Constant and Linear Functions

Now that we have defined the concept of a function, we'll spend the rest of Chapter **??** revisiting families of curves from prior courses in Algebra by viewing them through a 'function lens'. We start with lines and refer the reader to Section **??** for a review of the basic properties of lines. The simplest lines are vertical and horizontal lines. We leave it to the reader (see Exercise **??**) to think about why we eschew vertical lines in our discussion here, and begin with a functional description of horizontal lines.

Consider the horizontal lines graphed in the xy-plane as shown below. The Vertical Line Test, Theorem **??**, tells us that each describes y as a function of x so the question becomes how to represent these functions algebraically. The key here is to remember that the equation relating the independent variable x, the dependent variable y, and the function f is given by $y = f(x)$.

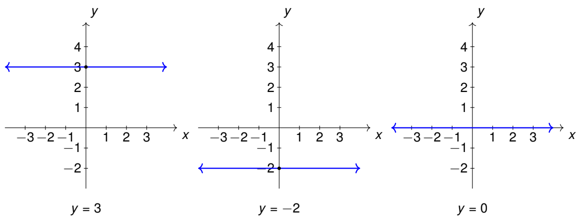


Figure 1: Graphs of constant functions

Alt text: Three examples of horizontal lines

In the graph on the left, y always equals 3 so we have $f(x) = 3$. Procedurally, '$f(x) = 3$' says that the rule f takes the input x, and, regardless of that input, gives the output 3. This is an example of what is called a **constant** function - a function which returns the **same** value regardless of the input. Likewise, the function represented by the graph in the middle is $f(x) = -2$, and the graph on the right (the x-axis) is the graph of $f(x) = 0$. In general, we have the following definition:

Definition 1. A **constant function** is a function of the form

$$f(x) = b$$

where b is real number. The domain of a constant function is $(-\infty, \infty)$.

Some remarks about Definition **??** are in order. First, note that we are using 'x' as the independent variable, 'f' as the function name, and the letter 'b' as a **parameter**. In this context, a parameter is a fixed, but arbitrary, constant used to describe a **family** of functions. Different values of b determine different constant functions. For example, $b = 3$ gives $f(x) = 3$, $b = -2$ gives $f(x) = -2$, and so on. Once b is chosen, however, it does not change as the independent variable, x, changes.

Also note that we are using the generic defaults for function names and independent variables, namely f and x, respectively. The functions $G(t) = \sqrt{\pi}$ and $Z(\rho) = 0$ are also fine examples of constant functions. Recall that inherent in the definition of a function is the notion of domain, so we record (as part of the definition) that a constant function has domain $(-\infty, \infty)$. The range of a constant function is the set $\{b\}$. The value b in this case is both the maximum and minimum of f, attained at each value in its domain.[1]

The next example showcases an application of constant functions and introduces the notion of a **piecewise-defined** function.

Example 1. The price of admission to see a matinee showing at a local movie theater is a function of the age of the ticket holder. If a person is aged A years, the price per ticket is $p(A)$ dollars and is given by:

$$p(A) = \begin{cases} 5.75 & \text{if } 0 \leq A < 6 \text{ or } A \geq 50, \\ 7.25 & \text{if } 6 \leq A < 50 \end{cases}$$

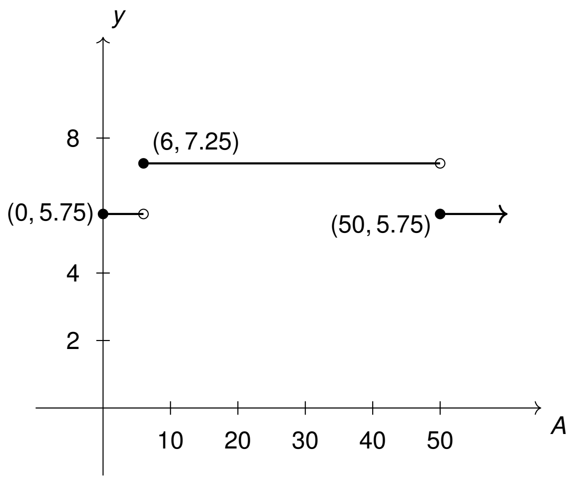
[1] It gets much weirder than that as we explore other more complicated functions. The key is to pay attention to the precision in the definitions of the terms involved in the discussion. Stay tuned!

1. Find and interpret $p(3)$, $p(6)$ and $p(62)$.
2. Explain the pricing structure verbally.
3. Graph p.

Explanation. The function p described above is an example of a **piecewise-defined** function because the **rule** to determine outputs, not just the value of the output, changes depending on the inputs.

1. To find $p(3)$, we note that the value $A = 3$ satisfies the inequality $0 \leq A < 6$ so we use the rule $p(A) = 5.75$. Hence, $p(3) = 5.75$ which means a ticket for a 3 year old is \$5.75. The next age, $A = 6$, just barely satisfies the inequality $6 \leq A < 50$ so we use the rule $p(A) = 7.25$, This yields $p(6) = 7.25$ which means a ticket for a 6 year old is \$7.25. Lastly, $A = 62$ satisfies the inequality $A \geq 50$, so we are back to the rule $p(A) = 5.75$. Thus $p(62) = 5.75$ which means someone 62 years young gets in for \$5.75.
2. Now that we've had some practice interpreting function values, we can begin to verbalize what the function is really saying. In the first 'piece' of the function, the inequality $0 \leq A < 6$ describes ticket holders under the age of 6 years and the inequality $A \geq 50$ describes ticket holders fifty years old or or older. For folks in these two age demographics, $p(A) = 5.75$ so the price per ticket is \$5.75. For everyone else, that is for folks at least 6 but younger than 50, the price is \$7.25 per ticket.
3. The independent variable here is specified as A, so we'll label our horizontal axis that way. The dependent variable remains unspecified so we can use the default y. The graph of $y = p(A)$ consists of three horizontal line pieces: the first is $y = 5.75$ for $0 \leq A < 6$, the second piece is $y = 7.25$ for $6 \leq A < 50$, and the last piece is $y = 5.75$ for $A \geq 50$.

   For the first piece, note that $A = 0$ is included in the inequality $0 \leq A < 6$ but $A = 6$ is not. For this reason, we have a point indicated at $(0, 5.75)$ but leave a hole[2] at $(6, 5.75)$. Similarly, to graph the second piece, we begin with a point at $(6, 7.25)$ and continue the horizontal line to a hole at $(50, 7.25)$. Lastly, we finish the graph with a point at $(50, 5.75)$ and continue to the right indefinitely.[3] Note the scaling on the horizontal axis compared to the vertical axis.



$$y = p(A) = \begin{cases} 5.75 & \text{if } 0 \leq A < 6 \text{ or } A \geq 50, \\ 7.25 & \text{if } 6 \leq A < 50 \end{cases}$$

Figure 2: Graphs of constant functions

[2] See our discussion about holes in graphs in Example **??** in Section **??**.

[3] The domain of p is $[0, \infty)$ by definition, even though few 327 year olds are out and about these days.

One of the favorite piecewise-defined functions in mathematical circles is the **greatest integer of *x***, denoted by $\lfloor x \rfloor$. In Section **??** we defined the set of **integers** as $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.[4] The value $\lfloor x \rfloor$ is defined to be the largest integer k with $k \leq x$. That is, $\lfloor x \rfloor$ is the unique integer k such that $k \leq x < k + 1$. Said differently, given any real number x, if x is an integer, then $\lfloor x \rfloor = x$. If not, then x lies in an interval between two integers, k and $k + 1$ and we choose $\lfloor x \rfloor = k$, the left endpoint.

Example 2. Let $\lfloor x \rfloor$ denote the greatest integer function.

1. Find $\lfloor 0.785 \rfloor$, $\lfloor 117 \rfloor$, $\lfloor -2.001 \rfloor$ and $\lfloor \pi + 6 \rfloor$
2. Explain how we can view $\lfloor x \rfloor$ as a piecewise-defined function and use this to graph $y = \lfloor x \rfloor$.

Explanation.

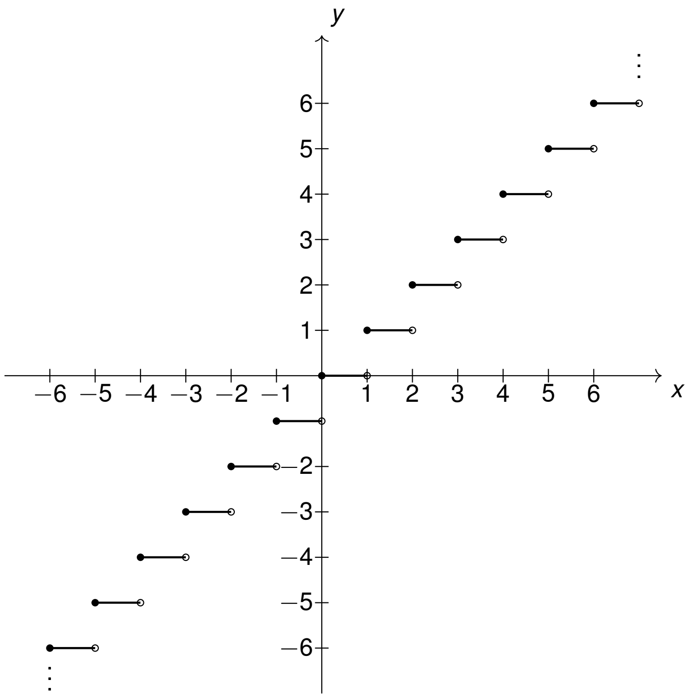
1. To find $\lfloor 0.785 \rfloor$, we note that $0 \leq 0.785 < 1$ so $\lfloor 0.785 \rfloor = 0$. Given that 117 is an integer, we have $\lfloor 117 \rfloor = 117$. To find $\lfloor -2.001 \rfloor$, we note that $-3 \leq -2.001 < -2$, so $\lfloor -2.001 \rfloor = -3$. Finally, with $\pi \approx 3.14$, we get $\pi + 6 \approx 9.14$ and $9 \leq \pi + 6 < 10$ so $\lfloor \pi + 6 \rfloor = 9$.
2. The first step in evaluating $\lfloor x \rfloor$ is to determine the interval $[k, k+1)$ containing x so it seems reasonable that these are the intervals which produce the 'pieces'. In this case, there happen to be infinitely many pieces. The inequality '$k \leq x < k + 1$' includes the left endpoint but excludes the right endpoint, so we have points at the left endpoints of our horizontal line segments while we have holes at the right endpoints.

A partial description of $\lfloor x \rfloor$ is given alongside a partial graph at the top of the next page. (A full description or a complete graph would require infinitely large paper!) We use the vertical dots $\vdots$ to indicate that both the rule and the graph continue indefinitely following the established pattern.[5]

$$\lfloor x \rfloor = \begin{cases} \vdots & \\ -5 & \text{if } -5 \leq x < -4 \\ -4 & \text{if } -4 \leq x < -3 \\ -3 & \text{if } -3 \leq x < -2 \\ -2 & \text{if } -2 \leq x < -1 \\ -1 & \text{if } -1 \leq x < 0 \\ 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } 1 \leq x < 2 \\ 2 & \text{if } 2 \leq x < 3 \\ 3 & \text{if } 3 \leq x < 4 \\ 4 & \text{if } 4 \leq x < 5 \\ 5 & \text{if } 5 \leq x < 6 \\ \vdots & \end{cases}$$

[4] The use of the letter $\mathbb{Z}$ for the integers is ostensibly because the German word **zahlen** means 'to count'.

[5] It is always dangerous to leave the rest of the pattern to the reader. See, for instance, this paper.



The graph of $y = \lfloor x \rfloor$.