos interactive below, adjusting the slider for t traces corresponding points on both $y = g(t)$ and $y = G(t)$. Dividing the y-coordinates on the graph of $y = G(t)$ by the y-coordinates of the corresponding points on the graph of $y = g(t)$ confirms a vertical scaling factor of 4.

Desmos link: `https://www.desmos.com/calculator/f9psccysng`

3. When comparing the formulas for $H(s) = 8s^3 - 12s^2$ and $h(s) = s^3 - 3s^2$, it doesn't appear as if any shifting or reflecting is going on (why not?)

We also note that since the coefficient of s^3 in the expression of $H(s)$ is 8 times that of the coefficient of s^3 in $h(s)$, but the coefficient of s^2 in $H(s)$ is only 4 times the coefficient of s^2 in $h(s)$, the change is not the result of a vertical scaling (again, why not?)

Hence, if anything, we are looking for a horizontal scaling. In other words, we are looking for a real number $b > 0$ so $h(bs) = H(s)$, that is, $(bs)^3 - 3(bs)^2 = b^3s^3 - 3b^2s^2 = 8s^3 - 12s^2$.

Matching up coefficients of s^3 gives $b^3 = 8$ so $b = 2$ which checks with the coefficients of $s^2 : 3b^2 = 3(2)^2 = 12$. Hence, we predict the graph of $y = H(s)$ to be the same as $y = h(s)$ except horizontally compressed by a factor of 2.

Using Desmos below, we graph both $y = h(s)$ and $y = H(s)$ on the same set of axes. The graph of H certainly appears to be a horizontally compressed version of h and when comparing the numerical values of the s-coordinates of the key points indicated, the ratio confirms a compression factor of 2.

Desmos link: https://www.desmos.com/calculator/fcwljnbnpq

4. (i) We first rewrite the expression for $F(x) = \frac{1 - f(x)}{2} = -\frac{1}{2}f(x) + \frac{1}{2}$ in order to use the theorems available to us. Note we have two modifications to the formula of $f(x)$ which correspond to three transformations.

Multiplying $f(x)$ by $-\frac{1}{2}$ indicates a vertical compression by a factor of 2 along with a reflection about the x-axis. Adding $\frac{1}{2}$ indicates a vertical shift up $\frac{1}{2}$ units.

As always the question is which to do first. Once again, we look to algebra for the answer. Picking the point $(1, 0)$ on the graph of $f(x)$, we know $f(1) = 0$. To see which point this corresponds to on the graph of $y = F(x)$, we find $F(1) = -\frac{1}{2}f(1) + \frac{1}{2} = -\frac{1}{2}(0) + \frac{1}{2} = 0 + \frac{1}{2} = \frac{1}{2}$.

Hence, we first multiplied the y-value 0 by $-\frac{1}{2}$. As above, we can think of $-\frac{1}{2} = (-1)\frac{1}{2}$ so that multiplying by $-\frac{1}{2}$ amounts to a vertical compression by a factor of 2 first, then the refection about the x-axis second. Lastly, adding the $\frac{1}{2}$ is the vertical shift up $\frac{1}{2}$ unit.

Beginning with the vertical scaling by a factor of $\frac{1}{2}$, we use Theorem **??** to graph $y = \frac{1}{2}f(x)$ starting from $y = f(x)$ by multiplying each of the y-coordinates of each of the points on the graph of $y = f(x)$ by $\frac{1}{2}$.

Next, we reflect the graph of $y = \frac{1}{2}f(x)$ across the x-axis to produce the graph of $y = -\frac{1}{2}f(x)$ by multiplying each of the y-coordinates of the points on the graph of $y = \frac{1}{2}f(x)$ by -1:

Finally, we shift the graph of $y = -\frac{1}{2}f(x)$ vertically up $\frac{1}{2}$ unit by adding $\frac{1}{2}$ to each of the y-coordinates of each of the points to obtain the graph of $y = -\frac{1}{2}f(x) + \frac{1}{2} = F(x)$.

In the Desmos interactive below, the graph of $y = f(x)$ is given as well as the option to click on each intermediary step, in sequence, to build up to the graph of $y = F(x)$.

Desmos link: https://www.desmos.com/calculator/hiumui8k8l

Note that as with horizontal scalings and reflections about the y-axis, the order of vertical scalings and reflections across the x-axis is interchangeable. Had we decided to think of the factor $-\frac{1}{2} = \frac{1}{2} \cdot (-1)$, we could have just as well started with the graph of $y = f(x)$ and produced the graph of $y = -f(x)$ first.

Next, we vertically scale the graph of $y = -f(x)$ by multiplying each of the y-coordinates of each of the points on the graph of $y = -f(x)$ by $\frac{1}{2}$ to obtain the graph of $y = -\frac{1}{2}f(x)$:

Notice we've reached the same graph of $y = -\frac{1}{2}f(x)$ that we had before, and, hence we arrive at the same final answer as before once we apply the vertical shift: $y == -\frac{1}{2}f(x) + \frac{1}{2} = F(x)$.

The Desmos interactive below builds $y = F(x)$ from $y = f(x)$ using this new sequence of transformations.

Desmos link: https://www.desmos.com/calculator/ycijrcos2v

We check our answer as we have so many times before. We start with the point $\left(1, \frac{1}{2}\right)$ and substitute $x = 1$ into $y = \frac{1 - f(x)}{2}$ to get $y = \frac{1 - f(1)}{2}$. From the graph of f, we know $f(1) = 0$, so we get $y = \frac{1-f(1)}{2} = \frac{1-0}{2} = \frac{1}{2}$. This proves $\left(1, \frac{1}{2}\right)$ is on the graph of $y = \frac{1-f(x)}{2}$. We invite the reader to check the remaining points.

Note that in the preceding example, since none of the transformations included adjusting the x-coordinates of points, the vertical asymptote, $x = 0$ remained in place.

(ii) As with the previous example, we first rewrite $F(x) = f\left(\frac{1-x}{2}\right) = F\left(-\frac{1}{2}x + \frac{1}{2}\right)$. Here again, we have two modifications to the formula $f(x)$, the $-\frac{1}{2}$ multiple indicating a horizontal scaling and a reflection across the y-axis and a horizontal shift.

Based on our experience from previous examples, we do the horizontal shift first, with the order of the scaling and reflection more or less irrelevant.

To produce the graph of $y = f\left(x + \frac{1}{2}\right)$ we subtract $\frac{1}{2}$ from each of the x-coordinates of each of the points on the graph of $y = f(x)$. This moves the graph to the left $\frac{1}{2}$ unit, including the vertical asymptote $x = 0$ which moves to $x = -\frac{1}{2}$.

Next, we graph $y = f\left(\frac{1}{2}x + \frac{1}{2}\right)$ starting with $y = f\left(x + \frac{1}{2}\right)$ by horizontally expanding the graph by a factor of 2. That is, we multiply each x-coordinates on the graph of $y = f\left(x + \frac{1}{2}\right)$ by 2, including the vertical asymptote, $x = -\frac{1}{2}$ which moves to $x = 2\left(-\frac{1}{2}\right) = -1$.

Finally, we reflect the graph of $y = f\left(\frac{1}{2}x + \frac{1}{2}\right)$ about the y-axis to graph $y = f\left(-\frac{1}{2}x + \frac{1}{2}\right)$. We accomplish this by multiplying each of the x-coordinates of each of the points on the graph of $y = f\left(\frac{1}{2}x + \frac{1}{2}\right)$ by -1. This includes the vertical asymptote which is moved to $x = (-1)(-1) = 1$.

As above, we can track these transformations, in sequence, using a Desmos interactive.

Desmos link: `https://www.desmos.com/calculator/5qhn69ljco`

To check our answer, we begin with the point $(-1, 0)$ and substitute $x = -1$ into $y = f\left(\frac{1-x}{2}\right)$. We get $y = f\left(\frac{1-(-1)}{2}\right) = f\left(\frac{2}{2}\right) = f(1)$. From the graph of f, we know $f(1) = 0$, hence we have $y = f(1) = 0$, proving $(-1, 0)$ is on the graph of $y = f\left(\frac{1-x}{2}\right)$. The reader is encouraged to check the remaining points.

As mentioned previously, instead of doing the horizontal scaling first, then the reflection, we could have done the reflection first, then the scaling. We leave this to the reader to check.

5. To write $g(x)$ in terms of $f(x)$, we assume we can find real numbers a, b, h, and k and choose signs $\pm$ so that $g(x) = \pm af(\pm bx - h) + k$.

   The most notable change we see is the vertical asymptote $x = 0$ has moved to $x = 2$. Moreover, instead of the graph increasing off to the right, it is decreasing coming in from the left. This suggests a horizontal shift of 2 units as well as a reflection across the y-axis.

   Since we always shift first then reflect, we have a shift *left* of 2 units followed by a reflection about the y-axis. In other words, $g(x) = \pm af(-x + 2) + k$.

   Comparing y-values, the y-values on the graph of g appear to be exactly twice the corresponding values on the graph of f, indicating a vertical stretch by a factor of 2. Hence, we get $g(x) = 2f(-x + 2)$. We leave it to the reader to check the graph of $y = 2f(-x + 2)$ matches the graph of $y = g(x)$.

To write $f(x)$ in terms of $g(x)$, we reverse the steps done in obtaining the graph of $g(x)$ from $f(x)$ in the reverse order.

Since to get from the graph of f to the graph of g, we: first, shifted left 2 units; second reflected across the y-axis; third, vertically stretched by a factor of 2, our first step in taking g back to f is to implement a vertical compression by a factor of 2. Hence, starting with the graph of $y = g(x)$, our first step results in the formula $y = \frac{1}{2}g(x)$.

Next, we need to undo the reflection abut the y-axis. If the point (a, b) is reflected about the y-axis, we obtain the point $(-a, b)$. To return to the point (a, b), we reflect $(-a, b)$ across the y-axis again: $(-(-a), b) = (a, b)$. Hence, we take the graph of $y = \frac{1}{2}g(x)$ and reflect it across the y-axis to obtain $y = \frac{1}{2}g(-x)$.

Our last step is to undo a horizontal shift to the left 2 units. The reverse of this process is shifting the graph to the *right* two units, so we get $y = \frac{1}{2}g(-(x-2)) = \frac{1}{2}g(-x+2)$.[14]

We leave it to the reader to start with the graph of $y = g(x)$ and check the graph of $y = \frac{1}{2}g(-x+2)$ matches the graph of $y = f(x)$. ■

4 Transformations in Sequence

Now that we have studied three basic classes of transformations: shifts, reflections, and scalings, we present a result below which provides one algorithm to follow to transform the graph of $y = f(x)$ into the graph of $y = af(bx - h) + k$ without the need of using Theorems **??**, **??**, **??**, **??** and **??** individually.

Theorem **??** is the ultimate generalization of Theorems **??**, **??**, **??**, **??**, **??** and **??**. We note the underlying assumption here is that regardless of the order or number of shifts, reflections and scalings applied to the graph of a function f, we can always represent the final result in the form $g(x) = af(bx - h) + k$. Since each of these transformations can ultimately be traced back to composing f with linear functions,[15] this fact is verified by showing compositions of linear functions results in a linear function.[16]

Theorem 6. Transformations in Sequence. Suppose f is a function. If $a, b \neq 0$, then to graph $g(x) = af(bx - h) + k$ start with the graph of $y = f(x)$ and follow the steps below.

1. Add h to each of the x-coordinates of the points on the graph of f.

   NOTE: This results in a horizontal shift to the left if $h < 0$ or right if $h > 0$.

2. Divide the x-coordinates of the points on the graph obtained in Step 1 by b.

   NOTE: This results in a horizontal scaling, but includes a reflection about the y-axis if $b < 0$.

3. Multiply the y-coordinates of the points on the graph obtained in Step 2 by a.

   NOTE: This results in a vertical scaling, but includes a reflection about the x-axis if $a < 0$.

4. Add k to each of the y-coordinates of the points on the graph obtained in Step 3.

   NOTE: This results in a vertical shift up if $k > 0$ or down if $k < 0$.

[14] To see this better, let $F(x) = \frac{1}{2}g(-x)$. Per Theorem **??**, the graph of $F(x-2) = \frac{1}{2}g(-(x-2)) = \frac{1}{2}g(-x+2)$ is the same as the graph of F but shifted 2 units to the right.

[15] See the remarks at the beginning of the section.

[16] See Exercise **??**.

Theorem **??** can be established by generalizing the techniques developed in this section. Suppose $(c, f(c))$ is on the graph of f. To match up the inputs of $f(bx-h)$ and $f(c)$, we solve $bx - h = c$ and solve.

We first add the h (causing the horizontal shift) and then divide by b. If b is a positive number, this induces only a horizontal scaling by a factor of $\frac{1}{b}$. If $b < 0$, then we have a factor of -1 in play, and dividing by it induces a reflection about the y-axis. So we have $x = \frac{c+h}{b}$ as the input to g which corresponds to the input $x = c$ to f.

We now evaluate $g\left(\frac{c+h}{b}\right) = af\left(b \cdot \frac{c+h}{b} - h\right) + K = af(c+h-h) = af(c) + k$. We notice that the output from f is first multiplied by a. As with the constant b, if $a > 0$, this induces only a vertical scaling. If $a < 0$, then the -1 induces a reflection across the x-axis. Finally, we add k to the result, which is our vertical shift.

A less precise, but more intuitive way to paraphrase Theorem **??** is to think of the quantity $bx - h$ is the 'inside' of the function f. What's happening inside f affects the inputs or x-coordinates of the points on the graph of f. To find the x-coordinates of the corresponding points on g, we undo what has been done to x in the same way we would solve an equation.

What's happening to the output can be thought of as things happening 'outside' the function, f. Things happening outside affect the outputs or y-coordinates of the points on the graph of f. Here, we follow the usual order of operations to simplify the new y-value: we first multiply by a then add k to find the corresponding y-coordinates on the graph of g.

It needs to be stressed that our approach to handling multiple transformations, as summarized in Theorem **??** is only one approach. Your instructor may have a different algorithm. As always, the more you understand, the less you'll ultimately need to memorize, so whatever algorithm you choose to follow, it is worth thinking through each step both algebraically and geometrically.

We make good use of Theorem **??** in the following example.

Example 4. Below is the complete graph of $y = f(x)$. Use Theorem **??** to graph $g(x) = \frac{4 - 3f(1-2x)}{2}$.

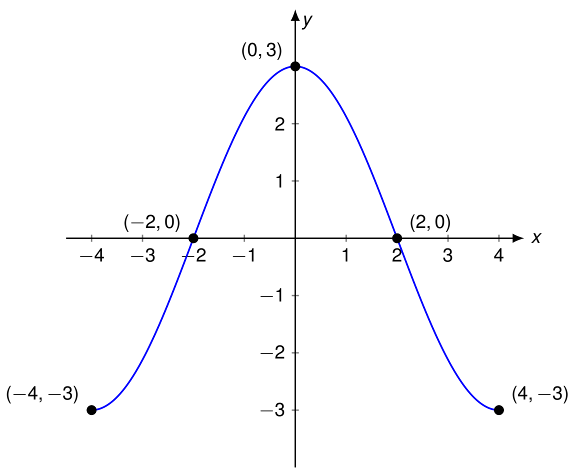

Figure 25: Graph of $y = f(x)$

Alt text: Graph of a smooth curve passing through the points $(-4,-3)$, $(-2,0)$, $(0,3)$, $(2,0)$, and $(4,-3)$. The curve rises from left to right, reaches a maximum at $(0,3)$, and then decreases symmetrically back to $(4,-3)$.

Solution. We use Theorem **??** to track the five 'key points' $(-4,-3)$, $(-2,0)$, $(0,3)$, $(2,0)$ and $(4,-3)$ indicated on the graph of f to their new locations.

We first rewrite $g(x)$ in the form presented in Theorem **??**, $g(x) = -\frac{3}{2}f(-2x+1)+2$. We set $-2x+1$ equal to the x-coordinates of the key points and solve.

For example, solving $-2x+1 = -4$, we first subtract 1 to get $-2x = -5$ then divide by -2 to get $x = \frac{5}{2}$. Subtracting the 1 is a horizontal shift to the left 1 unit. Dividing by -2 can be thought of as a two step process: dividing by 2 which compresses the graph horizontally by a factor of 2 followed by dividing (multiplying) by -1 which causes a reflection across the y-axis. We summarize the results in a table below on the left.

Next, we take each of the x values and substitute them into $g(x) = -\frac{3}{2}f(-2x+1)+2$ to get the corresponding y-values. Substituting $x = \frac{5}{2}$, and using the fact that $f(-4) = -3$, we get

$$g\left(\frac{5}{2}\right) = -\frac{3}{2}f\left(-2\left(\frac{5}{2}\right)+1\right)+2 = -\frac{3}{2}f(-4)+2 = -\frac{3}{2}(-3)+2 = \frac{9}{2}+2 = \frac{13}{2}$$

We see that the output from f is first multiplied by $-\frac{3}{2}$. Thinking of this as a two step process, multiplying by $\frac{3}{2}$ then by -1, we have a vertical stretching by a factor of $\frac{3}{2}$ followed by a reflection across the x-axis. Adding 2 results in a vertical shift up 2 units. Continuing in this manner, we get the table below on the right.

$(c, f(c))$	c	$-2x+1=c$	x
$(-4,-3)$	-4	$-2x+1=-4$	$x=\frac{5}{2}$
$(-2,0)$	-2	$-2x+1=-2$	$x=\frac{3}{2}$
$(0,3)$	0	$-2x+1=0$	$x=\frac{1}{2}$
$(2,0)$	2	$-2x+1=2$	$x=-\frac{1}{2}$
$(4,-3)$	4	$-2x+1=4$	$x=-\frac{3}{2}$

x	$g(x)$	$(x, g(x))$
$\frac{5}{2}$	$\frac{13}{2}$	$\left(\frac{5}{2},\frac{13}{2}\right)$
$\frac{3}{2}$	2	$\left(\frac{3}{2},2\right)$
$\frac{1}{2}$	$-\frac{5}{2}$	$\left(\frac{1}{2},-\frac{5}{2}\right)$
$-\frac{1}{2}$	2	$\left(-\frac{1}{2},2\right)$
$-\frac{3}{2}$	$\frac{13}{2}$	$\left(-\frac{3}{2},\frac{13}{2}\right)$

To graph g, we plot each of the points in the table above and connect them in the same order and fashion as the points to which they correspond.

To check, we can use the GeoGebra interactive below. Starting with graph of $y = f(x)$, we can click on each transformation, in sequence, and track not only the key points, but also see the graph at the end of each step. The graph of $y = g(x)$ is what results after the final step, Step 4.

GeoGebra link: `https://www.geogebra.org/m/vbjxnkwt`

■

The reader is strongly encouraged to graph the series of functions which shows the gradual transformation of the graph of f into the graph of g in Example **??**. We have outlined the sequence of transformations in the above exposition; all that remains is to plot the five intermediate stages. Our next example turns the tables and asks for the formula of a function given a desired sequence of transformations.

Example 5. Let $f(x) = x^2 - |x|$. Find and simplify the formula of the function $g(x)$ whose graph is the result of the graph of $y = f(x)$ undergoing the following sequence of transformations. Check your answer to each step using a graphing utility.

1. Vertical shift up 2 units.
2. Reflection across the x-axis.
3. Horizontal shift right 1 unit.
4. Horizontal compression by a factor of 2.
5. Vertical shift up 3 units.
6. Reflection across the y-axis.

Solution. To help keep us organized we will label each intermediary function. The function g_1 will be the result of applying the first transformation to f. The function g_2 will be the result of applying the first two transformations to f - which is also the result of applying the second transformation to g_1, and so on.[17]

1. Per Theorem **??**, $g_1(x) = f(x) + 2 = x^2 - |x| + 2$.
2. Per Theorem **??**, $g_2(x) = -g_1(x) = -[x^2 - |x| + 2] = -x^2 + |x| - 2$.
3. Per Theorem **??**, $g_3(x) = g_2(x - 1) = -(x - 1)^2 + |x - 1| - 2$.
4. Per Theorem **??**, $g_4(x) = g_3(2x) = -(2x - 1)^2 + |2x - 1| - 2$.
5. Per Theorem **??**, $g_5(x) = g_4(x) + 3 = -(2x - 1)^2 + |2x - 1| - 2 + 3 = -(2x - 1)^2 + |2x - 1| + 1$.
6. Per Theorem **??**, $g_6(x) = g_5(-x)$:

$$\begin{aligned} g_6(x) &= g_5(-x) \\ &= -(2(-x) - 1)^2 + |2(-x) - 1| + 1 \\ &= -(-2x - 1)^2 + |-2x - 1| + 1 \\ &= -[(-1)(2x + 1)]^2 + |[(-1)(2x + 1)| + 1 \\ &= -(-1)^2(2x + 1)^2 + |-1||2x + 1| + 1 \\ &= -(2x + 1)^2 + |2x + 1| + 1 \end{aligned}$$

   In the Desmos interactive below, the graph of $y = f(x)$ is given. By clicking on each of the functions g_1 through g_6, we are able to see each transformation, in sequence, as the graph of $y = f(x)$ is transformed into the graph of $y = g(x)$.

   Desmos link: https://www.desmos.com/calculator/7pyzjopxda

Hence, $g(x) = g_6(x) = -(2x + 1)^2 + |2x + 1| + 1$. ■

It is instructive to show that the expression $g(x)$ in Example **??** can be written as $g(x) = af(bx - h) + k$.

One way is to compare the graphs of f and g and work backwards. A more methodical way is to repeat the work of Example **??**, but never substitute the formula for $f(x)$ as follows:

1. Per Theorem **??**, $g_1(x) = f(x) + 2$.
2. Per Theorem **??**, $g_2(x) = -g_1(x) = -[f(x) + 2] = -f(x) - 2$.
3. Per Theorem **??**, $g_3(x) = g_2(x - 1) = -f(x - 1) - 2$.
4. Per Theorem **??**, $g_4(x) = g_3(2x) = -f(2x - 1) - 2$.
5. Per Theorem **??**, $g_5(x) = g_4(x) + 3 = -f(2x - 1) - 2 + 3 = -f(2x - 1) + 1$.
6. Per Theorem **??**, $g_6(x) = g_5(-x) = -f(2(-x) - 1) + 1 = -f(-2x - 1) + 1$.

[17] So, we can think of $g_0 = f$ and $g_6 = g$.

Hence $g(x) = -f(-2x - 1) + 1$. Note we can show f is even,[18] so $f(-2x - 1) = f(-(2x + 1)) = f(2x + 1)$ and obtain $g(x) = -f(2x + 1) + 1$.

At the beginning of this section, we discussed how all of the transformations we'd be discussing are the result of composing given functions with linear functions. Not all transformations, not even all rigid transformations,[19] fall into these categories.

For example, consider the graphs of $y = f(x)$ and $y = g(x)$ below.

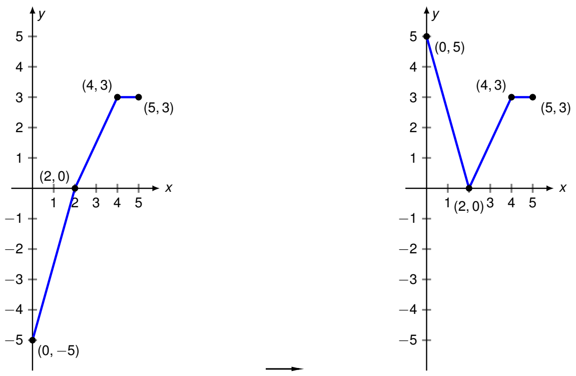

Figure 26: Graphs of $y = f(x)$ and $y = g(x)$

Alt text: The graph on the left is a piecewise linear function connecting the points $(0, -5)$, $(2, 0)$, $(4, 3)$, and $(5, 3)$. The graph rises from left to right and then remains horizontal from $(4, 3)$ to $(5, 3)$.

The graph on the right is a piecewise linear function connecting the points $(0, 5)$, $(2, 0)$, $(4, 3)$, and $(5, 3)$. The graph decreases from $(0, 5)$ to $(2, 0)$, then increases to $(4, 3)$, and remains horizontal from $(4, 3)$ to $(5, 3)$.

In Exercise **??**, we explore a non-linear transformation and revisit the pair of functions f and g then.

[18] Recall this means $f(-x) = f(x)$.

[19] See Section **??**.