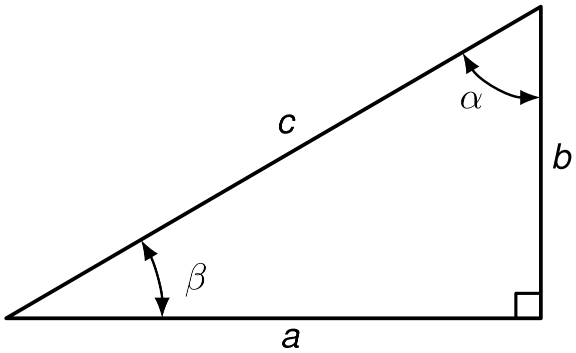
Right Triangle Trigonometry

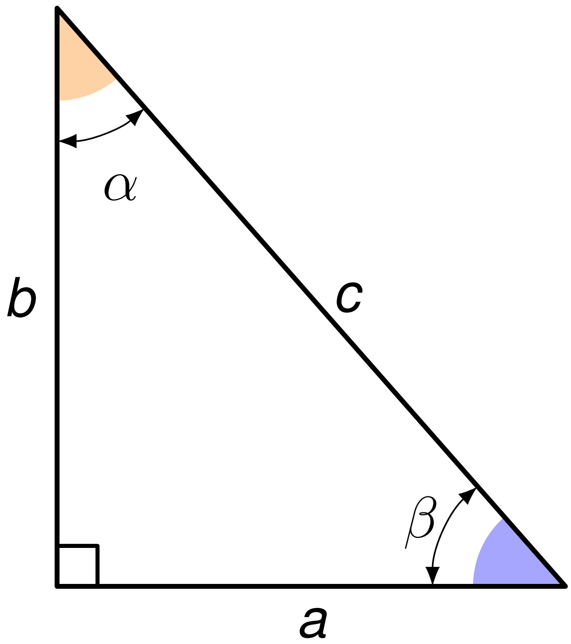
The word 'trigonometry' literally means 'measuring triangles,' so naturally most students' first introduction to trigonometry focuses on triangles. This section focuses on **right triangles**, triangles in which one angle measures 90°. Consider the right triangle below, where, as usual, a small square denotes the right angle, the labels '*a*,' '*b*,' and '*c*' denote the lengths of the sides of the triangle, and α and β represent the (measure of) the non-right angles. As you may recall, the side opposite the right angle is called the **hypotenuse** of the right triangle. Also note that since the sum of the measures of all angles in a triangle must add to 180°, we have $\alpha + \beta + 90^\circ = 180^\circ$, or $\alpha + \beta = 90^\circ$. Said differently, the non-right angles in a right triangle are *complements*.



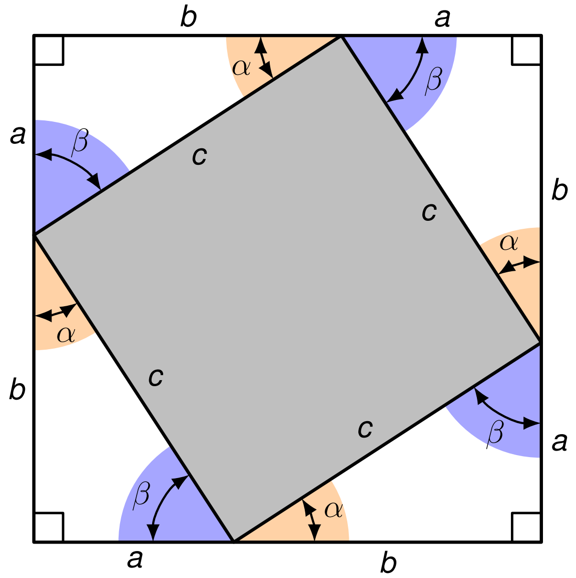
We now state and prove the most famous result about right triangles: **The Pythagorean Theorem**.

Theorem 1. (**The Pythagorean Theorem**) The square of the length of the hypotenuse of a right triangle is equal to the sums of the squares of the other two sides. More specifically, if c is the length of the hypotenuse of a right triangle and a and b are the lengths of the other two sides, then $a^2 + b^2 = c^2$.

There are several proofs of the Pythagorean Theorem,[1] but the one we choose to reproduce here showcases a nice interplay between algebra and geometry. Consider taking four copies of the right triangle below on the left and arranging them as seen below on the right.



[1] Including one by Mentor, Ohio native President James Garfield.



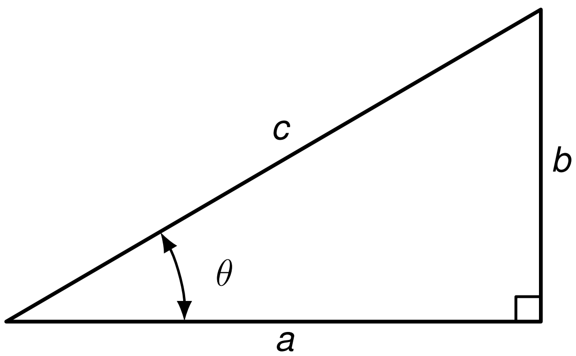
It should be clear that we have produced a large square with a side length of $(a+b)$. What is also true, but may not be obvious, is that the shaded quadrilateral is also a square. We can readily see the shaded quadrilateral has equal sides of length c. Moreover, since $\alpha+\beta = 90^\circ$, we get the interior angles of the shaded quadrilateral are each 90°. Hence, the shaded quadrilateral is indeed a square.

We finish the proof by computing the area of the of the large square in two ways. First, we square the length of its side: $(a+b)^2$. Next, we add up the areas of the four triangles, each having area $\frac{1}{2}ab$ along with the area of the shaded square, c^2. Equating these to expressions gives: $(a+b)^2 = 4\left(\frac{1}{2}ab\right) + c^2$. Since $(a+b)^2 = a^2 + 2ab + b^2$ and $4\left(\frac{1}{2}ab\right) = 2ab$, we have $a^2 + 2ab + b^2 = 2ab + c^2$ or $a^2 + b^2 = c^2$, as required.

It should be noted that the converse of the Pythagorean Theorem is also true. That is if a, b, and c are the lengths of sides of a triangle and $a^2 + b^2 = c^2$, then c the triangle is a right triangle.[2]

A list of integers (a, b, c) which satisfy the relationship $a^2 + b^2 = c^2$ is called a **Pythagorean Triple**. Some of the more common triples are: $(3, 4, 5)$, $(5, 12, 13)$, $(7, 24, 25)$, and $(8, 15, 17)$. We leave it to the reader to verify these integers satisfy the equation $a^2 + b^2 = c^2$ and suggest committing these triples to memory.

Next, we set about defining characteristic ratios associated with acute angles. Given any acute angle θ, we can imagine θ being an interior angle of a right triangle as seen below.



Focusing on the arrangement of the sides of the triangle with respect to the angle θ, we make the following

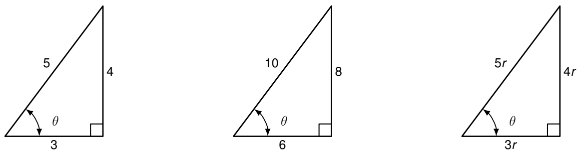
[2] We will prove this in Section **??** by generalizing the Pythagorean Theorem to a formula that works for *all* triangles.

definitions: the side with length a is called the side of the triangle which is **adjacent** to θ and the side with length b is called the side of the triangle **opposite** θ. As usual, the side labeled 'c' (the side opposite the right angle) is the hypotenuse. Using this diagram, we define three important **trigonometric ratios** of θ.

Definition 1. Suppose θ is an acute angle residing in a right triangle as depicted above.

- The **sine** of θ, denoted $\sin(\theta)$ is defined by the ratio: $\sin(\theta) = \dfrac{b}{c}$, or $\dfrac{\text{'length of opposite'}}{\text{'length of hypotenuse'}}$.
- The **cosine** of θ, denoted $\cos(\theta)$ is defined by the ratio: $\cos(\theta) = \dfrac{a}{c}$, or $\dfrac{\text{'length of adjacent'}}{\text{'length of hypotenuse'}}$.
- The **tangent** of θ, denoted $\tan(\theta)$ is defined by the ratio: $\tan(\theta) = \dfrac{b}{a}$, or $\dfrac{\text{'length of opposite'}}{\text{'length of adjacent'}}$.

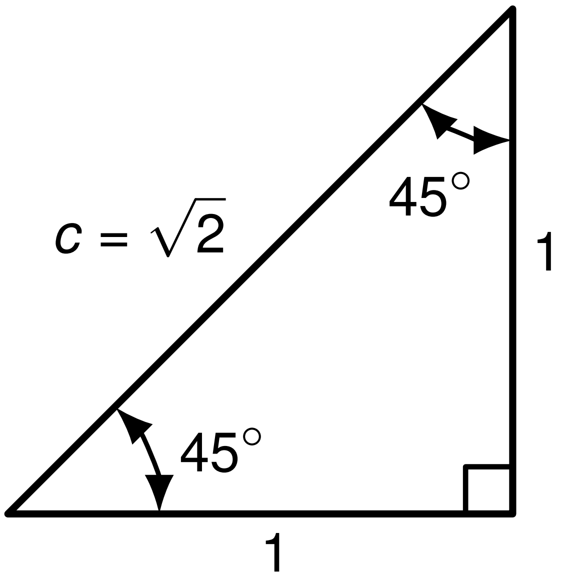
For example, consider the angle θ indicated in the triangle below on the left. Using Definition 1, we get $\sin(\theta) = \dfrac{4}{5}$, $\cos(\theta) = \dfrac{3}{5}$, and $\tan(\theta) = \dfrac{4}{3}$. One may well wonder if these trigonometric ratios we've found for θ change if the triangle containing θ changes. For example, if we scale all the sides of the triangle below on the left by a factor of 2, we produce the **similar triangle** below in the middle.[3] Using this triangle to compute our ratios for θ, we find $\sin(\theta) = \dfrac{8}{10} = \dfrac{4}{5}$, $\cos(\theta) = \dfrac{6}{10} = \dfrac{3}{5}$, and $\tan(\theta) = \dfrac{8}{6} = \dfrac{4}{3}$. Note that the scaling factor, here 2, is common to all sides of the triangle, and, hence, cancels from the numerator and denominator when simplifying each of the ratios.



In general, thanks to the Angle Angle Similarity Postulate, any two *right* triangles which contain our angle θ are similar which means there is a positive constant r so that the sides of the triangle are $3r$, $4r$, and $5r$ as seen above on the right. Hence, regardless of the right triangle in which we choose to imagine θ, $\sin(\theta) = \dfrac{4r}{5r} = \dfrac{4}{5}$, $\cos(\theta) = \dfrac{3r}{5r} = \dfrac{3}{5}$, and $\tan(\theta) = \dfrac{4r}{3r} = \dfrac{4}{3}$. Generalizing this same argument to any acute angle θ assures us that the ratios as described in Definition 1 are independent of the triangle we use.

Our next objective is to determine the values of $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$ for some of the more commonly used angles. We begin with 45°. In a right triangle, if one of the non-right angles measures 45°, then the other measures 45° as well. It follows that the two legs of the triangle must be congruent. Since we may choose any right triangle containing a 45° angle for our computations, we choose the length of one (hence both) of the legs to be 1. The Pythagorean Theorem gives the hypotenuse is: $c^2 = 1^2 + 1^2 = 2$, so $c = \sqrt{2}$. (We take only the positive square root here since c represents the length of the hypotenuse here, so, necessarily $c > 0$.) From this, we obtain the values below, and suggest committing them to memory.

[3] That is, a triangle with the same 'shape' - that is, the same angles.

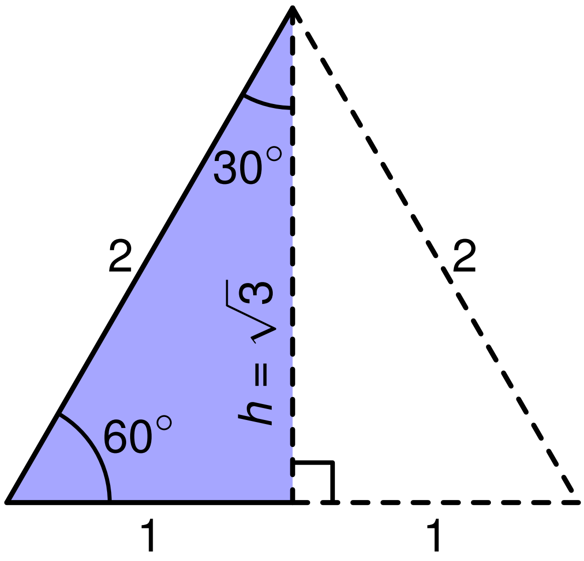


Trigonometric ratios for 45°:

- $\sin\left(45^\circ\right) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
- $\cos\left(45^\circ\right) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
- $\tan\left(45^\circ\right) = \frac{1}{1} = 1$

Note that we have 'rationalized' here to avoid the irrational number $\sqrt{2}$ appearing in the denominator. This is a common convention in trigonometry, and we will adhere to it unless extremely inconvenient.

Next, we investigate 60° and 30° angles. Consider the equilateral triangle below each of whose sides measures 2 units. Each of its interior angles is necessarily 60°, so if we drop an altitude, we produce two $30^\circ - 60^\circ - 90^\circ$ triangles each having a base measuring 1 unit and a hypotenuse of 2 units. Using the Pythagorean Theorem, we can find the height, h of these triangles: $1^2 + h^2 = 2^2$ so $h^2 = 3$ or $h = \sqrt{3}$. Using these, we can find the values of the trigonometric ratios for both 60° and 30°. Again, we recommend committing these values to memory.



Trigonometric ratios for 60°:

- $\sin\left(60^\circ\right) = \dfrac{\sqrt{3}}{2}$
- $\cos\left(60^\circ\right) = \dfrac{1}{2}$
- $\tan\left(60^\circ\right) = \dfrac{\sqrt{3}}{1} = \sqrt{3}$

Trigonometric ratios for 30°:

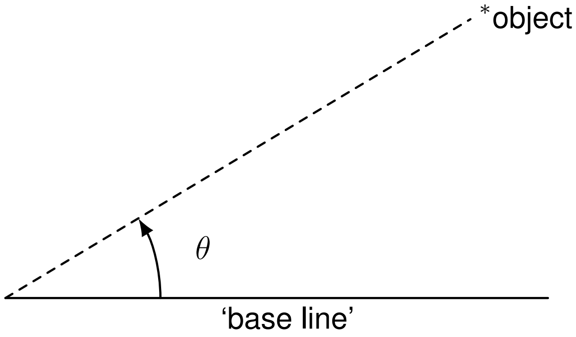
- $\sin\left(30^\circ\right) = \dfrac{1}{2}$
- $\cos\left(30^\circ\right) = \dfrac{\sqrt{3}}{2}$
- $\tan\left(30^\circ\right) = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

Note that since 30° and 60° are complements, the side *adjacent* to the 60° angle is the side *opposite* the 30° and the side *opposite* the 60° angle is the side *adjacent* to the 30° . Hence, the 'co'-sine of 30° is the sine of its 'co'mplement, 60° and vice-versa. This sort of 'swapping' is true of all complementary angles and will be generalized in Section **??**, Theorem **??**.

Note that the values of the trigonometric ratios we have derived for 30°, 45°, and 60° angles are the *exact* values of these ratios. For these angles, we can conveniently express the exact values of their sines, cosines, and tangents resorting, at worst, to using square roots. The reader may well wonder if, for instance, we can express the exact value of, say, $\sin\left(42^\circ\right)$ in terms of radicals. The answer in this case is 'yes' (see here), but, in general, we will not take the time to pursue such representations.[4] Hence, if a problem requests an 'exact' answer involving $\sin\left(42^\circ\right)$, we will leave it written as '$\sin\left(42^\circ\right)$' and use a calculator to produce a suitable approximation as the situation warrants.

Our first example requires the concept of an 'angle of inclination.' The angle of inclination (or angle of elevation) of an object refers to the angle whose initial side is some kind of base-line (say, the ground), and whose terminal side is the line-of-sight to an object above the base-line. Schematically:

[4] We will do a little of this in Section **??**.

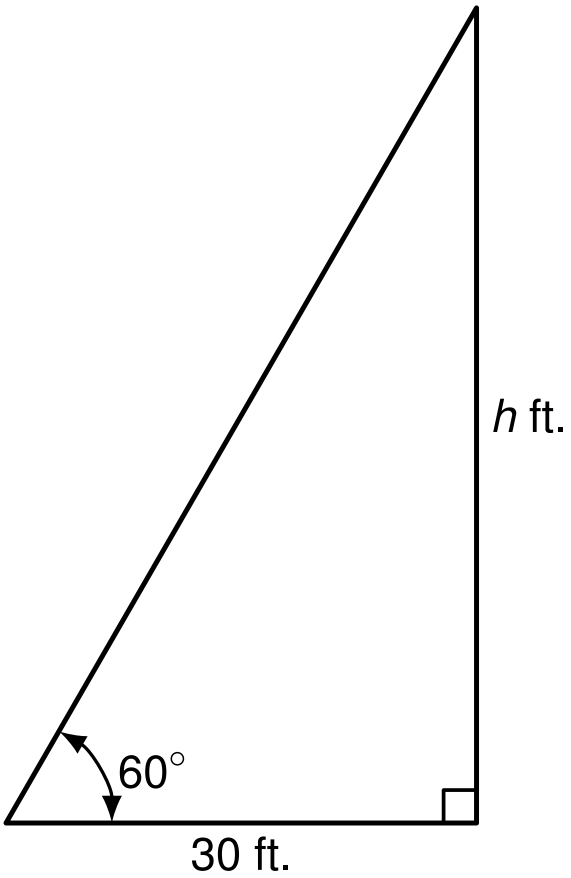


Example 1.

1. The angle of inclination from a point on the ground 30 feet away to the top of Lakeland's Armington Clocktower[5] is 60°. Find the height of the Clocktower to the nearest foot.
2. The Americans with Disabilities Act (ADA) stipulates the incline on an accessibility ramp be 5°. If a ramp is to be built so that it replaces stairs that measure 21 inches tall, how long does the ramp need to be? Round your answer to the nearest inch.
3. In order to determine the height of a California Redwood tree, two sightings from the ground, one 200 feet directly behind the other, are made. If the angles of inclination were 45° and 30°, respectively, how tall is the tree to the nearest foot?

Explanation. 1. We can represent the problem situation using a right triangle as shown below. If we let h denote the height of the tower, then we have $\tan\left(60^\circ\right) = \dfrac{h}{30}$.

[5]Named in honor of Raymond Q. Armington, Lakeland's Clocktower has been a part of campus since 1972.

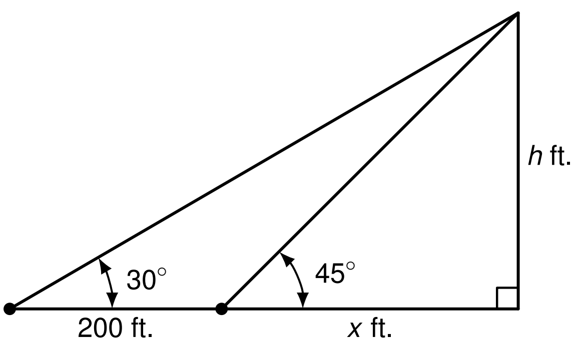


From this we get an exact answer of $h = 30\tan\left(60^\circ\right) = 30\sqrt{3}$ feet. Using a calculator, we get the approximation 51.96 which, when rounded to the nearest foot, gives us our answer of 52 feet.

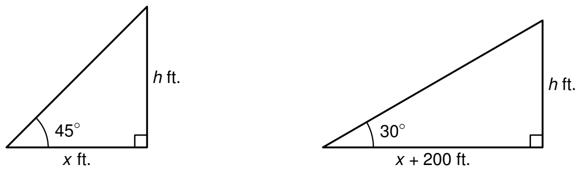
2. We diagram the situation below using ℓ to represent the unknown length of the ramp.

We have $\sin\left(5^\circ\right) = \dfrac{21}{\ell}$ so that $\ell = \dfrac{21}{\sin\left(5^\circ\right)} \approx 240.95$ inches. Hence, the ramp is 241 inches long.

3. Sketching the problem situation below, we find ourselves with two unknowns: the height h of the tree and the distance x from the base of the tree to the first observation point.



Luckily, we have two right triangles to help us find each unknown, as shown below. From the triangle below on the left, we get $\tan\left(45^\circ\right) = \dfrac{h}{x}$. From the triangle below on the right, we see $\tan\left(30^\circ\right) = \dfrac{h}{x+200}$.



Since $\tan\left(45^\circ\right) = 1$, the first equation gives $\dfrac{h}{x} = 1$, or $x = h$. Substituting this into the second equation gives $\dfrac{h}{h+200} = \tan\left(30^\circ\right) = \dfrac{\sqrt{3}}{3}$. Clearing fractions, we get $3h = (h+200)\sqrt{3}$. The result is a linear equation for h, so we expand the right hand side and gather all the terms involving h to one side.

$$\begin{aligned} 3h &= (h+200)\sqrt{3} \\ 3h &= h\sqrt{3} + 200\sqrt{3} \\ 3h - h\sqrt{3} &= 200\sqrt{3} \\ (3-\sqrt{3})h &= 200\sqrt{3} \\ h &= \frac{200\sqrt{3}}{3-\sqrt{3}} \approx 273.20 \end{aligned}$$

Hence, the tree is approximately 273 feet tall.

There are three more trigonometric ratios which are commonly used and they are defined in the same manner the ratios in Definition 1 are defined. They are listed below.

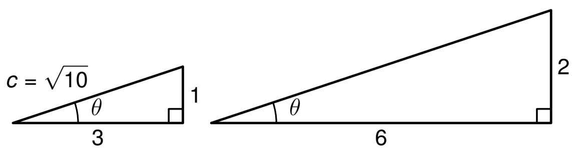
Definition 2. Suppose θ is an acute angle residing in a right triangle as depicted on page 2.

- The **cosecant** of θ, denoted $\csc(\theta)$ is defined by the ratio: $\csc(\theta) = \dfrac{c}{b}$, or $\dfrac{\text{‘length of hypotenuse’}}{\text{‘length of opposite’}}$.
- The **secant** of θ, denoted $\sec(\theta)$ is defined by the ratio: $\sec(\theta) = \dfrac{c}{a}$, or $\dfrac{\text{‘length of hypotenuse’}}{\text{‘length of adjacent’}}$.
- The **cotangent** of θ, denoted $\cot(\theta)$ is defined by the ratio: $\cot(\theta) = \dfrac{a}{b}$, or $\dfrac{\text{‘length of adjacent’}}{\text{‘length of opposite’}}$.

We practice these definitions in the following example.

Example 2. Suppose θ is an acute angle with $\cot(\theta) = 3$. Find the values of the remaining five trigonometric ratios: $\sin(\theta)$, $\cos(\theta)$, $\tan(\theta)$, $\csc(\theta)$, and $\sec(\theta)$.

Explanation. We are given $\cot(\theta) = 3$. So, to proceed, we construct a right triangle in which the length of the side adjacent to θ and the length of the side opposite of θ has a ratio of $3 = \dfrac{3}{1}$. Note there are infinitely many such right triangles - we have produced two below for reference. We will focus our attention on the triangle below on the left and encourage the reader to work through the details using the triangle below on the right to verify the choice of triangle doesn't matter.



From the diagram, we see immediately $\tan(\theta) = \frac{1}{3}$, but in order to determine the remaining four trigonometric ratios, we need to first find the value of the hypotenuse. The Pythagorean Theorem gives $1^2 + 3^2 = c^2$ so $c^2 = 10$ or $c = \sqrt{10}$. Rationalizing denominators, we find $\sin(\theta) = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$, $\cos(\theta) = \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10}$, $\csc(\theta) = \frac{\sqrt{10}}{1} = \sqrt{10}$ and $\sec(\theta) = \frac{\sqrt{10}}{3}$.

While we learned all about the trigonometric ratios of θ in Example 2, the identity of θ remains unknown. Since $\sin(\theta) = \frac{\sqrt{10}}{10} \approx 0.316$ is decidedly less than $\sin\left(30^\circ\right) = \frac{1}{2} = 0.5$, it stands to reason that $\theta < 30^\circ$. It turns out the calculator can provide for us a decimal approximation of θ by way of the '$\sin^{-1}(x)$' function. Here, the '-1' exponent denotes an inverse function (see Section **??**) does **not** mean reciprocal.[6] That is, $\sin^{-1}(x)$ (read 'sine-inverse of x') gives an angle whose sine is x. Hence, we may write $\theta = \sin^{-1}\left(\frac{\sqrt{10}}{10}\right) \approx 18.43^\circ$. The functions $\cos^{-1}(x)$ and $\tan^{-1}(x)$ work similarly. Indeed,

$$\theta = \sin^{-1}\left(\frac{\sqrt{10}}{10}\right) = \cos^{-1}\left(\frac{3\sqrt{10}}{10}\right) = \tan^{-1}\left(\frac{1}{3}\right),$$

and the reader is encouraged to use a calculator to verify these statements.

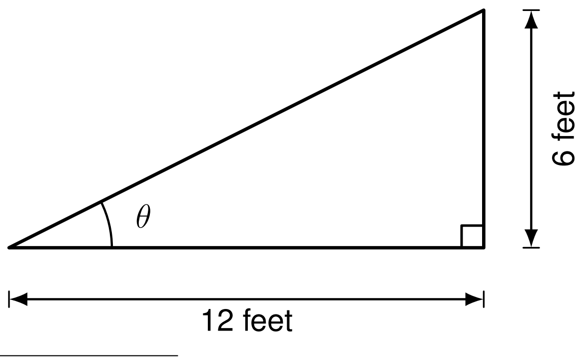
Please note there is **much** more to these inverse functions than the 'angle finder' description use here.[7] That being said, we finish this section showcasing a use for the $\tan^{-1}(x)$ function below.

Example 3. [8] The roof on the house below has a '6/12 pitch'. This means that when viewed from the side, the roof line has a rise of 6 feet over a run of 12 feet. Find the angle of inclination from the bottom of the roof to the top of the roof. Round your answer to the nearest hundredth of a degree.

Front View

Side View

Explanation. If we divide the side view of the house down the middle, we find that the roof line forms the hypotenuse of a right triangle with legs of length 6 feet and 12 feet as depicted below.



[6] That is, $\sin^{-1}(x) \neq \frac{1}{\sin(x)}$. That being said, $(\sin(x))^{-1} = \frac{1}{\sin(x)} = \csc(x)$.

[7] See Section **??** for all of the pedantic details.

[8] The authors would like to thank Dan Stitz for this problem and associated graphics.

The angle of inclination, θ, satisfies $\tan(\theta) = \frac{6}{12} = \frac{1}{2}$. Hence, $\theta = \tan^{-1}\left(\frac{1}{2}\right) \approx 26.56^\circ$.