LINEAR FUNCTIONS

Now that we've discussed the functions which correspond to horizontal lines, $y = b$, we move to discussing the functions which can be represented by lines of the form $y = mx + b$ where $m \neq 0$. These functions are called **linear** functions and are described below.

Definition 1. A **linear function** is a function of the form

$$f(x) = mx + b,$$

where m and b are real numbers with $m \neq 0$. The domain of a linear function is $(-\infty, \infty)$.

As with Definition **??**, in Definition **??**, x is the independent variable, f is the function name, and both m and b are parameters. Notice that m is restricted by $m \neq 0$ for if $m = 0$ then the function $f(x) = mx + b$ would reduce to the constant function $f(x) = b$. The domain of linear functions, like that of constant functions, is specified as $(-\infty, \infty)$

Recall[1] that the form of the line $y = mx + b$ is called the slope-intercept form of the line and the slope, m, and the y-intercept $(0, b)$, are easily determined when the line is written this way. Likewise, the form of the function in Definition **??**, $f(x) = mx + b$, is often called the **slope-intercept form** of a linear function.

The graph of a linear function is the graph of the line $y = mx + b$. Lines are uniquely determined by two points, and two points of geometric interest are the axis intercepts. We've already reminded you of the y-intercept, $(0, b)$, which is obtained by setting $x = 0$. Similarly, to find the x-intercept, we set $y = 0$ and solve $mx + b = 0$ for x. We leave this to the reader in Exercise **??**. In addition to having special graphical significance, axis intercepts quite often play important roles in applications involving both linear and non-linear functions. For that reason, we take the time to define them here using function notation.

Definition 2. Suppose f is a function represented by the graph of $y = f(x)$.

- If 0 is in the domain of f then the point $(0, f(0))$ is the **y-intercept** of the graph of $y = f(x)$.

  That is, $(0, f(0))$ is where the graph meets the y-axis.

- If 0 is in the range of f then the solutions to $f(x) = 0$ are called the **zeros** of f. If c is a zero of f then the point $(c, 0)$ is an **x-intercept** of the graph of $y = f(x)$.

  That is, $(c, 0)$ is where the graph meets the x-axis.

As is customary in this text, Definition **??** uses the default independent variable x, function name f, and dependent variable y, so these letters will change depending on the context. Also note that the 'zeros' of a function are the solutions to $f(x) = 0$ - so they are **real numbers**. The x-intercepts are, on the other hand, **points** on the graph. As a quick example, consider $f(x) = x - 3$. The zeros of f are found by solving $f(x) = 0$, or $x - 3 = 0$. We get one solution, $x = 3$. Therefore, $x = 3$ is the **zero** of f that corresponds graphically to the x-**intercept** $(3, 0)$.

We now turn our attention to slope. The role of slope, or more generally a 'rate of change', in Science and Mathematics cannot be overstated.[2] As you may recall, or quickly read about on page **??**, the slope of a line that has been graphed in the xy-plane is defined geometrically as follows:

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x},$$

where the capital Greek letter 'Δ' denotes 'change in.'[3] In this course, it is vital that we regard the slope of a linear function as a rate of change of **function outputs** to **function inputs**. That is, given the graph of a linear function $y = f(x) = mx + b$:

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{\Delta[f(x)]}{\Delta x} = \frac{\Delta\text{outputs}}{\Delta\text{inputs}}.$$

[1] or see Section **??**

[2] The first half of any introductory Calculus course is about slope.

[3] More specifically, if (x_0, y_0) and (x_1, y_1) are two distinct points in the plane, then $\Delta x = x_1 - x_0$ and $\Delta y = y_1 - y_0$.

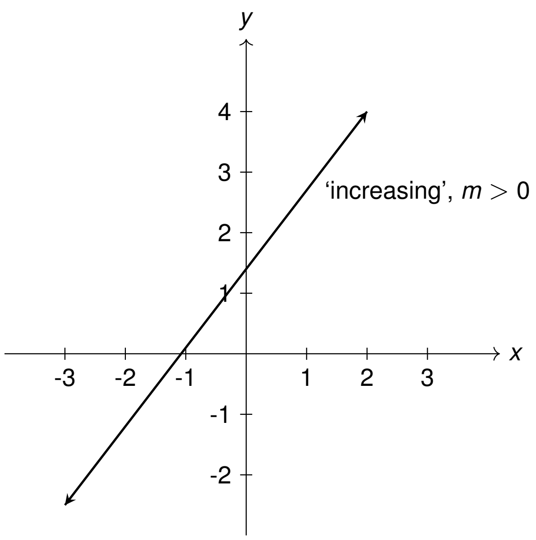
What is important to note here is that for linear functions, the rate of change m is constant for all values in the domain.[4] We'll see the importance of this statement in the upcoming examples.

Geometrically, the sign of the slope has a profound impact on the graph of the line. Recall that if the slope $m > 0$, the line rises as we read from left to right; if $m < 0$, the line falls as we read from left to right; if $m = 0$, we have a horizontal line and the graph plateaus. We define these notions more precisely for general functions in the following definition.

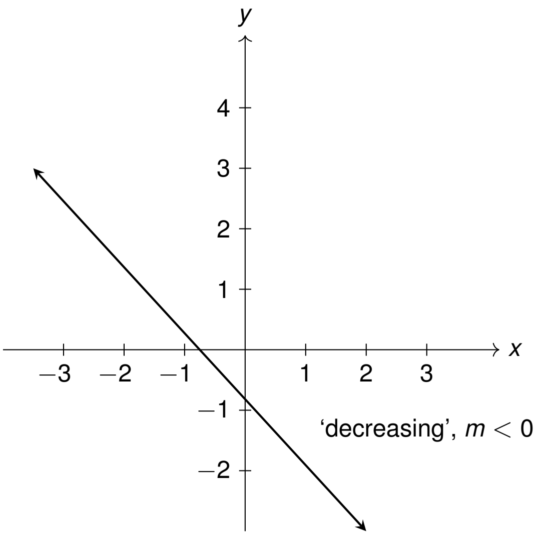
Definition 3. Let f be a function defined on an interval I. Then f is said to be:

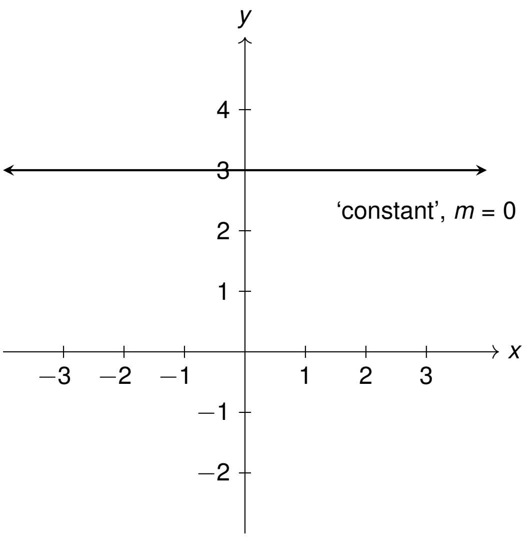
- **increasing** on I if, whenever $a < b$, then $f(a) < f(b)$. (i.e., as inputs increase, outputs **increase**.)
  NOTE: The graph of an increasing function **rises** as one moves from left to right.
- **decreasing** on I if, whenever $a < b$, then $f(a) > f(b)$. (i.e., as inputs increase, outputs **decrease**.)
  NOTE: The graph of a decreasing function **falls** as one moves from left to right.
- **constant** on I if $f(a) = f(b)$ for all a, b in I. (i.e., outputs don't change with inputs.)
  NOTE: The graph of a function that is constant over an interval is a horizontal line.

Again, as with Definition **??**, Definition **??** applies to any function, not just linear and constant functions. Also, note that, like Definition **??**, Definition **??** blurs the line between the function, f, and its outputs, $f(x)$, because the verbiage 'f is increasing' is really a statement about the outputs, $f(x)$. Finally, when we ask 'where' a function is increasing, decreasing or constant, we are looking for an interval of **inputs**. We'll have more to say about this in later sections, but for now, we summarize these ideas graphically below.



[4]See Exercise **??** for more details.





From the graphs above, we see that regardless if $m > 0$ or $m < 0$, the range of linear functions is $(-\infty, \infty)$. Therefore, linear functions have no maximum or minimum.[5]

Example 1. The cost, in dollars, to produce x PortaBoy[6] game systems for a local retailer is given by $C(x) = 80x + 150$ for $x \geq 0$.

1. Find and interpret $C(0)$ and $C(5)$ and use these to graph $y = C(x)$.
2. Explain the significance of the restriction on the domain, $x \geq 0$.
3. Interpret the slope of $y = C(x)$ geometrically and as a rate of change.
4. How many PortaBoys can be produced for $15,000?

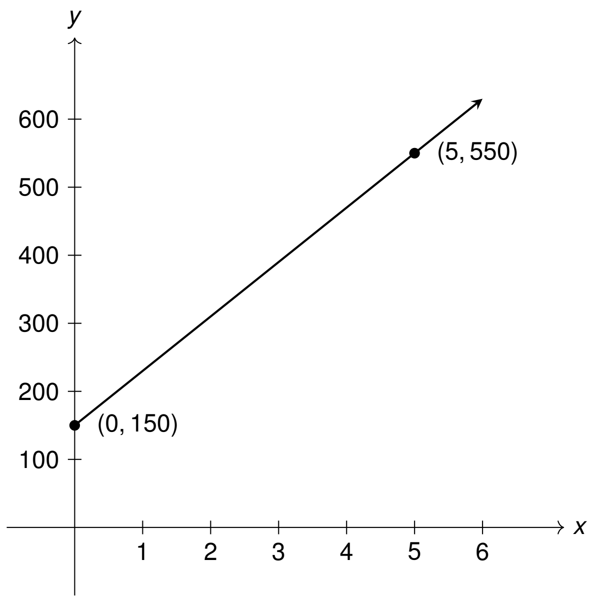
Solution.

1. To find $C(0)$, we substitute 0 for x in the formula $C(x)$ and obtain: $C(0) = 80(0) + 150 = 150$. Given that x represents the number of PortaBoys produced and $C(x)$ represents the cost to produce said

[5]This is one of the more pedantic reasons why we distinguish between constant and linear functions. See the discussion concerning the range of a constant function on page **??**.

[6]The similarity of this name to PortaJohn is deliberate.

PortaBoys, $C(0) = 150$ means it costs \$150 even if we don't produce any PortaBoys at all. At first, this may not seem realistic, but that \$150 is often called the **fixed** or **start-up** cost of the venture. Things like re-tooling equipment, leasing space, or any other 'up front' costs get lumped into the fixed cost. To find $C(5)$, we substitute 5 for x in the formula $C(x)$: $C(5) = 80(5) + 150 = 550$. This means it costs \$550 to produce 5 PortaBoys for the local retailer. These two computations give us two points on the graph: $(0, C(0))$ and $(5, C(5))$. Along with the domain restriction $x \geq 0$, we get:



$y = C(x)$

2. In this context, x represents the number of PortaBoys produced. It makes no sense to produce a negative quantity of game systems,[7] so $x \geq 0$.

3. The cost function $C(x) = 80x+150$ is in slope-intercept form so we recognize the slope as the coefficient of x, $m = 80$. With $m > 0$, the function C is always increasing. This means that it costs more money to make more game systems. To interpret the slope as a rate of change, we note that the output, $C(x)$, is the cost in dollars, while the input, x, is the number of PortaBoys produced:

$$m = 80 = \frac{80}{1} = \frac{\Delta[C(x)]}{\Delta x} = \frac{\$80}{1 \text{ PortaBoy produced}}.$$

Hence, the cost to produce PortaBoys is increasing at a rate of \$80 per PortaBoy produced. This is often called the **variable cost** for the venture.

4. To find how many PortaBoys can be produced for \$15,000, we solve $C(x) = 15000$, which means $80x + 150 = 15000$. This yields $x = 185.625$. We can produce only a whole number amount of PortaBoys so we are left with two options: produce 185 or 186 PortaBoys. Given that $C(185) = 14950$ and $C(186) = 15030$, we would be over budget if we produced 186 PortaBoys. Hence, we can produce 185 PortaBoys for \$15,000 (with \$50 to spare).

A couple of remarks about Example **??** are in order. First, if x represents the number of PortaBoy game systems being produced, then x can really only take on whole number values. We will revisit this scenario in Section **??** where we will see how the approach presented here allows us to use more elegant techniques when analyzing the situation than a discrete data set would allow.[8]

[7] Actually, it makes no sense to produce a fractional part of a game system, either, which we'll discuss later in this example.

[8] This is an example of using a 'continuous' variable to model a 'discrete' scenario. Contrast this with the discussion following Example **??** in Section **??**.

Second, once we know that the variable cost is \$80 per PortaBoy, we can revisit a computation we did earlier in the example. We computed $C(185) = 14950$ and needed to compute $C(186)$. With 186 being just one more PortaBoy than 185, we can use the variable cost to get

$$C(186) = C(185) + 80(1) = 14950 + 80 = 15030,$$

which agrees with our earlier computation.[9] If we wanted to find $C(300)$, we could do something similar. Using $300 - 185 = 115$, we can find $C(300)$ as follows:

$$C(300) = C(185) + 80(115) = 14950 + 9200 = 24150.$$

In general, we could rewrite $C(x) = C(185) + 80(x - 115)$. This same reasoning shows that for any x_0 in the domain of C, we have $C(x) = C(x_0) + 80(x - x_0)$ - a fact we invite the reader to verify.[10]

Indeed, the computations above are at the heart of what it means to be a linear function: linear functions change at a constant rate known as the slope. To better see this algebraically, recall that given a point (x_0, y_0) on a line along with the slope, m, the **point-slope form of the line** is: $y - y_0 = m(x - x_0)$.[11] Rewriting, we get $y = y_0 + m(x - x_0)$ and setting $y = f(x)$ and $y_0 = f(x_0)$ yields:

Formula 1. The **point-slope form** of a linear function is

$$f(x) = f(x_0) + m(x - x_0)$$

A few remarks are in order. First note that if the point $(x_0, f(x_0))$ is the y-intercept $(0, b)$, Equation **??** immediately reduces to the slope-intercept form of the line: $f(x) = f(x_0) + m(x - x_0) = b + m(x - 0) = mx + b$, so you can use Equation **??** exclusively from this point forward.[12]

Second, if we write $\Delta x = x - x_0$, then $x = x_0 + \Delta x$ so we can rewrite Equation **??** as follows:

$$\begin{array}{ccccc} f(x_0 + \Delta x) & = & f(x_0) & + & m\Delta x \\ \text{(new output)} & = & \text{(known output)} & + & \text{(change in outputs)} \end{array}$$

In other words, changing the **input** by Δx results in changing the **output** by $m\Delta x$. This tracks since

$$m\Delta x = \frac{\Delta[f(x)]}{\Delta x}\Delta x = \Delta[f(x)] = \Delta\text{outputs}.$$

The fact that we can write $\Delta\text{outputs} = m\Delta x$ for any choice of x_0 is another way to see that for linear functions, the rate of change is constant. That is, the rate of change, m, is the same for all values x_0 in the domain. We'll put Equation **??** to good use in the next example.

Example 2. The local retailer in Example **??** is trying to mathematically model the relationship between the number of PortaBoy systems sold and the price per system. Suppose 20 systems were sold when the price was \$220 per system but when the systems went on sale for \$190 each, sales doubled.

1. Find a formula for a linear function p which represents the price $p(x)$ as a function of the number of systems sold, x. Graph $y = p(x)$, find and interpret the intercepts, and determine a reasonable domain for p.

2. Interpret the slope of $p(x)$ in terms of price and game system sales.

3. If the retailer wants to sell 150 PortaBoys next week, what should the price be?

4. How many systems would sell if the price per system were set at \$150?

[9] The cost to produce 'just one more item' is called the **marginal cost**. The difference between variable and marginal costs in this case are the units used: the variable cost is \$80 per Portaboy whereas the marginal cost is simply \$80.

[10] In the case $x_0 = 0$, this formula reduces to $C(x) = C(0) + 80(x - 0) = 150 + 80x = 80x + 150$. To show the formula in general, consider $C(x_0) = 80x_0 + 150 \ldots$

[11] See Section **??** for a review of this form.

[12] In other words, the slope intercept form of a line is just a special case of the point-slope form.

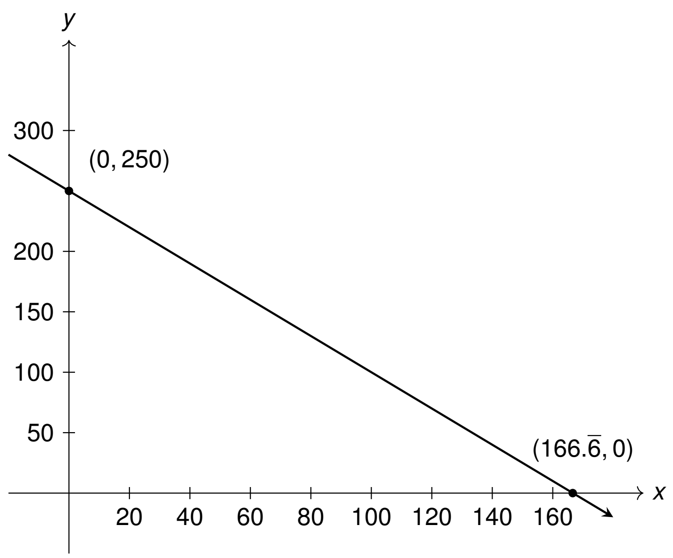
Explanation. 1. We are asked to find a linear function $p(x)$ ostensibly because the retailer has only two data points and two points are all that is needed to determine a unique line. We know that 20 PortaBoys were sold when the price was 220 dollars and double that, so 40 units, were sold when the price was 190 dollars. Using the language of function notation, these statements translate to $p(20) = 220$ and $p(40) = 190$, respectively. We first find the slope

$$m = \frac{\Delta[p(x)]}{\Delta x} = \frac{190 - 220}{40 - 20} = \frac{-30}{20} = -1.5$$

and then substitute it and a pair $(x_0, p(x_0))$ into the point-slope formula. We have two choices: $x_0 = 20$ and $p(x_0) = 220$ or $x_0 = 40$ and $p(x_0) = 190$. We'll choose the former and invite the reader to use the latter - both will result in the same simplified expression. The point-slope formula yields

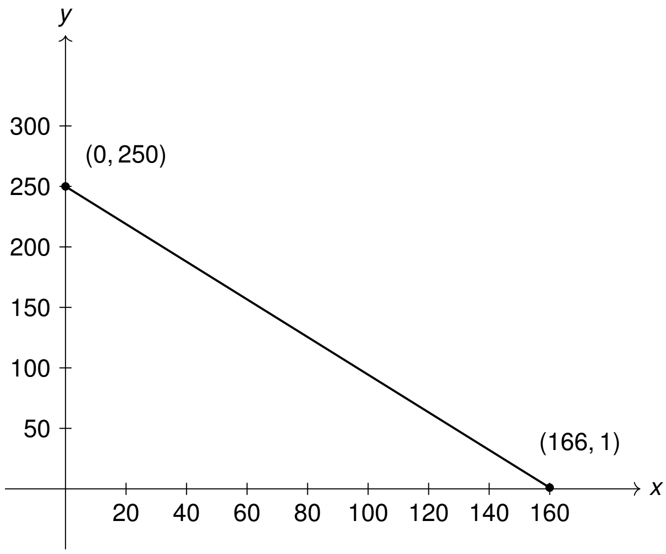
$$p(x) = p(x_0) + m(x - x_0) = 220 + (-1.5)(x - 40)$$

which simplifies to $p(x) = -1.5x + 250$. (To check this algebraically, we can verify that $p(20) = 220$ and $p(40) = 190$.) To find the y-intercept of the graph, we substitute $x = 0$ and find $p(0) = 250$. Hence our y-intercept is $(0, 250)$. To find the x-intercept, we set $p(x) = 0$. Solving $-1.5x + 250 = 0$ gives $x = 166.\overline{6}$, so our x-intercept is $(166.\overline{6}, 0)$.[13] The graph on the left is that of the line $y = -1.5x + 250$.



$y = -1.5x + 250$

[13] The exact value is $x = \frac{500}{3}$. Recall that the bar over the 6 indicates that the decimal repeats. See page ?? for details.



$y = p(x)$

To determine a reasonable domain for p, we certainly require $x \geq 0$, because we can't sell a negative number of game systems.[14] Next, we require $p(x) \geq 0$, otherwise we'd be **paying** customers to 'buy' PortaBoys. Solving $-1.5x + 250 \geq 0$ results in $x \leq 166.\overline{6}$. This shouldn't be too surprising since our graph passes through the x-axis at $(166.\overline{6}, 0)$, going from positive y-values (hence, positive $p(x)$ values) to negative y (hence negative $p(x)$ values).[15]

Given that x represents the number of PortaBoys sold, we need to choose to end the domain at either $x = 166$ or $x = 167$. We have that $p(166) = 1 > 0$ but $p(167) = -0.5 < 0$ so we settle on the domain $[0, 166]$. Our final answer is $p(x) = -1.5x + 250$ restricted to $0 \leq x \leq 166$ which is graphed above on the right.

2. The slope $m = -1.5$ represents the rate of change of the price of a system with respect to sales of PortaBoys. The slope is negative so we have that the price is **decreasing** at a rate of \$1.50 per PortaBoy sold. (Said differently, you can sell one more PortaBoy for every \$1.50 drop in price.)

3. To determine the price which will move 150 PortaBoys, we find $p(150) = -1.5(150) + 250 = 25$. That is, the price would have to be \$25 per system.

4. If the price of a PortaBoy were set at \$150, we'd have $p(x) = 150$, or $-1.5x + 250 = 150$. This yields $-1.5x = -100$ or $x = 66.\overline{6}$. Again our algebraic solution lies between two whole numbers, so we find $p(66) = 151$ and $p(67) = 149.5$. If the price were set at \$150, we'd sell 66 systems, since to sell 67 systems, we'd have to drop the price just under \$150.

The function p in Example **??** is called the **price-demand** function (or, sometimes called more simply a 'demand function') because it returns the price $p(x)$ associated with a certain demand x - that is, how many products will sell.[16] These functions, along with cost functions like the one in Example **??**, will be revisited in Example **??**.

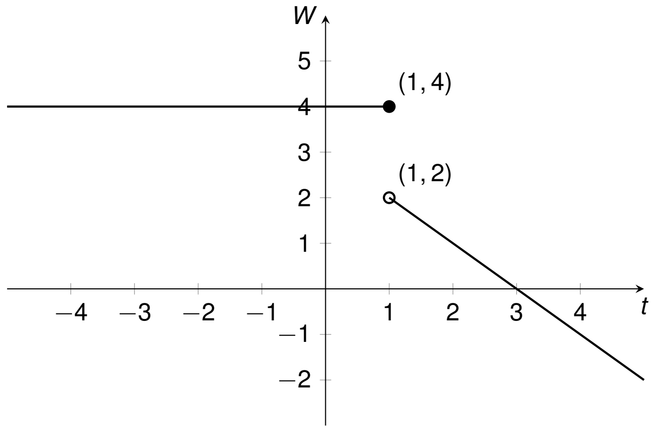
Our next two examples focus on writing formulas for piecewise-defined functions, the second of which models a real-world situation.

Example 3. Find a formula for the function L graphed below.

[14] ignoring returns, that is.

[15] We'll discuss these sorts of connections in greater depth in Section **??**.

[16] It may seem counter-intuitive to express price as a function of demand. Shouldn't the price determine how many systems people will buy? We will address this issue later.



Explanation. From the graph of $W = L(t)$ we see that there are two distinct pieces. Taking note of the point at $(1,4)$, we get $L(t) = 4$ for $t \leq 1$. To represent L for $t > 1$, we use the point-slope form of a linear function: $L(t) = L(t_0) + m(t - t_0)$. The only 'point' labeled with this part of the graph is the hole at $(1,2)$ and it isn't technically part of the graph, so we will avoid using it.[17] Instead, we infer from the graph two other points: $(2,1)$ and $(3,0)$. We get the slope to be

$$m = \frac{\Delta W}{\Delta t} = \frac{\Delta[L(t)]}{\Delta t} = \frac{3-2}{0-1} = -1.$$

Next, we choose a point to plug into $L(t) = L(t_0) + m(t - t_0)$. We have two options: $t_0 = 2$ and $L(t_0) = 1$ or $t_0 = 3$ and $L(t_0) = 0$. Using the latter, we get $L(t) = 0 + (-1)(t - 3)$, or $L(t) = -t + 3$. Putting this together with the first part, we get:

$$L(t) = \begin{cases} 4 & \text{if} \quad t \leq 1 \\ -t+3 & \text{if} \quad t > 1 \end{cases}$$

Note that when $t = 1$ is substituted into the expression $-t + 3$, we get 2, so the hole at $(1,2)$ checks.[18]

Example 4. A popular Fōn-i smartphone carrier offers the following smartphone data plan: use any amount of data up to and including 4 gigabytes for \$60 per month with an 'overage' charge of \$5 per gigabyte. Determine a formula that computes the cost in dollars as a function of using g gigabytes of data per month. Graph your answer.

Explanation. It is clear from context that we are to use the variable g (for 'g'igabytes) as the independent variable. We are asked to compute the cost so it seems natural to name the function C. Hence, we are after a formula for $C(g)$. Knowing that g represents the amount of data used each month, we must have $g \geq 0$. In order to get a feel for the formula for $C(g)$, we can choose some specific values for g and determine the cost, $C(g)$. For example, if we use no data at all, 1 gigabyte of data, or 3.796 gigabytes of data, the cost is the same: \$60. Indeed, per the plan, for any amount of data up to and including 4 gigabytes, the cost is \$60.

Translating this to function notation means $C(0) = 60$, $C(1) = 60$, $C(3.796) = 60$, and, in general, $C(g) = 60$ for $0 \leq g \leq 4$. What happens if we use more than 4 gigabytes? Let's say we use 6 gigabytes. Per the plan, we are charged \$60 for the first 4 and then \$5 for each gigabyte over 4. Using 6 gigabytes means that we are 2 gigabytes over and our overage charge is (\$5)(2) = \$10. The total cost is the base plus the overages or \$60 + \$10 = \$70. In general, if $g > 4$, the expression $(g - 4)$ computes the amount of data used over 4 gigabytes. Our base plus overage then comes to: $60 + 5(g - 4) = 5g + 40$. Putting this together with our previous work, we get

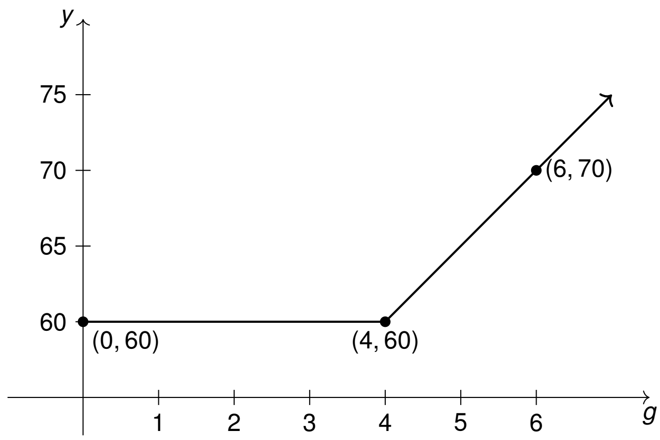
$$C(g) = \begin{cases} 60 & \text{if} \quad 0 \leq g \leq 4 \\ 5g+40 & \text{if} \quad g > 4 \end{cases}$$

[17] We actually could use the point $(1,2)$ to find the equation of the **line** containing $(1,2)$ and, say $(3,0)$, which is $y = -t + 3$. It's just that the graph of $L(t)$ and the line $y = -t + 3$ only agree for $t > 1$, so it would be incorrect to write $L(1) = 2$.

[18] Alternatively, for t values larger than 1 but getting closer and closer to 1, $L(t) \approx 2$.

To graph C, we graph $y = C(g)$. For $0 \leq g \leq 4$, we have the horizontal line $y = 60$ from $(0, 60)$ to $(4, 60)$. For $g > 4$, we have the line $y = 5g + 40$. Even though the inequality $g > 4$ is strict, we nevertheless substitute $g = 4$ into the formula $y = 5g + 40$ and get $y = 60$. Normally, this would produce a hole at $(4, 60)$, but in this case, the point $(4, 60)$ is already on the graph from the first piece of the function. Essentially, the point $(4, 60)$ from $C(g) = 60$ for $0 \leq g \leq 4$ 'plugs' the hole from $C(g) = 5g + 40$ when $g > 4$.

We are graphing a line so we need to plot just one more point to determine the graph. From our work above, we know $C(6) = 70$, so we use $(6, 70)$ as our second point. Our graph is below. As with the graphs shown on page **??** from Example **??**, we use '≍' to denote a break in the vertical axis in order to better display the graph.



$y = C(g)$