1 Functions as Mappings

Mathematics can be thought of as the study of patterns. In most disciplines, Mathematics is used as a language to express, or codify, relationships between quantities - both algebraically and geometrically - with the ultimate goal of solving real-world problems. The fact that the same algebraic equation which models the growth of bacteria in a petri dish is also used to compute the account balance of a savings account or the potency of radioactive material used in medical treatments speaks to the universal nature of Mathematics. Indeed, Mathematics is more than just about solving a specific problem in a specific situation, it's about abstracting problems and creating universal tools which can be used by a variety of scientists and engineers to solve a variety of problems.

This power of abstraction has a tendency to create a language that is initially intimidating to students. Mathematical definitions are precise and adherence to that precision is often a source of confusion and frustration. It doesn't help matters that more often than not very common words are used in Mathematics with slightly different definitions than is commonly expected. The first 'universal tool' we wish to highlight - the concept of a 'function' - is a perfect example of this phenomenon in that we redefine a word that already has multiple meanings in English.

Definition 1. Given two sets[1] A and B, a **function** from A to B is a process by which each element of A is matched with (or 'mapped to') one and only one element of B.

The grammar here '**from** A **to** B' is important. Thinking of a function as a process, we can view the elements of the set A as our starting materials, or **inputs** to the process. The function processes these inputs according to some specified rule and the result is a set of **outputs** - elements of the set B. In terms of inputs and outputs, Definition 1 says that a function is a process in which each **input** is matched to one and only one **output**.

For example, let's take a look at some of the pets in the Stitz household. Taylor's pets include White Paw and Cooper (both cats), Bingo (a lizard) and Kennie (a turtle). Let N be the set of pet names: $N = \{\text{White Paw, Cooper, Bingo, Kennie}\}$, and let T be the set of pet types: $T = \{\text{cat, lizard, turtle}\}$. Let f be the process that takes each pet's name as the input and returns that pet's type as the output. Let g be the reverse of f: that is, g takes each pet type as the input and returns the names of the pets of that type as the output. Note that both f and g are codifying the **same** given information about Taylor's pets, but one of them is a function and the other is not.

To help identify which process f or g is a function and why the other is not, we create **mapping diagrams** for f and g below. In each case, we organize the inputs in a column on the left and the outputs in a column on the right. We draw an arrow connecting each input to its corresponding output(s). Note that the arrows communicate the grammatical bias: the arrow originates at the input and points to the output.

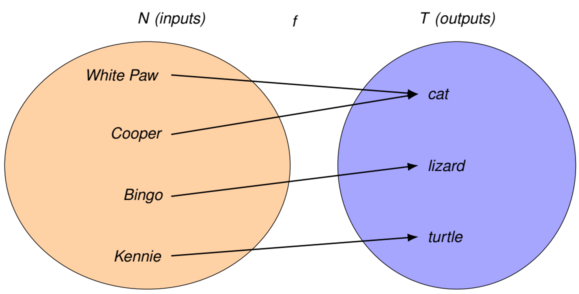


Figure 1: Mapping diagram for f

[1] Please refer to Section **??** for a review of this terminology.

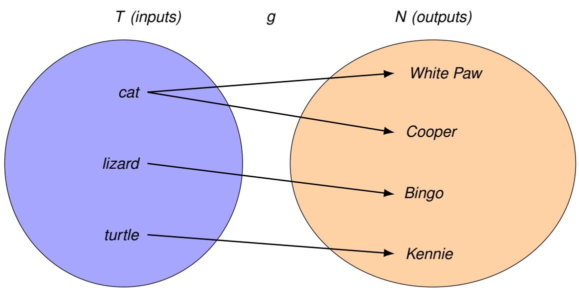


Figure 2: Mapping diagram for g

Alt text: Mapping diagrams illustrating a function and its reverse. In the top diagram, the function f maps the inputs White Paw, Cooper, Bingo, and Kennie to the outputs cat, cat, lizard, and turtle, respectively. In the bottom diagram, the function g maps cat to White Paw and Cooper, lizard to Bingo, and turtle to Kennie.

The process f is a function since f matches each of its inputs (each pet name) to just one output (the pet's type). The fact that different inputs (White Paw and Cooper) are matched to the same output (cat) is fine. On the other hand, g matches the input 'cat' to the two different outputs 'White Paw' and 'Cooper', so g is not a function. Functions are favored in mathematical circles because they are processes which produce only one answer (output) for any given query (input). In this scenario, for instance, there is only one answer to the question: 'What type of pet is White Paw?' but there is more than one answer to the question 'Which of Taylor's pets are cats?'

As you might expect, with functions being such an important concept in Mathematics, we need to build a vocabulary to assist us when discussing them. To that end, we have the following definitions.[2]

Definition 2. Suppose f is a function from A to B.

- If $a \in A$, we write $f(a)$ (read 'f of a') to denote the unique element of B to which f matches a.

  That is, if we view 'a' as the input to f, then '$f(a)$' is the output from f.
- The set A is called the **domain**.

  Said differently, the domain of a function is the set of inputs to the function.
- The set $\{f(a) \mid a \in A\}$ is called the **range** of f.

  Said differently, the range of a function is the set of outputs from the function.

Some remarks about Definition 2 are in order. First, and most importantly, the notation '$f(a)$' in Definition 2 introduces yet another mathematical use for parentheses. Parentheses are used in some cases as grouping symbols, to represent ordered pairs, and to delineate intervals of real numbers. More often than not, the use of parentheses in expressions like '$f(a)$' is confused with multiplication. As always, paying attention to the context is key. If f is a function and 'a' is in the domain of f, then '$f(a)$' is the output from f when you input a. The diagram below provides a nice generic picture to keep in mind when thinking of a function as a mapping process with input 'a' and output '$f(a)$'.

[2]Please refer to Section **??** for a review of the terminology used in these definitions.

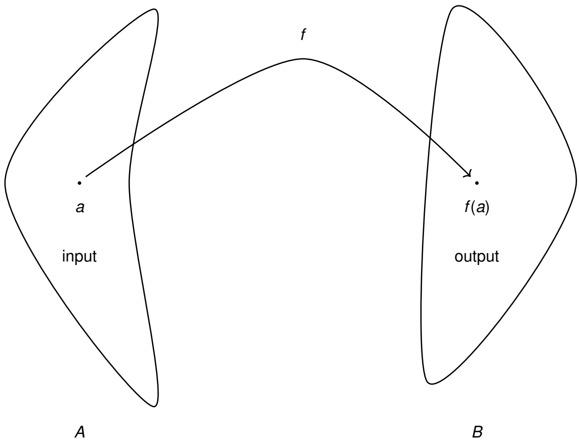


Figure 3: Diagrammatic representation of a function

Alt text: Diagram illustrating the concept of a function. An element labeled a in set A is mapped by the function f to the element labeled $f(a)$ in set B. The element a is identified as the input, and $f(a)$ is identified as the output.

In the preceding pet example, the symbol f(Bingo), read 'f of Bingo', is asking what type of pet Bingo is, so f(Bingo) = lizard. The fact that f is a function means f(Bingo) is unambiguous because f matches the name 'Bingo' to only one pet type, namely 'lizard'. In contrast, if we tried to use the notation 'g(cat)' to indicate what pet name g matched to 'cat', we have **two** possibilities, White Paw and Cooper, with no way to determine which one (or both) is indicated.

Continuing to apply Definition 2 to our pet example, we find that the domain of the function f is N, the set of pet names. Finding the range takes a little more work, mostly because it's easy to be caught off guard by the notation used in the definition of 'range'. The description of the range as '$\{f(a) \mid a \in A\}$' is an example of 'set-builder' notation. In English, '$\{f(a) \mid a \in A\}$' reads as 'the set of $f(a)$ such that a is in A'. In other words, the range consists of all of the outputs from f - all of the $f(a)$ values - as a varies through each of the elements in the domain A. Note that while every element of the set A is, by definition, an element of the domain of f, not every element of the set B is necessarily part of the range of f.[3]

In our pet example, we can obtain the range of f by looking at the mapping diagram or by constructing the set $\{f(\text{White Paw}), f(\text{Cooper}), f(\text{Bingo}), f(\text{Kennie})\}$ which lists all of the outputs from f as we run through all of the inputs to f. Keep in mind that we list each element of a set only once so the range of f is:[4]

$$\{f(\text{White Paw}), f(\text{Cooper}), f(\text{Bingo}), f(\text{Kennie})\} = \{\text{cat}, \text{lizard}, \text{turtle}\} = T.$$

If we let n denote a generic element of N then $f(n)$ is some element t in T, so we write $t = f(n)$. In this equation, n is called the **independent variable** and t is called the **dependent variable**.[5] Moreover, we say 't is a function of n', or, more specifically, 'the type of pet is a function of the pet name' meaning that every pet name n corresponds to one, and only one, pet type t. Even though f and t are different things,[6] it is very common for the function and its outputs to become more-or-less synonymous, even in what are otherwise precise mathematical definitions.[7] We will endeavor to point out such ambiguities as we move through the text.

[3]For purposes of completeness, the set B is called the **codomain** of f. For us, the concepts of domain and range suffice since our codomain will most always be the set of real numbers, $\mathbb{R}$.

[4]If instead of mapping N into T, we could have mapped N into $U = \{\text{cat}, \text{lizard}, \text{turtle}, \text{dog}\}$ in which case the range of f would not have been the entire codomain U.

[5]These adjectives stem from the fact that the value of t **depends** entirely on our (independent) choice of n.

[6]Specifically, f is a function so it requires and domain, a range and a rule of assignment whereas t is simply the output from f.

[7]In fact, it is not uncommon to see the name of the function as the same as the dependent variable. For example, writing '$y = y(x)$' would be a way to communicate the idea that 'y is a function of x'.

While the concept of a function is very general in scope, we will be focusing primarily on functions of real numbers because most disciplines use real numbers to quantify data. Our next example explores a function defined using a table of numerical values.

Example 1. Suppose Skippy records the outdoor temperature every two hours starting at 6 a.m. and ending at 6 p.m. and summarizes the data in the table below:

time (hours after 6 a.m.)	outdoor temperature in degrees Fahrenheit
0	64
2	67
4	75
6	80
8	83
10	83
12	82

1. Explain why the recorded outdoor temperature is a function of the corresponding time.
2. Is time a function of the outdoor temperature? Explain.
3. Let f be the function which matches time to the corresponding recorded outdoor temperature.
   Find and interpret the following:
   - $f(2)$
   - $f(4)$
   - $f(2+4)$
   - $f(2)+f(4)$
   - $f(2)+4$
4. Solve and interpret $f(t) = 83$.
5. State the range of f. What is lowest recorded temperature of the day? The highest?

Explanation. 1. The outdoor temperature is a function of time because each time value is associated with only one recorded temperature.

2. Time is not a function of the outdoor temperature because there are instances when different times are associated with a given temperature. For example, the temperature 83 corresponds to both of the times 8 and 10.
3. - To find $f(2)$, we look in the table to find the recorded outdoor temperature that corresponds to when the time is 2. We get $f(2) = 67$ which means that 2 hours after 6 a.m. (i.e., at 8 a.m.), the temperature is 67°F.
   - Per the table, $f(4) = 75$, so the recorded outdoor temperature at 10 a.m. (4 hours after 6 a.m.) is 75°F.
   - From the table, we find $f(2+4) = f(6) = 80$, which means that at noon (6 hours after 6 a.m.), the recorded outdoor temperature is 80°F.
   - Using results from above we see that $f(2)+f(4) = 67+75 = 142$. When adding $f(2)+f(4)$, we are adding the recorded outdoor temperatures at 8 a.m. (2 hours after 6 a.m.) and 10 a.m. (4 hours after 6 AM), respectively, to get 142°F.
   - We compute $f(2)+4 = 67+4 = 71$. Here, we are adding 4°F to the outdoor temperature recorded at 8 a.m..

4. Solving $f(t) = 83$ means finding all of the input (time) values t which produce an output value of 83. From the data, we see that the temperature is 83 when the time is 8 or 10, so the solution to $f(t) = 83$ is $t = 8$ or $t = 10$. This means the outdoor temperature is 83°F at 2 p.m. (8 hours after 6 a.m.) and at 4 p.m. (10 hours after 6 a.m.).

5. The range of f is the set of all of the outputs from f, or in this case, the outside recorded temperatures. Based on the data, we get $\{64, 67, 75, 80, 82, 83\}$. (Here again, we list elements of a set only once.) The lowest recorded temperature of the day is 64°F and the highest recorded temperature of the day is 83°F.

A few remarks about Example 1 are in order. First, note that $f(2 + 4)$, $f(2) + f(4)$ and $f(2) + 4$ all work out to be numerically different, and more importantly, all represent different things.[8] One of the common mistakes students make is to misinterpret expressions like these, so it's important to pay close attention to the syntax here.

Next, when solving $f(t) = 83$, the variable 't' is being used as a convenient 'dummy' variable or placeholder in the sense that solving $f(t) = 83$ produces the same solutions as solving $f(x) = 83$, $f(w) = 83$, or even $f(?) = 83$. All of these equations are asking for the same thing: what inputs to f produce an output of 83. The choice of the letter 't' here makes sense since the inputs are **t**ime values. Throughout the text, we will endeavor to use meaningful labels when working in applied situations, but the fact remains that the choice of letters (or symbols) is completely arbitrary.

Finally, given that the range in this example was a finite set of real numbers, we could find the smallest and largest elements of it. Here, they correspond to the coolest and warmest temperatures of the day, respectively, but the meaning would change if the function related different quantities. In many applications involving functions, the end goal is to find the minimum or maximum values of the outputs of those functions (called **optimizing** the function) so for that reason, we have the following definition.

Definition 3. Suppose f is a function whose range is a set of real numbers containing m and M.

- The value m is called the **minimum**[9] of f if $m \leq f(x)$ for all x in the domain of f.

  That is, the minimum of f is the smallest output from f, if it exists.

- The value M is called the **maximum**[10] of f if $f(x) \leq M$ for all x in the domain of f.

  That is, the maximum of f is the largest output from f, if it exists.

- Taken together, the values m and M (if they exist) are called the **extrema**[11] of f.

Definition 3 is an example where the name of the function, f, is being used almost synonymously with its outputs in that when we speak of 'the minimum and maximum of the **function** f' we are really talking about the minimum and maximum values of the **outputs** $f(x)$ as x varies through the domain of f. Thus we say that the maximum of f is 83 and the minimum of f is 64 when referring to the highest and lowest recorded temperatures in the previous example.

2 Algebraic Representations of Functions

By focusing our attention to functions that involve real numbers, we gain access to all of the structures and tools from prior courses in Algebra. In this subsection, we discuss how to represent functions algebraically using formulas and begin with the following example.

Example 2.

[8] You may be wondering why one would ever compute these quantities. Rest assured that we will use expressions like these in examples throughout the text. For now, it suffices just to know that they are different.

[9] also called 'absolute' or 'global' minimum

[10] also called 'absolute' or 'global' maximum

[11] also called the 'absolute' or 'global' extrema or the 'extreme values'

1. Let f be the function which takes a real number and performs the following sequence of operations:
   - Step 1: add 2
   - Step 2: multiply the result of Step 1 by 3
   - Step 3: subtract 1 from the result of Step 2.

   The for this function f:

   (i) Find and simplify $f(-5)$.

   (ii) Find and simplify a formula for $f(x)$.

2. Let $h(t) = -t^2 + 3t + 4$.

   (i) Find and simplify the following:
      i. $h(-1)$, $h(0)$ and $h(2)$.
      ii. $h(2x)$ and $2h(x)$.
      iii. $h(t + 2)$, $h(t) + 2$ and $h(t) + h(2)$.

   (ii) Solve $h(t) = 0$.

Explanation. 1. (i) We take -5 and follow it through each step:
- Step 1: adding 2 gives us $-5 + 2 = -3$.
- Step 2: multiplying the result of Step 1 by 3 yields $(-3)(3) = -9$.
- Step 3: subtracting 1 from the result of Step 2 produces $-9 - 1 = -10$.

Hence, $f(-5) = -10$.

(ii) To find a formula for $f(x)$, we repeat the above process but use the variable 'x' in place of the number -5:
- Step 1: adding 2 gives us the quantity $x + 2$.
- Step 2: multiplying the result of Step 1 by 3 yields $(x + 2)(3) = 3x + 6$.
- Step 3: subtracting 1 from the result of Step 2 produces $(3x + 6) - 1 = 3x + 5$.

Hence, we have codified f using the formula $f(x) = 3x + 5$. In other words, the function f matches each real number 'x' with the value of the expression '$3x + 5$'. As a partial check of our answer, we use this formula to find $f(-5)$. We compute $f(-5)$ by substituting $x = -5$ into the formula $f(x)$ and find $f(-5) = 3(-5) + 5 = -10$ as before.

2. As before, representing the function h as $h(t) = -t^2 + 3t + 4$ means that h matches the real number t with the value of the expression $-t^2 + 3t + 4$.

   (i) To find $h(-1)$, we substitute -1 for t in the expression $-t^2 + 3t + 4$. It is highly recommended that you be generous with parentheses here in order to avoid common mistakes:

$$\begin{aligned} h(-1) &= -(-1)^2 + 3(-1) + 4 \\ &= -(1) + (-3) + 4 \\ &= 0. \end{aligned}$$

   Similarly, $h(0) = -(0)^2 + 3(0) + 4 = 4$, and $h(2) = -(2)^2 + 3(2) + 4 = -4 + 6 + 4 = 6$.

   (ii) To find $h(2x)$, we substitute $2x$ for t:

$$\begin{aligned} h(2x) &= -(2x)^2 + 3(2x) + 4 \\ &= -(4x^2) + (6x) + 4 \\ &= -4x^2 + 6x + 4. \end{aligned}$$

   The expression $2h(x)$ means that we multiply the expression $h(x)$ by 2. We first get $h(x)$ by substituting x for t: $h(x) = -x^2 + 3x + 4$. Hence,

$$\begin{aligned} 2h(x) &= 2\left(-x^2 + 3x + 4\right) \\ &= -2x^2 + 6x + 8. \end{aligned}$$

(iii) To find $h(t+2)$, we substitute the quantity $t+2$ in place of t:

$$\begin{aligned} h(t+2) &= -(t+2)^2 + 3(t+2) + 4 \\ &= -\left(t^2 + 4t + 4\right) + (3t+6) + 4 \\ &= -t^2 - 4t - 4 + 3t + 6 + 4 \\ &= -t^2 - t + 6. \end{aligned}$$

To find $h(t)+2$, we add 2 to the expression for $h(t)$

$$\begin{aligned} h(t) + 2 &= \left(-t^2 + 3t + 4\right) + 2 \\ &= -t^2 + 3t + 6. \end{aligned}$$

From our work above, we see that $h(2) = 6$ so

$$\begin{aligned} h(t) + h(2) &= \left(-t^2 + 3t + 4\right) + 6 \\ &= -t^2 + 3t + 10. \end{aligned}$$

3. We know $h(-1) = 0$ from above, so $t = -1$ should be one of the answers to $h(t) = 0$. In order to see if there are any more, we set $h(t) = -t^2 + 3t + 4 = 0$. Factoring[12] gives $-(t+1)(t-4) = 0$, so we get $t = -1$ (as expected) along with $t = 4$.

A few remarks about Example 2 are in order. First, note that $h(2x)$ and $2h(x)$ are different expressions. In the former, we are multiplying the **input** by 2; in the latter, we are multiplying the **output** by 2. The same goes for $h(t+2)$, $h(t)+2$ and $h(t)+h(2)$. The expression $h(t+2)$ calls for adding 2 to the input t and then performing the function h. The expression $h(t)+2$ has us performing the process h first, then adding 2 to the output $h(t)$. Finally, $h(t)+h(2)$ directs us to first find the outputs $h(t)$ and $h(2)$ and then add the results. As we saw in Example 1, we see here again the importance paying close attention to syntax.[13]

Let us return for a moment to the function f in Example 2 which we ultimately represented using the formula $f(x) = 3x+5$. If we introduce the dependent variable y, we get the equation $y = f(x) = 3x+5$, or, more simply $y = 3x+5$. To say that the equation $y = 3x+5$ describes y as a function of x means that for each choice of x, the formula $3x+5$ determines only one associated y-value.

We could turn the tables and ask if the equation $y = 3x+5$ describes x as a function of y. That is, for each value we pick for y, does the equation $y = 3x+5$ produce only one associated x value? One way to proceed is to solve $y = 3x+5$ for x and get $x = \frac{1}{3}(y-5)$. We see that for each choice of y, the expression $\frac{1}{3}(y-5)$ evaluates to just one number, hence, x is a function of y. If we give this function a name, say g, we have $x = g(y) = \frac{1}{3}(y-5)$, where in this equation, y is the independent variable and x is the dependent variable. We explore this idea in the next example.

Example 3.

1. Consider the equation $x^3 + y^2 = 25$.
   (i) Does this equation represent y as a function of x? Explain.
   (ii) Does this equation represent x as a function of y? Explain.
2. Consider the equation $u^4 + t^3 u = 16$.
   (i) Does this equation represent t as a function of u? Explain.
   (ii) Does this equation represent u as a function of t? Explain.

Solution.

[12] You may need to review Section **??**.

[13] As was mentioned before, we will give meanings to the these quantities in other examples throughout the text.

1. (i) To say that $x^3 + y^2 = 25$ represents y as a function of x, we need to show that for each x we choose, the equation produces only one associated y-value. To help with this analysis, we solve the equation for y in terms of x.

$$\begin{array}{rcl} x^3 + y^2 & = & 25 \\ y^2 & = & 25 - x^3 \\ y & = & \pm\sqrt{25 - x^3} \end{array}$$

The presence of the '$\pm$' indicates that there is a good chance that for some x-value, the equation will produce **two** corresponding y-values. Indeed, $x = 0$ produces $y = \pm\sqrt{25 - 0^3} = \pm 5$. Hence, $x^3 + y^2 = 25$ equation does **not** represent y as a function of x because $x = 0$ is matched with more than one y-value. If you need to review solving equations like this, see Section **??** for a review.

(ii) To see if $x^3 + y^2 = 25$ represents x as a function of y, we solve the equation for x in terms of y:

$$\begin{array}{rcl} x^3 + y^2 & = & 25 \\ x^3 & = & 25 - y^2 \\ x & = & \sqrt[3]{25 - y^2} \end{array}$$

In this case, each choice of y produces only **one** corresponding value for x, so $x^3 + y^2 = 25$ represents x as a function of y.

2. (i) To see if $u^4 + t^3 u = 16$ represents t as a function of u, we proceed as above and solve for t in terms of u:

$$\begin{array}{rcll} u^4 + t^3 u & = & 16 & \\ t^3 u & = & 16 - u^4 & \\ t^3 & = & \dfrac{16 - u^4}{u} & \text{assumes } u \neq 0 \\ t & = & \sqrt[3]{\dfrac{16 - u^4}{u}} & \text{extract cube roots.} \end{array}$$

Although it's a bit cumbersome, as long as $u \neq 0$ the expression $\sqrt[3]{\dfrac{16 - u^4}{u}}$ will produce just one value of t for each value of u. What if $u = 0$? In that case, the equation $u^4 + t^3 u = 16$ reduces to $0 = 16$ - which is never true - so we don't need to worry about that case.[14] Hence, $u^4 + t^3 u = 16$ represents t as a function of u.

(ii) In order to determine if $u^4 + t^3 u = 16$ represents u as a function of t, we could attempt to solve $u^4 + t^3 u = 16$ for u in terms of t, but we won't get very far.[15] Instead, we take a different approach and experiment with looking for solutions for u for specific values of t. If we let $t = 0$, we get $u^4 = 16$ which gives $u = \pm\sqrt[4]{16} = \pm 2$. Hence, $t = 0$ corresponds to more than one u-value which means $u^4 + t^3 u = 16$ does not represent u as a function of t.

We'll have more to say about using equations to describe functions in Section **??**. For now, we turn our attention to a geometric way to represent functions.

3 Geometric Representations of Functions

In this section, we introduce how to graph functions. As we'll see in this and later sections, visualizing functions geometrically can assist us in both analyzing them and using them to solve associated application problems. Our playground, if you will, for the Geometry in this course is the Cartesian Coordinate Plane. The reader would do well to review Section **??** as needed.

[14] Said differently, $u = 0$ is not in the domain of the function represented by the equation $u^4 + t^3 u = 16$.

[15] Try it for yourself!

Our path to the Cartesian Plane requires ordered pairs. In general, we can represent every function as a set of ordered pairs. Indeed, given a function f with domain A, we can represent $f = \{(a, f(a)) \mid a \in A\}$. That is, we represent f as a set of ordered pairs $(a, f(a))$, or, more generally, (input, output). For example, the function f which matches Taylor's pet's names to their associated pet type can be represented as:

$$f = \{(\text{White Paw}, \text{cat}), (\text{Cooper}, \text{cat}), (\text{Bingo}, \text{lizard}), (\text{Kennie}, \text{turtle})\}$$

Moving on, we next consider the function f from Example 1 which relates time to temperature. In this case, $f = \{(0, 64), (2, 67), (4, 75), (6, 80), (8, 83), (10, 83), (12, 82)\}$. This function has numerical values for both the domain and range so we can identify these ordered pairs with points in the Cartesian Plane. The first coordinates of these points (the abscissae) represent time values so we'll use t to label the horizontal axis. Likewise, we'll use T to label the vertical axis since the second coordinates of these points (the ordinates) represent temperature values. Note that labeling these axes in this way determines our independent and dependent variable names, t and T, respectively.

The plot of these points is called 'the **graph** of f'. More specifically, we could describe this plot as 'the graph of $f(t)$', because we have decided to name the independent variable t. Most specifically, we could describe the plot as 'the graph of $T = f(t)$', given that we have named the independent variable t and the dependent variable T.

Below we present two plots, both of which are graphs of the function f. In both cases, the vertical axis has been scaled in order to save space. In our first graph, the same increment on the horizontal axis to measure 1 unit measures 10 units on the vertical axis.

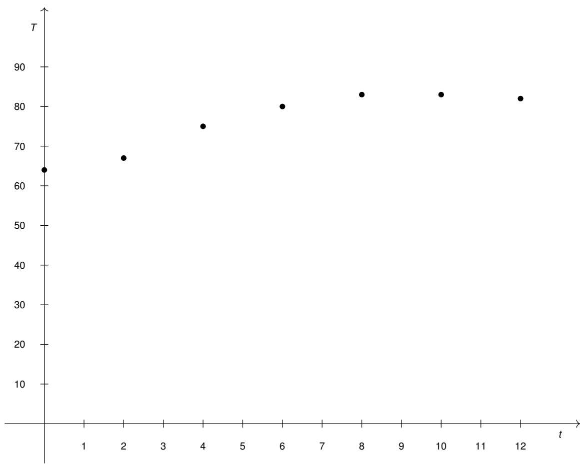


Figure 4: The graph of $T = f(t)$

Alt text: Scatter plot of temperature T versus time t. The plotted points are approximately $(0, 64)$, $(2, 67)$, $(4, 75)$, $(6, 80)$, $(8, 83)$, $(10, 83)$, and $(12, 82)$. The temperature generally increases over time, reaches a high value near 83, and then levels off.

In our second graph, the same increment which measures 1 unit on the horizontal axis measures 2 on the vertical axis. The '$\asymp$' symbol on the vertical axis in the graph below is used to indicate a jump in the vertical labeling.

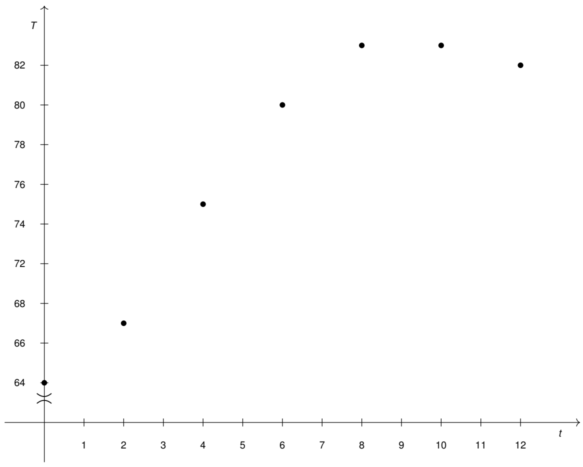


Figure 5: The graph of $T = f(t)$

Alt text: Scatter plot of temperature T versus time t. The plotted points are $(0, 64)$, $(2, 67)$, $(4, 75)$, $(6, 80)$, $(8, 83)$, $(10, 83)$, and $(12, 82)$. The temperature increases from 64 to a maximum of 83, remains near 83, and then decreases slightly to 82.

Both are perfectly accurate data plots, but they have different visual impacts. Note here that the extrema of f, 64 and 83, correspond to the lowest and highest points on the graph, respectively: $(0, 64)$, $(8, 83)$ and $(10, 83)$. More often than not, we will use the graph of a function to help us optimize that function.[16]

If you found yourself wanting to connect the dots in the graphs above, you're not alone. As it stands, however, the function f matches only seven inputs to seven outputs, so those seven points - and just those seven points - comprise the graph of f. That being said, common everyday experience tells us that while the data Skippy collected in his table gives some good information about the relationship between time and temperature on a given day, it is by no means a complete description of the relationship.

For example, Skippy's data cannot tell us what the temperature was at 7 a.m. or 12:13 p.m, although we are pretty sure there were outdoor temperatures at those times. Also, given that at some point it was 64°F and later on it was 83°F, it seems reasonable to assume that at some point it was 70°F or even 79.923°F.

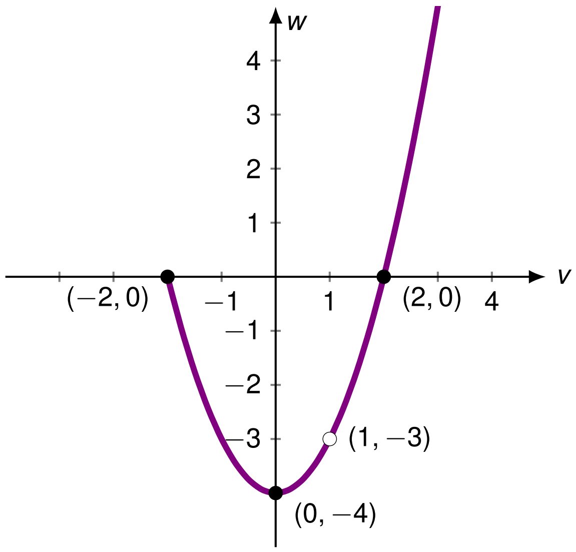
Skippy's temperature function f is an example of a **discrete** function in the sense that each of the data points are 'isolated' with measurable gaps in between. The idea of 'filling in' those gaps is a quest to find a **continuous** function to model this same phenomenon.[17] We'll return to this example in Sections **??** and **??** in an attempt to do just that.

In the meantime, our next example involves a function whose domain is (almost) an **interval** of real numbers and whose graph consists of a (mostly) **connected** arc.

Example 4. Consider the graph below.

[16] One major use of Calculus is to optimize functions analytically - that is, without a graph.

[17] Roughly speaking, a **continuous variable** is a variable which takes on values over an **interval** of real numbers as opposed to values in a discrete list. In this case we would think of time as a 'continuum' - an interval of real numbers as opposed to 7 or so isolated times. A **continuous function** is a function which takes an interval of real numbers and maps it in such a way that its graph is a connected curve with no holes or gaps. This is technically a Calculus idea, but we'll need to discuss the notion of continuity a few times in the text.



Alt text: Graph of a smooth U-shaped curve passing through the points $(-2, 0)$, $(0, -4)$, and $(2, 0)$. The curve reaches its lowest point at $(0, -4)$. An open circle at $(1, -3)$ indicates that this point is excluded from the graph.

1. (i) Explain why this graph suggests that w is a function of v, $w = F(v)$.
   (ii) Find $F(0)$ and solve $F(v) = 0$.
   (iii) Find the domain and range of F using interval notation.[18] Find the extrema of F, if any exist.
2. Does this graph suggest v is a function of w? Explain.

Solution. The challenge in working with only a graph is that unless points are specifically labeled (as some are in this case), we are forced to approximate values. In addition to the labeled points, there are other interesting features of the graph; a gap or 'hole' labeled $(1, -3)$ and an arrow on the upper right hand part of the curve. We'll have more to say about these two features shortly.

1. (i) In order for w to be a function of v, each v-value on the graph must be paired with only one w-value. What if this weren't the case? We'd have at least two points with the **same** v-coordinate with **different** w-coordinates. Graphically, we'd have two points on graph on the same vertical line, one above the other. This never happens so we may conclude that w is a function of v.
   (ii) The value $F(0)$ is the output from F when $v = 0$. The points on the graph of F are of the form $(v, F(v))$ thus we are looking for the w-coordinate of the point on the graph where $v = 0$. Given that the point $(0, -4)$ is labeled on the graph, we can be sure $F(0) = -4$.
   To solve $F(v) = 0$, we are looking for the v-values where the output, or associated w value, is 0. Hence, we are looking for points on the graph with a w-coordinate of 0. We find two such points, $(-2, 0)$ and $(2, 0)$, so our solutions to $F(v) = 0$ are $v = \pm 2$. Pictures highlighting the relevant graphical features are given at the top of the next page.
   (iii) The domain of F is the set of inputs to F. With v as the input here, we need to describe the set of v-values on the graph. We can accomplish this by **projecting** the graph to the v-axis and seeing what part of the v-axis is covered. Using the GeoGebra interactive below, adjust the slider to visualize the graph of F being projected to the v-axis. Each point on the graph of F is 'pushed' to the v-axis, and the portion which is covered by this process is the domain of F.

GeoGebra link: `https://www.geogebra.org/m/xrjvhdwu`

[18] Please consult Section **??** for a review of interval notation if need be.

The leftmost point on the graph is $(-2, 0)$, so we know that the domain starts at $v = -2$. The graph continues to the right until we encounter the 'hole' labeled at $(1, -3)$. This indicates one and only one point, namely $(1, -3)$ is missing from the curve which for us means $v = 1$ is not in the domain of F. The graph continues to the right and the arrow on the graph indicates that the graph goes upwards to the right indefinitely. Hence, our domain is $\{v \mid v \geq -2,\ v \neq 1\}$ which, in interval notation, is $[-2, 1) \cup (1, \infty)$.

To find the range of F, we need to describe the set of outputs - in this case, the w-values on the graph. Here, we project the graph to the w-axis. Once again, we are fortunate to have a GeoGebra interactive to help us visualize this process.

GeoGebra link: `https://www.geogebra.org/m/z2pvgbmr`

Vertically, the graph starts at $(0, -4)$ so our range starts at $w = -4$. Note that even though there is a hole at $(1, -3)$, the w-value -3 is covered by what **appears** to be the point $(-1, -3)$ on the graph.[19] The arrow indicates that the graph extends upwards indefinitely so the range of F is $\{w \mid w \geq -4\}$ or, in interval notation, $[-4, \infty)$. Regarding extrema, F has a minimum of -4 when $v = 0$, but given that the graph extends upwards indefinitely, F has no maximum.

2. Finally, to determine if v is a function of w, we look to see if each w-value is paired with only one v-value on the graph. We have points on the graph, namely $(-2, 0)$ and $(2, 0)$, that clearly show us that $w = 0$ is matched with the **two** v-values $v = 2$ and $v = -2$. Hence, v is not a function of w.

It cannot be stressed enough that when given a graphical representation of a function, certain assumptions must be made. In the previous example, for all we know, the minimum of the graph is at $(0.001, -4.0001)$ instead of $(0, -4)$. If we aren't given an equation or table of data, or if specific points aren't labeled, we really have no way to tell. We also are assuming that the graph depicted in the example, while ultimately made of infinitely many points, has no gaps or holes other than those noted. This allows us to make such bold claims as the existence of a point on the graph with a w-coordinate of -3.

Before moving on to our next example, it is worth noting that the geometric argument made in Example 4 to establish that w is a function of v can be generalized to any graph. This result is the celebrated Vertical Line Test and it enables us to detect functions geometrically. Note that the statement of the theorem resorts to the 'default' x and y labels on the horizontal and vertical axes, respectively.

Theorem 1. The Vertical Line Test: A graph in the xy-plane[20] represents y as a function of x if and only if no vertical line intersects the graph more than once.

Let's take a minute to discuss the phrase 'if and only if' used in Theorem 1. The statement 'the graph represents y as a function of x **if and only if** no vertical line intersects the graph more than once' is actually saying two things. First, it's saying 'the graph represents y as a function of x **if** no vertical line intersects the graph more than once' and, second, 'the graph represents y as a function of x **only if** no vertical line intersects the graph more than once'.

Logically, these statements are saying two different things. The first says that if no vertical line crosses the graph more than once, then the graph represents y as a function of x. But the question remains: could a graph represent y as a function of x and yet there be a vertical line that intersects the graph more than once? The answer to this is 'no' because the second statement says that the **only** way the graph represents y as a function of x is the case when no vertical line intersects the graph more than once.

Applying the Vertical Line Test to the graph given in Example 4, we see below that all of the vertical lines meet the graph at most once (several are shown for illustration) showing w is a function of v. Notice that some of the lines ($x = -3$ and $x = 1$, for example) don't hit the graph at all. This is fine because the Vertical Line Test is looking for lines that hit the graph more than once. It does not say **exactly** once so missing the graph altogether is permitted.

[19] For all we know, it could be $(-0.992, -3)$.

[20] That is, the horizontal axis is labeled with 'x' and the vertical axis is labeled with 'y'.

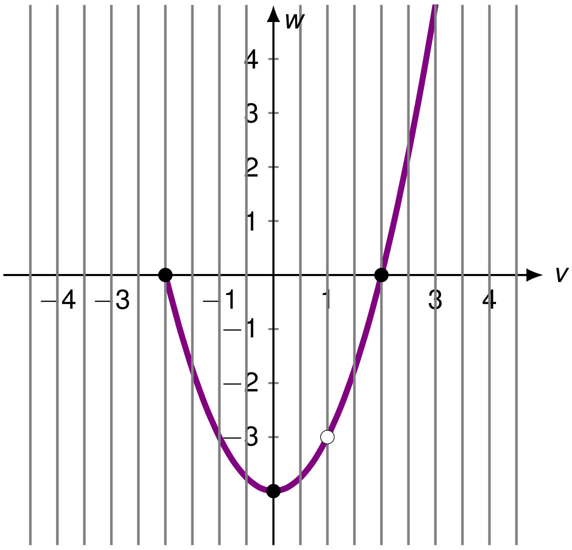


Figure 6: Vertical line test

Alt text: Graph of the function F. The graph forms a U-shaped curve with v-intercepts at $(-2, 0)$ and $(2, 0)$ and a minimum at $(0, -4)$. An open circle at $(1, -3)$ indicates that this point is excluded from the graph. Several vertical lines are drawn across the graph to illustrate the Vertical Line Test. Each vertical line intersects the graph at most once.

There is also a geometric test to determine if the graph above represents v as a function of w. We introduce this aptly-named **Horizontal Line Test** in Exercise **??** and revisit it in Sections **??** and **??**.

Our next example revisits the function h from Example 2 from a graphical perspective.

Example 5. With the help of a graphing utility graph $h(t) = -t^2 + 3t + 4$. From your graph, state the domain, range and extrema, if any exist.

Solution. The dependent variable wasn't specified so we use the default 'y' label for the vertical axis and set about graphing $y = h(t)$. From our work in Example 2, we already know $h(-1) = 0$, $h(0) = 4$, $h(2) = 6$ and $h(4) = 0$. These give us the points $(-1, 0)$, $(0, 4)$, $(2, 6)$ and $(4, 0)$, respectively. Using these as a guide, we can use Desmos to produce the graph below.[21]

Desmos link: `https://www.desmos.com/calculator/lamgypzhv3`

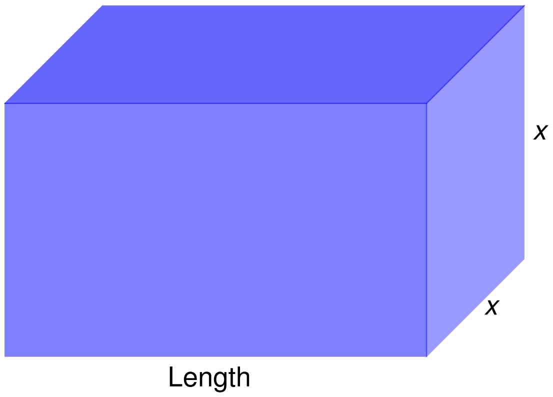
As nice as the graph is, it is still technically incomplete. There is no restriction stated on the independent variable t so the domain of h is all real numbers. However, the graph as presented shows only the behavior of h between roughly $t = -2.5$ and $t = 4.25$. By zooming out, we see that the graph extends downwards indefinitely which we could indicate by adding the arrows to the graph. Hence, we record the domain as $(-\infty, \infty)$ and the range as $(-\infty, 6.25]$. There is no minimum, but the maximum of h is 6.25 and it occurs at $t = 1.5$.

Our last example of the section uses the interplay between algebraic and graphical representations of a function to solve a real-world problem.

Example 6. The United States Postal Service mandates that when shipping parcels using 'Parcel Select' service, the sum of the length (the longest dimension) and the girth (the distance around the thickest part of the parcel perpendicular to the length) must not exceed 130 inches.[22] Suppose we wish to ship a rectangular box whose girth forms a square measuring x inches per side as shown below.

[21] The curve in this example is called a 'parabola'. In Section **??**, we'll learn how to graph these accurately **by hand**.

[22] See here.



Alt text: A rectangular box whose base has dimensions $x \times x$.

It turns out[23] that the volume of a box, $V(x)$, measured in cubic inches, whose length plus girth is exactly 130 inches is given by the formula: $V(x) = x^2(130 - 4x)$ for $0 < x \leq 26$.

1. Find and interpret $V(5)$.
2. Make a table of values and use these along with a graphing utility to graph $y = V(x)$.
3. What is the largest volume box that can be shipped? What value of x maximizes the volume? Round your answers to two decimal places.

Solution.

1. To find $V(5)$, we substitute $x = 5$ into the expression $V(x)$: $V(5) = (5)^2(130 - 4(5)) = 25(110) = 2750$. Our result means that when the length and width of the square measure 5 inches, the volume of the resulting box is 2750 cubic inches.[24]
2. The domain of V is specified by the inequality $0 < x \leq 26$, so we can begin graphing V by sampling V at finitely many x-values in this interval to help us get a sense of the range of V. This, in turn, will help us determine an adequate viewing window on our graphing utility when the time comes.

   It seems natural to start with what's happening near $x = 0$. Even though the expression $x^2(130 - 4x)$ is defined when we substitute $x = 0$ (it reduces very quickly to 0), it would be incorrect to state $V(0) = 0$ because $x = 0$ is not in the domain of V. However, there is nothing stopping us from evaluating $V(x)$ at values x 'very close' to $x = 0$. A table of such values is given below.

x	$V(x)$
0.1	1.296
0.01	0.012996
0.001	0.000129996
10^{-23}	$\approx 1.3 \times 10^{-44}$

   There is no such thing as a 'smallest' positive number,[25] so we will have points on the graph of V to the right of $x = 0$ leading to the point $(0, 0)$. We indicate this behavior by putting a hole at $(0, 0)$.[26]

[23] We'll skip the explanation for now because we want to focus on just the different representations of the function. Rest assured, you'll be asked to construct this very model in Exercise **??** in Section **??**.

[24] Note that we have $V(5)$ and 25(110) in the same string of equality. The first set of parentheses is function notation and directs us to substitute 5 for x in the expression $V(x)$ while the second indicates multiplying 25 by 110. Context is key!

[25] If p is any positive real number, $0 < 0.5p < p$, so we can always find a smaller positive real number.

[26] What's really needed here is the precise definition of 'closeness' discussed in Calculus. This hand-waving will do for now.

Moving forward, we start with $x = 5$ and sample V at steps of 5 in its domain. Our goal is to graph $y = V(x)$, so we list the ordered pairs $(x, V(x))$. The right endpoint, $x = 26$, is included in the domain $0 < x \leq 26$, so we end our table with $(26, V(26)) = (26, 17{,}576)$.

x	$V(x)$	$(x, V(x))$
≈ 0	≈ 0	hole at $(0, 0)$
5	2750	$(5, 2750)$
10	9000	$(10, 9000)$
15	15,750	$(15, 15{,}750)$
20	20,000	$(20, 20{,}000)$
25	18,750	$(25, 18{,}750)$
26	17,576	$(26, 17{,}576)$

To graph the points, $(x, V(x))$, we use the domain as a guide to help us set the horizontal bounds (i.e., the bounds on x) and the sample values from the range to help us set the vertical bounds (i.e., the bounds on y). With some help from Desmos, we create the graph below.

Desmos link: `https://www.desmos.com/calculator/vq29svps5g`

3. The largest volume in this case refers to the maximum of V. The biggest y-value in our table of data is 20,000 cubic inches which occurs at $x = 20$ inches, but the graph produced by Desmos indicates that there are points on the graph of V with y-values (hence $V(x)$ values) greater than 20,000. Indeed, the graph continues to rise to the right of $x = 20$ and the graphing utility reports the maximum y-value to be $y \approx 20{,}342.593$ when $x \approx 21.667$.

   Desmos link: `https://www.desmos.com/calculator/eaijwrnfa5`

   Rounding to two decimal places, we find the maximum volume obtainable under these conditions is about 20,342.59 cubic inches which occurs when the length and width of the square side of the box are approximately 21.67 inches.[27]

   It is worth noting that while the function V has a maximum, it did not have a minimum. Even though $V(x) > 0$ for all x in its domain,[28] the presence of the hole at $(0, 0)$ means that 0 is not in the range of V. Hence, based on our model, we can never make a box with a 'smallest' volume.[29]

Example 6 typifies the interplay between Algebra and Geometry which lies ahead. Both the algebraic description of V: $V(x) = x^2(130 - 4x)$ for $0 < x \leq 26$, and the graph of $y = V(x)$ were useful in describing aspects of the physical situation at hand. Wherever possible, we'll use the algebraic representations of functions to **analytically** produce **exact** answers to certain problems and use the graphical descriptions to check the reasonableness of our answers.

That being said, we'll also encounter problems which we simply **cannot** answer analytically (such as determining the maximum volume in the previous example), so we will be forced to resort to using technology (specifically graphing technology) in order to find **approximate** solutions. The most important thing to keep in mind is that while technology may **suggest** a result, it is ultimately Mathematics that **proves** it.

We close this section with a summary of the different ways to represent functions.

Ways to Represent a Function

Suppose f is a function with domain A. Then f can be represented:

[27] We could also find the length of the box in this case as well. The sum of length and girth is 130 inches so the length is 130 minus the girth, or $130 - 4x \approx 130 - 4(21.67) = 43.32$ inches.

[28] said differently, the values of $V(x)$ are **bounded below** by 0.

[29] How realistic is this?

- verbally; that is, by describing how the inputs are matched with their outputs.
- using a mapping diagram.
- as a set of ordered pairs of the form (input, output): $\{(a, f(a)) \mid a \in A\}$.

If f is a function whose domain and range are subsets of real numbers, then f can be represented:

- algebraically as a formula for $f(a)$.
- graphically by plotting the points $\{(a, f(a)) \mid a \in A\}$ in the plane.

Note: An important consequence of the last bulleted item is that the point (a, b) is on the graph of $y = f(x)$ if and only if $f(a) = b$.