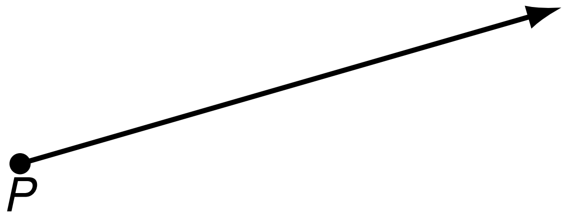
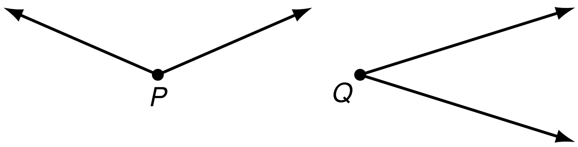
Angles in Degrees

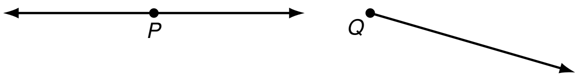
This section serves as a review of the concept of 'angle' and the use of the degree system to measure angles. Recall that a **ray** is usually described as a 'half-line' and can be thought of as a line segment in which one of the two endpoints is pushed off infinitely distant from the other, as pictured below. The point from which the ray originates is called the **initial point** of the ray.



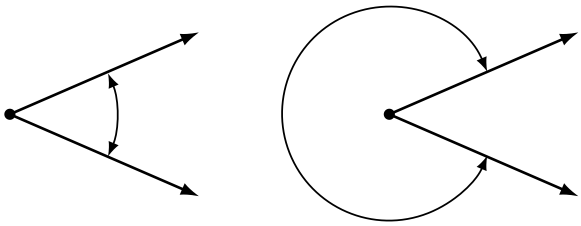
When two rays share a common initial point they form an **angle** and the common initial point is called the **vertex** of the angle. Two examples of what are commonly thought of as angles are



However, the two figures below also depict angles - albeit these are, in some sense, extreme cases. In the first case, the two rays are directly opposite each other forming what is known as a **straight angle**; in the second, the rays are identical so the 'angle' is indistinguishable from the ray itself.

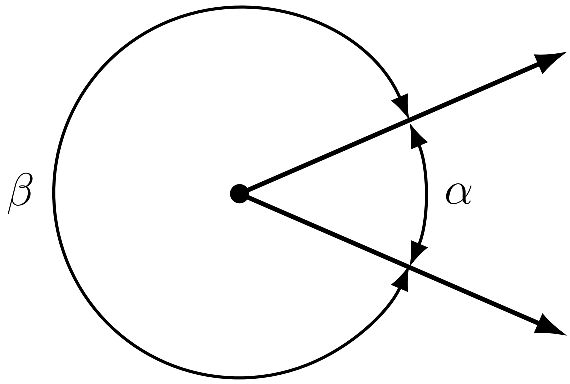


The **measure of an angle** is a number which indicates the amount of rotation that separates the rays of the angle. There is one immediate problem with this, as pictured below.

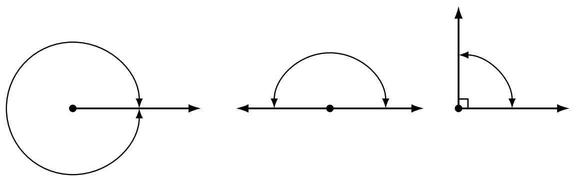


Which amount of rotation are we attempting to quantify? What we have just discovered is that we have at least two angles described by this diagram.[1] Clearly these two angles have different measures because one appears to represent a larger rotation than the other, so we must label them differently. In this book, we use lower case Greek letters such as α (alpha), β (beta), γ (gamma) and θ (theta) to label angles. So, for instance, we have

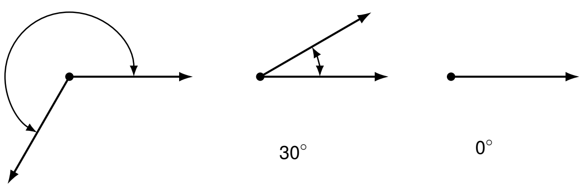
[1] The phrase 'at least' will be justified in short order.



One system to measure angles is **degree measure**. Quantities measured in degrees are denoted by the symbol '°.' One complete revolution as shown below is 360°, and parts of a revolution are measured proportionately.[2] Thus half of a revolution (a straight angle) measures $\frac{1}{2}\left(360^\circ\right) = 180^\circ$, a quarter of a revolution (a **right angle**) measures $\frac{1}{4}\left(360^\circ\right) = 90^\circ$ and so on.



Note that in the above figure, we have used the small square '■' to denote a right angle, as is commonplace in Geometry. Recall that if an angle measures strictly between 0° and 90° it is called an **acute angle** and if it measures strictly between 90° and 180° it is called an **obtuse angle**. It is important to note that, theoretically, we can know the measure of any angle as long as we know the proportion it represents of entire revolution.[3] For instance, the measure of an angle which represents a rotation of $\frac{2}{3}$ of a revolution would measure $\frac{2}{3}\left(360^\circ\right) = 240^\circ$, the measure of an angle which constitutes only $\frac{1}{12}$ of a revolution measures $\frac{1}{12}\left(360^\circ\right) = 30^\circ$ and an angle which indicates no rotation at all is measured as 0°.



Using our definition of degree measure, we have that 1° represents the measure of an angle which constitutes $\frac{1}{360}$ of a revolution. Even though it may be hard to draw, it is nonetheless not difficult to imagine an angle with measure smaller than 1°. There are two ways to subdivide degrees. The first, and most familiar, is **decimal degrees**. For example, an angle with a measure of 30.5° would represent a rotation halfway between 30°

[2]The choice of '360' is most often attributed to the Babylonians.

[3]This is how a protractor is graded.

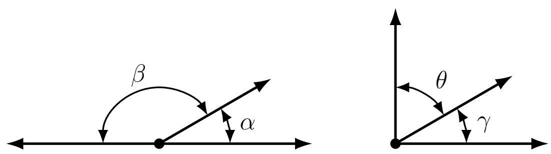
and 31°, or equivalently, $\dfrac{30.5}{360} = \dfrac{61}{720}$ of a full rotation. This can be taken to the limit using Calculus so that measures like $\sqrt{2}^\circ$ make sense.[4] The second way to divide degrees is the **Degree - Minute - Second** (**DMS**) system. In this system, one degree is divided equally into sixty minutes, and in turn, each minute is divided equally into sixty seconds.[5] In symbols, we write $1^\circ = 60'$ and $1' = 60''$, from which it follows that $1^\circ = 3600''$. To convert a measure of 42.125° to the DMS system, we start by noting that $42.125^\circ = 42^\circ + 0.125^\circ$. Converting the partial amount of degrees to minutes, we find $0.125^\circ \left(\dfrac{60'}{1^\circ}\right) = 7.5' = 7' + 0.5'$. Converting the partial amount of minutes to seconds gives $0.5'\left(\dfrac{60''}{1'}\right) = 30''$. Putting it all together yields

$$\begin{aligned} 42.125^\circ &= 42^\circ + 0.125^\circ \\ &= 42^\circ + 7.5' \\ &= 42^\circ + 7' + 0.5' \\ &= 42^\circ + 7' + 30'' \\ &= 42^\circ 7' 30'' \end{aligned}$$

On the other hand, to convert $117^\circ 15' 45''$ to decimal degrees, we first compute $15'\left(\dfrac{1^\circ}{60'}\right) = \dfrac{1}{4}^\circ$ and $45''\left(\dfrac{1^\circ}{3600''}\right) = \dfrac{1}{80}^\circ$. Then we find

$$\begin{aligned} 117^\circ 15' 45'' &= 117^\circ + 15' + 45'' \\ &= 117^\circ + \frac{1}{4}^\circ + \frac{1}{80}^\circ \\ &= \frac{9381}{80}^\circ \\ &= 117.2625^\circ \end{aligned}$$

Recall that two acute angles are called **complementary angles** if their measures add to 90°. Two angles, either a pair of right angles or one acute angle and one obtuse angle, are called **supplementary angles** if their measures add to 180°. In the diagram below, the angles α and β are supplementary angles while the pair γ and θ are complementary angles.



Supplementary Angles   Complementary Angles

In practice, the distinction between the angle itself and its measure is blurred so that the sentence 'α is an angle measuring 42°' is often abbreviated as '$\alpha = 42^\circ$.' It is now time for an example.

Example 1. Let $\alpha = 111.371^\circ$ and $\beta = 37^\circ 28' 17''$.

1. Convert α to the DMS system. Round your answer to the nearest second.
2. Convert β to decimal degrees. Round your answer to the nearest thousandth of a degree.
3. Sketch α and β.

[4] Awesome math pun aside, this is the same idea behind defining irrational exponents in Section **??**.
[5] Does this kind of system seem familiar?

4. Find a supplementary angle for α.
5. Find a complementary angle for β.

Solution.

1. To convert α to the DMS system, we start with $111.371^\circ = 111^\circ + 0.371^\circ$. Next we convert $0.371^\circ\left(\frac{60'}{1^\circ}\right) = 22.26'$. Writing $22.26' = 22' + 0.26'$, we convert $0.26'\left(\frac{60''}{1'}\right) = 15.6''$. Hence,

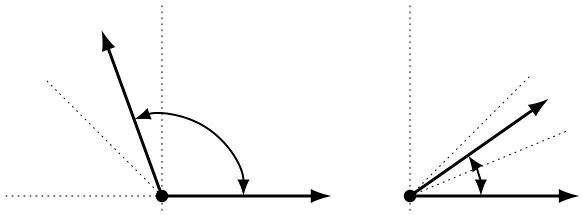
$$\begin{aligned} 111.371^\circ &= 111^\circ + 0.371^\circ \\ &= 111^\circ + 22.26' \\ &= 111^\circ + 22' + 0.26' \\ &= 111^\circ + 22' + 15.6'' \\ &= 111^\circ 22' 15.6'' \end{aligned}$$

Rounding to seconds, we obtain $\alpha \approx 111^\circ 22' 16''$.

2. To convert β to decimal degrees, we convert $28'\left(\frac{1^\circ}{60'}\right) = \left(\frac{7}{15}\right)^\circ$ and $17''\left(\frac{1^\circ}{3600'}\right) = \left(\frac{17}{3600}\right)^\circ$. Putting it all together, we have

$$\begin{aligned} 37^\circ 28' 17'' &= 37^\circ + 28' + 17'' \\ &= 37^\circ + \left(\frac{7}{15}\right)^\circ + \left(\frac{17}{3600}\right)^\circ \\ &= \left(\frac{134897}{3600}\right)^\circ \\ &\approx 37.471^\circ \end{aligned}$$

3. To sketch α, we first note that $90^\circ < \alpha < 180^\circ$. Dividing this range in half, we get $90^\circ < \alpha < 135^\circ$, and once more, we have $90^\circ < \alpha < 112.5^\circ$. This gives us a pretty good estimate for α, as shown below.[6] Proceeding similarly for β, we find $0^\circ < \beta < 90^\circ$, then $0^\circ < \beta < 45^\circ$, $22.5^\circ < \beta < 45^\circ$, and lastly, $33.75^\circ < \beta < 45^\circ$.



Angle α Angle β

4. To find a supplementary angle for α, we seek an angle θ so that $\alpha + \theta = 180^\circ$. We get $\theta = 180^\circ - \alpha = 180^\circ - 111.371^\circ = 68.629^\circ$.

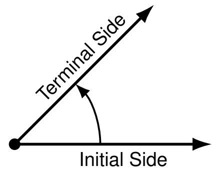
5. To find a complementary angle for β, we seek an angle γ so that $\beta + \gamma = 90^\circ$. We get $\gamma = 90^\circ - \beta = 90^\circ - 37^\circ 28' 17''$. While we could reach for the calculator to obtain an approximate answer, we choose instead to do a bit of sexagesimal[7] arithmetic. We first rewrite $90^\circ = 90^\circ 0' 0'' = 89^\circ 60' 0'' = 89^\circ 59' 60''$. In essence, we are 'borrowing' $1^\circ = 60'$ from the degree place, and then borrowing $1' = 60''$ from the minutes place.[8] This yields, $\gamma = 90^\circ - 37^\circ 28' 17'' = 89^\circ 59' 60'' - 37^\circ 28' 17'' = 52^\circ 31' 43''$. ■

[6] If this process seems hauntingly familiar, it should. Compare this method to the Bisection Method introduced in Section **??**.

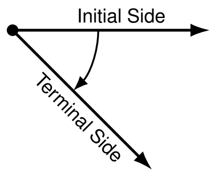
[7] Like 'latus rectum,' this is also a real math term.

[8] This is the exact same kind of 'borrowing' you used to do in Elementary School when trying to find $300 - 125$. Back then, you were working in a base ten system; here, it is base sixty.

Up to this point, we have discussed only angles which measure between 0° and 360°, inclusive. Ultimately, we want to use the arsenal of Algebra which we have stockpiled in Chapters **??** through **??** to not only solve geometric problems involving angles, but also to extend their applicability to other real-world phenomena. A first step in this direction is to extend our notion of 'angle' from merely measuring an extent of rotation to quantities which indicate an amount of rotation along with a **direction**. To that end, we introduce the concept of an **oriented angle**. As its name suggests, in an oriented angle, the direction of the rotation is important. We imagine the angle being swept out starting from an **initial side** and ending at a **terminal side**, as shown below. When the rotation is counter-clockwise[9] from initial side to terminal side, we say that the angle is **positive**; when the rotation is clockwise, we say that the angle is **negative**.

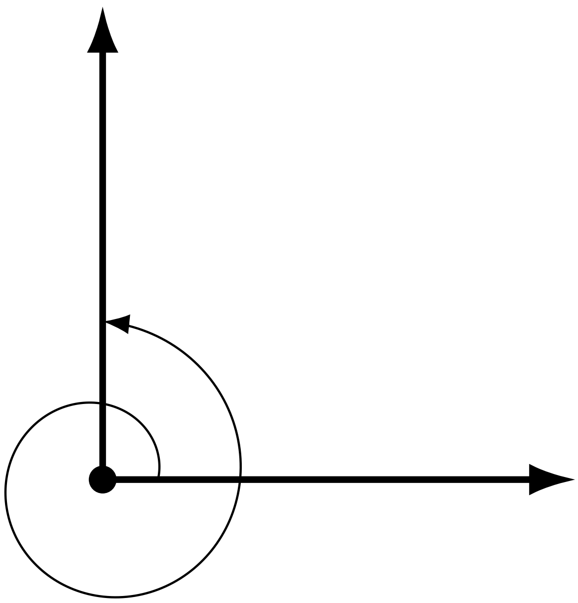


A positive angle, 45°



A negative angle, -45°

At this point, we also extend our allowable rotations to include angles which encompass more than one revolution. For example, to sketch an angle with measure 450° we start with an initial side, rotate counter-clockwise one complete revolution (to take care of the 'first' 360°) then continue with an additional 90° counter-clockwise rotation, as seen below.

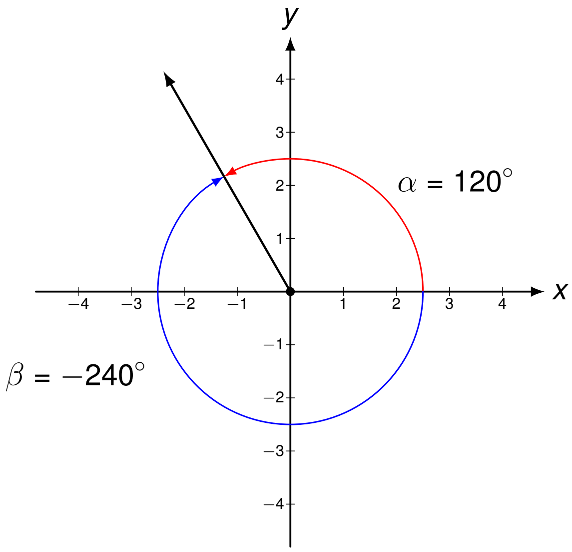


450°

To further connect angles with the Algebra which has come before, we shall often overlay an angle diagram on the coordinate plane. An angle is said to be in **standard position** if its vertex is the origin and its initial side coincides with the positive horizontal (usually labeled as the x-) axis. Angles in standard position are classified according to where their terminal side lies. For instance, an angle in standard position whose terminal side

[9]'widdershins'

lies in Quadrant I is called a 'Quadrant I angle'. If the terminal side of an angle lies on one of the coordinate axes, it is called a **quadrantal angle**. Two angles in standard position are called **coterminal** if they share the same terminal side.[10] In the figure below, $\alpha = 120^\circ$ and $\beta = -240^\circ$ are two coterminal Quadrant II angles drawn in standard position. Note that $\alpha = \beta + 360^\circ$, or equivalently, $\beta = \alpha - 360^\circ$. We leave it as an exercise to the reader to verify that coterminal angles always differ by a multiple of 360°.[11] More precisely, if α and β are coterminal angles, then $\beta = \alpha + 360^\circ \cdot k$ where k is an integer.[12]



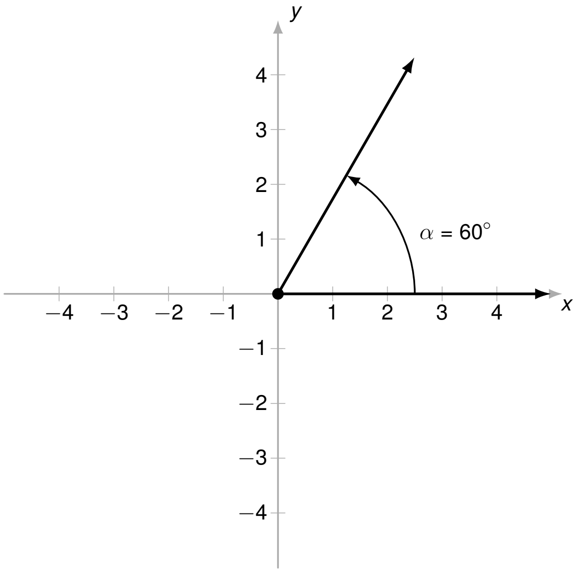
Example 2. Graph each of the (oriented) angles below in standard position and classify them according to where their terminal side lies. Find three coterminal angles, at least one of which is positive and one of which is negative.

1. $\alpha = 60^\circ$
2. $\beta = -225^\circ$
3. $\gamma = 540^\circ$
4. $\phi = -750^\circ$

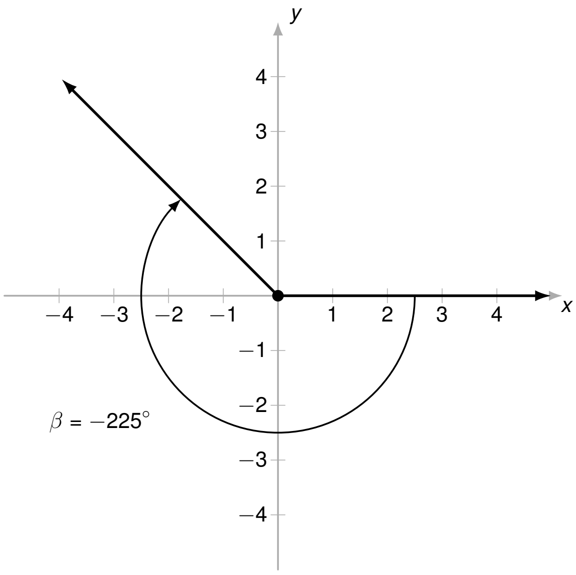
Solution.

1. To graph $\alpha = 60^\circ$, we draw an angle with its initial side on the positive x-axis and rotate counter-clockwise $\dfrac{60^\circ}{360^\circ} = \dfrac{1}{6}$ of a revolution. We see that α is a Quadrant I angle. To find angles which are coterminal, we look for angles θ of the form $\theta = \alpha + 360^\circ \cdot k$, for some integer k. When $k = 1$, we get $\theta = 60^\circ + 360^\circ = 420^\circ$. Substituting $k = -1$ gives $\theta = 60^\circ - 360^\circ = -300^\circ$. Finally, if we let $k = 2$, we get $\theta = 60^\circ + 720^\circ = 780^\circ$.

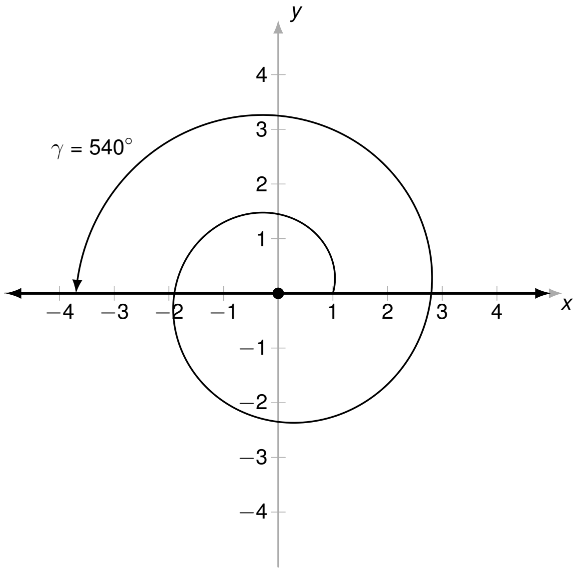
[10] Note that by being in standard position they automatically share the same initial side which is the positive x-axis.
[11] It is worth noting that all of the pathologies of Analytic Trigonometry result from this innocuous fact.
[12] Recall that this means $k = 0, \pm 1, \pm 2, \ldots$.



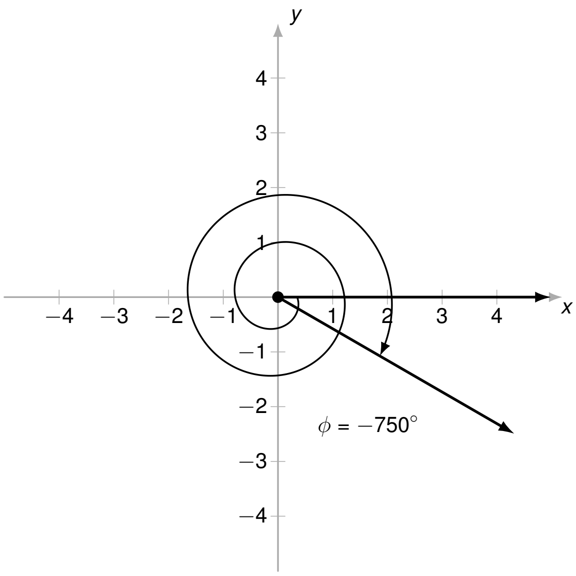
2. Since $\beta = -225^\circ$ is negative, we start at the positive x-axis and rotate *clockwise* $\dfrac{225^\circ}{360^\circ} = \dfrac{5}{8}$ of a revolution. We see that β is a Quadrant II angle. To find coterminal angles, we proceed as before and compute $\theta = -225^\circ + 360^\circ \cdot k$ for integer values of k. We find 135°, -585° and 495° are all coterminal with -225°.



3. Since $\gamma = 540^\circ$ is positive, we rotate counter-clockwise from the positive x-axis. One full revolution accounts for 360°, with 180°, or $\dfrac{1}{2}$ of a revolution remaining. Since the terminal side of γ lies on the negative x-axis, γ is a quadrantal angle. All angles coterminal with γ are of the form $\theta = 540^\circ + 360^\circ \cdot k$, where k is an integer. Working through the arithmetic, we find three such angles: 180°, -180° and 900°.



4. The Greek letter ϕ is pronounced 'fee' or 'fie' and since ϕ is negative, we begin our rotation clockwise from the positive x-axis. Two full revolutions account for 720°, with just 30° or $\frac{1}{12}$ of a revolution to go. We find that ϕ is a Quadrant IV angle. To find coterminal angles, we compute $\theta = -750^\circ + 360^\circ \cdot k$ for a few integers k and obtain -390°, -30° and 330°.



Note that since there are infinitely many integers, any given angle has infinitely many coterminal angles, and the reader is encouraged to plot the few sets of coterminal angles found in Example 2 to see this.

As we'll see in Section **??** and throughout Chapter **??**, degree measure is very popular for many applications involving geometry and modeling physical forces. In Section **??**, we'll introduce a different method of measuring angles, **radian measure**, which is tied directly to arc length and is useful in other applications involving circular motion and periodic phenomenon.