1 Graphs of Quadratic Functions

You may recall studying quadratic equations in a previous Algebra course. If not, you may wish to refer to **??** to revisit this topic. In this section, we review these equations in the context of our next family of functions: the quadratic functions.

Definition 1. A **quadratic function** is a function of the form

$$f(x) = ax^2 + bx + c,$$

where a, b and c are real numbers with $a \neq 0$. The domain of a quadratic function is $(-\infty, \infty)$.

The independent variable in Definition **??** is x while the values a, b and c are parameters. Note that $a \neq 0$ - otherwise we would have a LINEAR FUNCTION. Alt text: A **linear function** is a function of the form

$$f(x) = mx + b,$$

where m and b are real numbers with $m \neq 0$.

The most basic quadratic function is $f(x) = x^2$, the squaring function. Using a table of values, we construct the graph below.

x	$f(x) = x^2$
-2	4
$-\frac{3}{2}$	$\frac{9}{4}$
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4

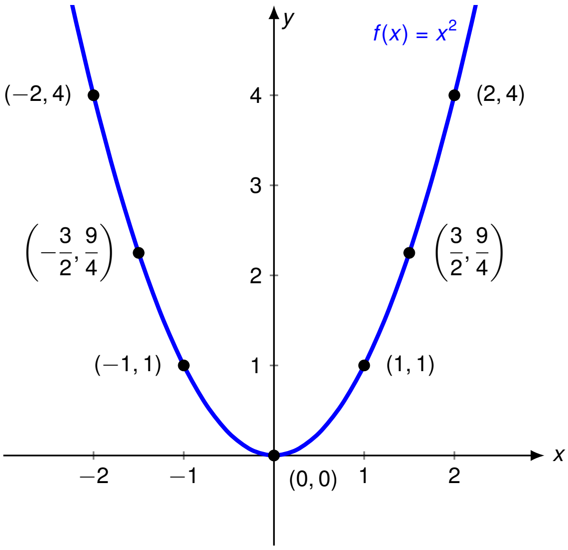


Figure 1: The caption: a parabola

Alt text: The alt text: a blue parabola, with marked points

Its shape may look familiar from your previous studies in Algebra – it is called a **parabola**. The point $(0, 0)$ is called the **vertex** of the parabola because it is the sole point where the function obtains its extreme value, in this case, a minimum of 0 when $x = 0$.

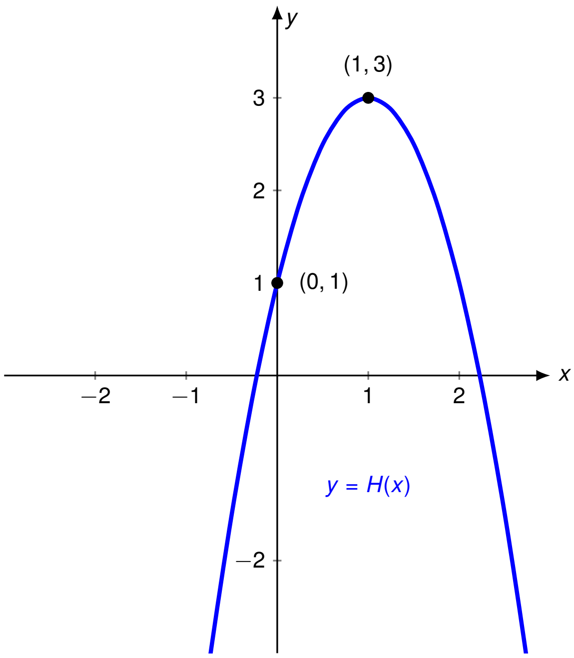
Indeed, the range of $f(x) = x^2$ appears to be $[0, \infty)$ from the graph. We can substantiate this algebraically since for all x, $f(x) = x^2 \geq 0$. This tells us that the range of f is a subset of $[0, \infty)$. To show that the range of f actually equals $[0, \infty)$, we need to show that every real number c in $[0, \infty)$ is in the range of f. That is, for every $c \geq 0$, we have to show c is an output from f. In other words, we have to show there is a real number x so that $f(x) = x^2 = c$. Choosing $x = \sqrt{c}$, we find $f(x) = f(\sqrt{c}) = (\sqrt{c})^2 = c$, as required.[1]

The techniques we used to graph many of the absolute value functions in Section **??** can be applied to quadratic functions, too. In fact, knowing the graph of $f(x) = x^2$ enables us to graph **every** quadratic function, but there's some extra work involved. We start with the following theorem:

Theorem 1. For real numbers a, h and k with $a \neq 0$, the graph of $F(x) = a(x - h)^2 + k$ is a parabola with vertex (h, k). If $a > 0$, the graph resembles '$\smile$.' If $a < 0$, the graph resembles '$\frown$.' Moreover, the vertical line $x = h$ is the **axis of symmetry** of the graph of $y = F(x)$.

To prove Theorem **??** the reader is encouraged to revisit the discussion following the proof of Theorem **??**, replacing every occurrence of absolute value notation with the squared exponent.[2] Alternatively, the reader can skip ahead and read the statement and proof of Theorem **??** in Section **??**. In the meantime we put Theorem **??** to good use in the next two examples.

Example 1. Use Theorem **??** to write a possible formula for $H(x)$ whose graph is given below:



Explanation. We are instructed to use Theorem **??**, so we know $H(x) = a(x - h)^2 + k$ for some choice of parameters a, h and k. The vertex is $(1, 3)$ so we know $h = 1$ and $k = 3$, and hence $H(x) = a(x - 1)^2 + 3$. To find the value of a, we use the fact that the y-intercept, as labeled, is $(0, 1)$. This means $H(0) = 1$, or $a(0 - 1)^2 + 3 = 1$. This reduces to $a + 3 = 1$ or $a = -2$. Our final answer[3] is $H(x) = -2(x - 1)^2 + 3$.

Example 2.

Graph the following functions using Theorem **??**. Find the vertex, zeros and axis-intercepts (if any exist). Find the extrema and then list the intervals over which the function is increasing, decreasing or constant.

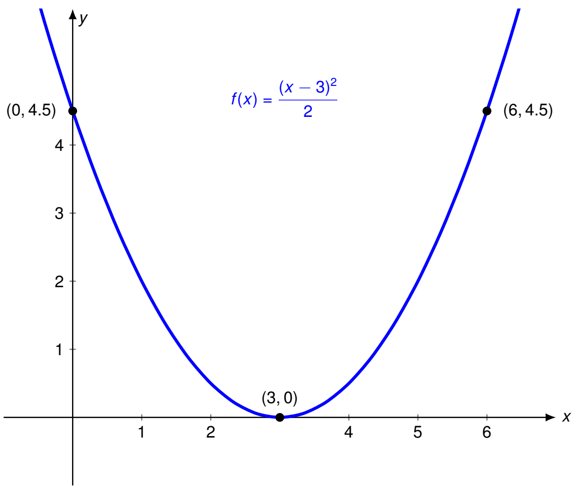
[1] We'll revisit this argument (with an inequality) in Section **??**.

[2] i.e., replace $|x|$ with x^2, $|c|$ with c^2, $|x - h|$ with $(x - h)^2$.

[3] The reader is encouraged to compare this example with number **??** of Example **??**.

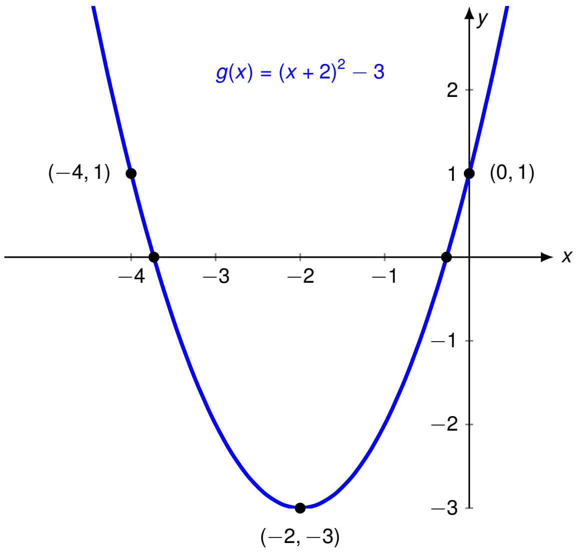
1. $f(x) = \dfrac{(x-3)^2}{2}$
2. $g(x) = (x+2)^2 - 3$
3. $h(t) = -2(t-3)^2 + 1$
4. $i(t) = \dfrac{(3-2t)^2+1}{2}$

Explanation. 1. For $f(x) = \dfrac{(x-3)^2}{2} = \dfrac{1}{2}(x-3)^2 + 0$, we identify $a = \dfrac{1}{2}$, $h = 3$ and $k = 0$. Thus the vertex is $(3,0)$ and the parabola opens upwards. The only x-intercept is $(3,0)$. Since $f(0) = \dfrac{1}{2}(0-3)^2 = \dfrac{9}{2}$, our y-intercept is $\left(0, \dfrac{9}{2}\right)$. To help us graph the function, it would be nice to have a third point and we'll use symmetry to find it. The y-value three units to the **left** of the vertex is 4.5, so the y-value must be 4.5 three units to the **right** of the vertex as well. Hence, we have our third point: $\left(6, \dfrac{9}{2}\right)$ and our graph below.



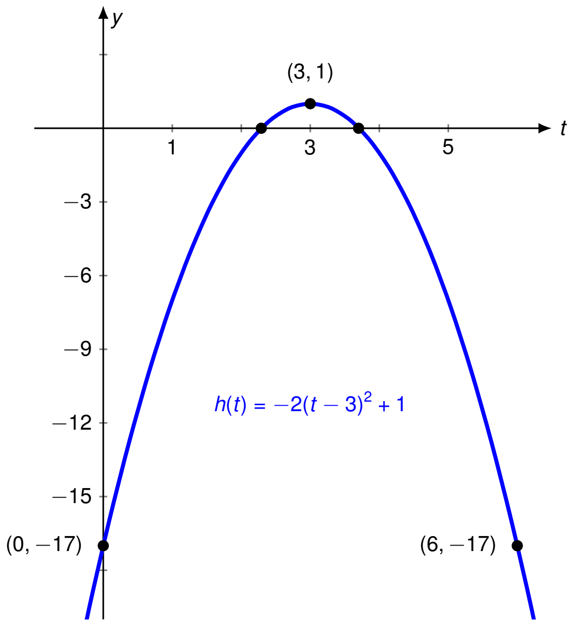
From the graph, we get that the range is $[0,\infty)$ and see that f has the minimum value of 0 at $x = 3$ and no maximum. Also, f is decreasing on $(-\infty, 3]$ and increasing on $[3,\infty)$.

2. For $g(x) = (x+2)^2 - 3 = (1)(x-(-2))^2 + (-3)$, we identify $a = 1$, $h = -2$ and $k = -3$. This means that the vertex is $(-2,-3)$ and the parabola opens upwards. Thus we have two x-intercepts. To find them, we set $y = g(x) = 0$ and solve. Doing so yields the equation $(x+2)^2 - 3 = 0$, or $(x+2)^2 = 3$. Extracting square roots gives us the two zeros of g: $x + 2 = \pm\sqrt{3}$, or $x = -2 \pm \sqrt{3}$. Our x-intercepts are $(-2-\sqrt{3}, 0) \approx (-3.73, 0)$ and $(-2+\sqrt{3}, 0) \approx (-0.27, 0)$. We find $g(0) = (0+2)^2 - 3 = 1$ so our y-intercept is $(0,1)$. Using symmetry, we get $(-4,1)$ as another point to help us graph.



The range of g is $[-3, \infty)$. The minimum of g is -3 at $x = -2$, and g has no maximum. Moreover, g is decreasing on $(-\infty, -2]$ and g is increasing on $[-2, \infty)$.

3. Given $h(t) = -2(t-3)^2 + 1$, we identify $a = -2$, $h = 3$ and $k = 1$. Hence the vertex of the graph is $(3, 1)$ and the parabola opens downwards. Solving $h(t) = -2(t-3)^2 + 1 = 0$ gives $(t-3)^2 = \frac{1}{2}$. Extracting square roots[4] gives $t - 3 = \pm\frac{\sqrt{2}}{2}$, so that when we add 3 to each side,[5] we get $t = \frac{6 \pm \sqrt{2}}{2}$. Hence, our t-intercepts are $\left(\frac{6-\sqrt{2}}{2}, 0\right) \approx (2.29, 0)$ and $\left(\frac{6+\sqrt{2}}{2}, 0\right) \approx (3.71, 0)$. To find the y-intercept, we compute $h(0) = -2(0-3)^2 + 1 = -17$. Thus the y-intercept is $(0, -17)$. Using symmetry, we also have that $(6, -17)$ is on the graph.



[4] and rationalizing denominators!

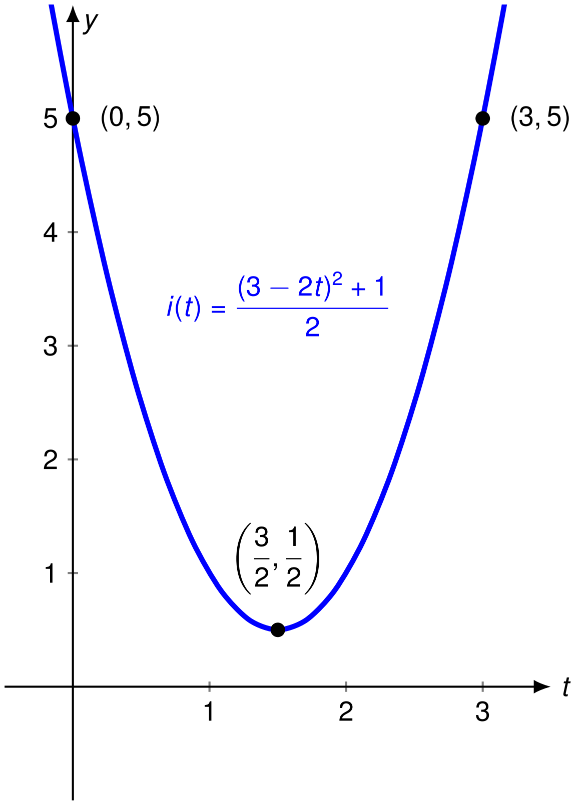
[5] and get common denominators!

From the graph, we get the range of h is $(-\infty, 1]$. The maximum of h is 1 at $x = 3$, and h has no minimum. We see h is increasing on $(-\infty, 3]$ and h is decreasing on $[3, \infty)$.

4. We have some work ahead of us to put $i(t)$ into a form we can use to exploit Theorem **??**:

$$\begin{aligned} i(t) = \frac{(3-2t)^2+1}{2} &= \frac{1}{2}(-2t+3)^2 + \frac{1}{2} &&= \frac{1}{2}\left[-2\left(t-\frac{3}{2}\right)\right]^2 + \frac{1}{2} \\ &= \frac{1}{2}(-2)^2\left(t-\frac{3}{2}\right)^2 + \frac{1}{2} &&= 2\left(t-\frac{3}{2}\right)^2 + \frac{1}{2} \end{aligned}$$

We identify $a = 2$, $h = \frac{3}{2}$ and $k = \frac{1}{2}$. Hence our vertex is $\left(\frac{3}{2}, \frac{1}{2}\right)$ and the parabola opens upwards, meaning there are no t-intercepts. Since $i(0) = \frac{(3-2(0))^2+1}{2} = 5$, we get $(0, 5)$ as the y-intercept. Using symmetry, this means we also have $(3, 5)$ on the graph.



The range is $\left[\frac{1}{2}, \infty\right)$ with the minimum of i, $\frac{1}{2}$, occurring when $t = \frac{3}{2}$. Also, i is decreasing on $\left(-\infty, \frac{3}{2}\right]$ and increasing on $\left[\frac{3}{2}, \infty\right)$.

A few remarks about Example **??** are in order. First note that none of the functions are in the form of Definition **??**. However, if we took the time to perform the indicated operations and simplify, we'd find:

- $f(x) = \frac{(x-3)^2}{2} = \frac{1}{2}x^2 - 3x + \frac{9}{2}$
- $g(x) = (x+2)^2 - 3 = x^2 + 4x + 1$
- $h(t) = -2(t-3)^2 + 1 = -2t^2 + 12t - 17$
- $i(t) = \frac{(3-2t)^2+1}{2} = 2t^2 - 6t + 5$

While the y-intercepts of the graphs of the each of the functions are easier to see when the formulas for the functions are written in the form of Definition **??**, the vertex is not. For this reason, the form of the functions presented in Theorem **??** are given a special name.

Definition 2. General and Vertex Form of Quadratic Functions:

- The **general form** of the quadratic function f is $f(x) = ax^2 + bx + c$, where a, b and c are real numbers with $a \neq 0$.
- The **vertex form** of the quadratic function f is $f(x) = a(x - h)^2 + k$, where a, h and k are real numbers with $a \neq 0$.

If we proceed as in the remarks following Example **??**, we can convert any quadratic function given to us in vertex form and convert to general form by performing the indicated operation and simplifying:

$$\begin{aligned} f(x) &= a(x-h)^2 + k \\ &= a\left(x^2 - 2hx + h^2\right) + k \\ &= ax^2 - 2ahx + ah^2 + k \\ &= ax^2 + (-2ah)x + (ah^2 + k). \end{aligned}$$

With the identifications $b = -2ah$ and $c = ah^2 + k$, we have written $f(x)$ in the form $f(x) = ax^2 + bx + c$. Likewise, through a process known as 'completing the square', we can take any quadratic function written in general form and rewrite it in vertex form. We briefly review this technique in the following example – for a more thorough review the reader should see Section **??**.

Example 3. Graph the following functions. Find the vertex, zeros and axis-intercepts, if any exist. Find the extrema and then list the intervals over which the function is increasing, decreasing or constant.

1. $f(x) = x^2 - 4x + 3$.
2. $g(t) = 6 - 4t - 2t^2$

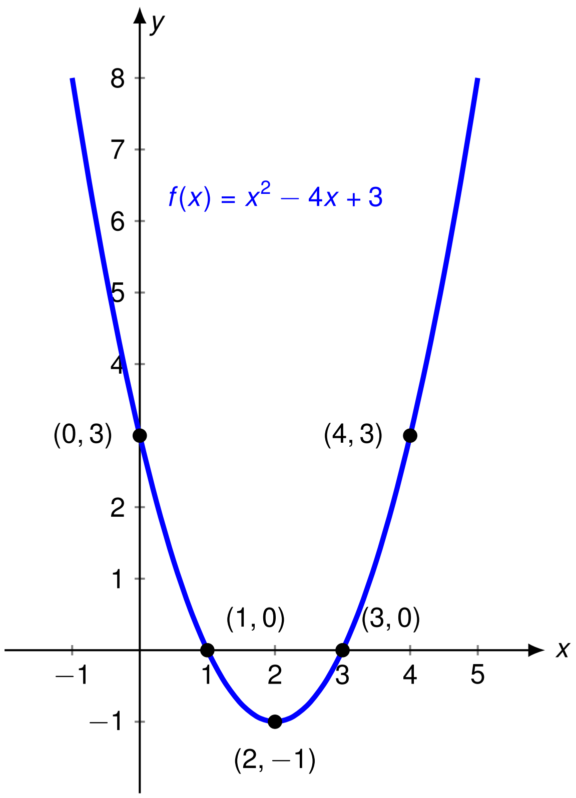
Explanation. 1. We follow the procedure for completing the square in Section **??**. The only difference here is instead of the quadratic equation being set to 0, it is equal to $f(x)$. This means when we are finished completing the square, we need to solve for $f(x)$.

$$\begin{aligned} f(x) &= x^2 - 4x + 3 && \\ f(x) - 3 &= x^2 - 4x && \text{Subtract 3 from both sides.} \\ f(x) - 3 + (-2)^2 &= x^2 - 4x + (-2)^2 && \text{Add } \left(\frac{1}{2}(-4)\right)^2 \text{ to both sides.} \\ f(x) + 1 &= (x-2)^2 && \text{Factor the perfect square trinomial.} \\ f(x) &= (x-2)^2 - 1 && \text{Solve for } f(x). \end{aligned}$$

The reader is encouraged to start with $f(x) = (x-2)^2 - 1$, perform the indicated operations and simplify the result to $f(x) = x^2 - 4x + 3$. From the vertex form, $f(x) = (x-2)^2 - 1$, we see that the vertex is $(2, 1)$ and that the parabola opens upwards. To find the zeros of f, we set $f(x) = 0$.

We have two equivalent expressions for $f(x)$ so we could use either the general form or vertex form. We solve the former and leave it to the reader to solve the latter to see that we get the same results either way. To solve $x^2 - 4x + 3 = 0$, we factor: $(x-3)(x-1) = 0$ and obtain $x = 1$ and $x = 3$. We get two x-intercepts, $(1, 0)$ and $(3, 0)$.

To find the y-intercept, we need $f(0)$. We use the general form and find that the y-intercept is $(0, 3)$. From symmetry, we know the point $(4, 3)$ is also on the graph.



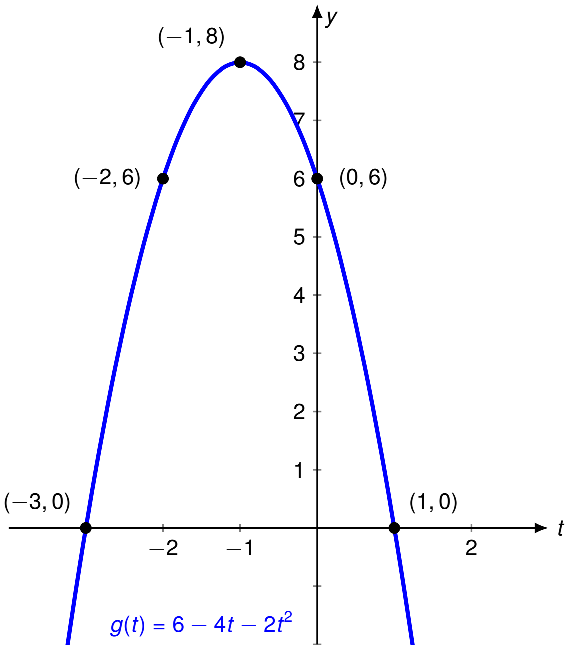
The range of f is $[-1, \infty)$ with the minimum -1 at $x = 2$. Finally, f is decreasing on $(-\infty, 2]$ and increasing from $[2, \infty)$.

2. We rewrite $g(t) = 6 - 4t - 2t^2$ as $g(t) = -2t^2 - 4t + 6$ and proceed to complete the square:

$$\begin{array}{rcll}
g(t) & = & -2t^2 - 4t + 6 & \\
g(t) - 6 & = & -2t^2 - 4t & \text{Subtract 6 from both sides.} \\
\dfrac{g(t) - 6}{-2} & = & \dfrac{-2t^2 - 4t}{-2} & \text{Divide both sides by } -2. \\
\dfrac{g(t) - 6}{-2} + (1)^2 & = & t^2 + 2t + (1)^2 & \text{Add } \left(\frac{1}{2}(2)\right)^2 \text{ to both sides.} \\
\dfrac{g(t) - 6}{-2} + 1 & = & (t + 1)^2 & \text{Factor the prefect square trinomial.} \\
\dfrac{g(t) - 6}{-2} & = & (t + 1)^2 - 1 & \\
g(t) - 6 & = & -2\left[(t + 1)^2 - 1\right] & \\
g(t) & = & -2(t + 1)^2 + 2 + 6 & \\
g(t) & = & -2(t + 1)^2 + 8 & \text{Solve for } g(t).
\end{array}$$

We can check our answer by expanding $-2(t + 1)^2 + 8$ and show that it simplifies to $-2t^2 - 4t + 6$. From the vertex form, we find that the vertex is $(-1, 8)$ and that the parabola opens downwards. Setting $g(t) = -2t^2 - 4t + 6 = 0$, we factor to get $-2(t - 1)(t + 3) = 0$ so $t = -3$ and $t = 1$. Hence, our two t-intercepts are $(-3, 0)$ and $(1, 0)$.

Since $g(0) = 6$, we get the y-intercept to be $(0, 6)$. Using symmetry, we also have the point $(-2, 6)$ on the graph.



The range is $(-\infty, 8]$ with a maximum of 8 when $t = -1$. Finally we note that g is increasing on $(-\infty, -1]$ and decreasing on $[-1, \infty)$.

We now generalize the procedure demonstrated in Example **??**. Let $f(x) = ax^2 + bx + c$ for $a \neq 0$:

$$\begin{array}{rcll}
f(x) & = & ax^2 + bx + c & \\
f(x) - c & = & ax^2 + bx & \text{Subtract } c \text{ from both sides.} \\
\dfrac{f(x) - c}{a} & = & \dfrac{ax^2 + bx}{a} & \text{Divide both sides by } a \neq 0. \\
\dfrac{f(x) - c}{a} & = & x^2 + \dfrac{b}{a}x & \\
\dfrac{f(x) - c}{a} + \left(\dfrac{b}{2a}\right)^2 & = & x^2 + \dfrac{b}{a}x + \left(\dfrac{b}{2a}\right)^2 & \text{Add } \left(\dfrac{b}{2a}\right)^2 \text{ to both sides.} \\
\dfrac{f(x) - c}{a} + \dfrac{b^2}{4a^2} & = & \left(x + \dfrac{b}{2a}\right)^2 & \text{Factor the perfect square trinomial.} \\
\dfrac{f(x) - c}{a} & = & \left(x + \dfrac{b}{2a}\right)^2 - \dfrac{b^2}{4a^2} & \text{Solve for } f(x). \\
f(x) - c & = & a\left[\left(x + \dfrac{b}{2a}\right)^2 - \dfrac{b^2}{4a^2}\right] & \\
f(x) - c & = & a\left(x + \dfrac{b}{2a}\right)^2 - a\dfrac{b^2}{4a^2} & \\
f(x) & = & a\left(x + \dfrac{b}{2a}\right)^2 - \dfrac{b^2}{4a} + c & \\
f(x) & = & a\left(x + \dfrac{b}{2a}\right)^2 + \dfrac{4ac - b^2}{4a} & \text{Get a common denominator.}
\end{array}$$

By setting $h = -\dfrac{b}{2a}$ and $k = \dfrac{4ac - b^2}{4a}$, we have written the function in the form $f(x) = a(x - h)^2 + k$. This establishes the fact that every quadratic function can be written in vertex form.[6] Moreover, writing a quadratic function in vertex form allows us to identify the vertex rather quickly, and so our work also shows us that the

[6]To avoid completing the square, we could solve the equations $b = -2ah$ and $c = ah^2 + k$ for h and k. See Exercise **??**.

vertex of $f(x) = ax^2 + bx + c$ is $\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$. It is not worth memorizing the expression $\frac{4ac - b^2}{4a}$ especially since we can write this as $f\left(-\frac{b}{2a}\right)$. (Think about this last statement for a moment.)

We summarize the information detailed above in the following:

Equation 0.1. Vertex Formulas for Quadratic Functions:

Suppose a, b, c, h and k are real numbers where $a \neq 0$.

- If $f(x) = a(x - h)^2 + k$ then the vertex of the graph of $y = f(x)$ is the point (h, k).
- If $f(x) = ax^2 + bx + c$ then the vertex of the graph of $y = f(x)$ is the point $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

Completing the square is also the means by which we may derive the celebrated Quadratic Formula, a formula which returns the solutions to $ax^2 + bx + c = 0$ for $a \neq 0$. Before we state it here for reference, we wish to encourage the reader to pause a moment and read the derivation if the Quadratic Formula found in Section **??**. The work presented in this section transforms the general form of a quadratic **function** into the vertex form whereas the work in Section **??** finds a formula to solve an **equation**. There is great value in understanding the similarities and differences between the two approaches.

Equation 0.2. The Quadratic Formula: The zeros of the quadratic function $f(x) = ax^2 + bx + c$ are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

It is worth pointing out the symmetry inherent in Equation **??**. We may rewrite the zeros as:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a},$$

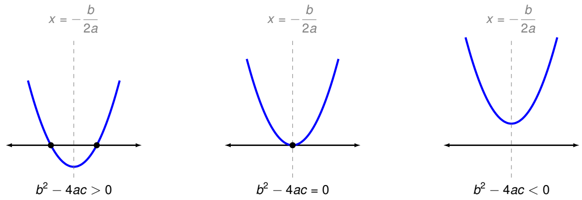
so that, if there are real zeros, they (like the rest of the parabola) are symmetric about the line $x = -\frac{b}{2a}$. Another way to view this symmetry is that the x-coordinate of the vertex is the average of the zeros. We encourage the reader to verify this fact in all of the preceding examples, where applicable.

Next, recall that if the quantity $b^2 - 4ac$ is strictly negative then we do not have any real zeros. This quantity is called the **discriminant** and is useful in determining the number and nature of solutions to a quadratic equation. We remind the reader of this below.

Equation 0.3. The Discriminant of a Quadratic Function: Given a quadratic function in general form $f(x) = ax^2 + bx + c$, the **discriminant** is the quantity $b^2 - 4ac$.

- If $b^2 - 4ac > 0$ then f has two unequal (distinct) real zeros.
- If $b^2 - 4ac = 0$ then f has one (repeated) real zero.
- If If $b^2 - 4ac < 0$ then f has two unequal (distinct) non-real zeros.

We'll talk more about what we mean by a 'repeated' zero and how to compute 'non-real' zeros in Chapter **??**. For us, the discriminant has the graphical implication that if $b^2 - 4ac > 0$ then we have two x-intercepts; if $b^2 - 4ac = 0$ then we have just one x-intercept, namely, the vertex; and if $b^2 - 4ac < 0$ then we have no x-intercepts because the parabola lies entirely above or below the x-axis. We sketch each of these scenarios below assuming $a > 0$. (The sketches for $a < 0$ are similar - see Exercise **??**.)



We now revisit the economic scenario first described in Examples **??** and **??** where we were producing and selling PortaBoy game systems. Recall that the cost to produce x PortaBoys is denoted by $C(x)$ and the price-demand function, that is, the price to charge in order to sell x systems is denoted by $p(x)$. We introduce two more related functions below: the **revenue** and **profit** functions.

Definition 3. Revenue and Profit: Suppose $C(x)$ represents the cost to produce x units and $p(x)$ is the associated price-demand function. Under the assumption that we are producing the same number of units as are being sold:

- The **revenue** obtained by selling x units is $R(x) = x\,p(x)$.
  That is, revenue = (number of items sold) · (price per item).
- The **profit** made by selling x units is $P(x) = R(x) - C(x)$.
  That is, profit = (revenue) − (cost).

Said differently, the **revenue** is the amount of money **collected** by selling x items whereas the **profit** is how much money is **left over** after the costs are paid.

Example 4. In Example **??** the cost to produce x PortaBoy game systems for a local retailer was given by $C(x) = 80x + 150$ for $x \geq 0$ and in Example **??** the price-demand function was found to be $p(x) = -1.5x + 250$, for $0 \leq x \leq 166$.

1. Find formulas for the associated revenue and profit functions; include the domain of each.
2. Find and interpret $P(0)$.
3. Find and interpret the zeros of P.
4. Graph $y = P(x)$. Find the vertex and axis intercepts.
5. Interpret the vertex of the graph of $y = P(x)$.
6. What should the price per system be in order to maximize profit?
7. Find and interpret the average rate of change of P over the interval $[0, 57]$.

Explanation.

1. The formula for the revenue function is $R(x) = x\,p(x) = x(-1.5x + 250) = -1.5x^2 + 250x$. Since the domain of p is restricted to $0 \leq x \leq 166$, so is the domain of R. To find the profit function $P(x)$, we subtract $P(x) = R(x) - C(x) = \left(-1.5x^2 + 250x\right) - (80x + 150) = -1.5x^2 + 170x - 150$. The cost function formula is valid for $x \geq 0$, but the revenue function is valid when $0 \leq x \leq 166$. Hence, the domain of P is likewise restricted to $[0, 166]$.
2. We find $P(0) = -1.5(0)^2 + 170(0) - 150 = -150$. This means that if we produce and sell 0 PortaBoy game systems, we have a profit of $-\$150$. Since profit = (revenue) − (cost), this means our costs exceed our revenue by \$150. This makes perfect sense, since if we don't sell any systems, our revenue is \$0 but our fixed costs (see Example **??**) are \$150.

3. To find the zeros of P, we set $P(x) = 0$ and solve $-1.5x^2 + 170x - 150 = 0$. Factoring here would be challenging to say the least, so we use the Quadratic Formula, Equation **??**. Identifying $a = -1.5$, $b = 170$ and $c = -150$, we obtain

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-170 \pm \sqrt{170^2 - 4(-1.5)(-150)}}{2(-1.5)} \\ &= \frac{-170 \pm \sqrt{28000}}{-3} \\ &= \frac{170 \pm 20\sqrt{70}}{3} \\ &\approx 0.89,\ 112.44. \end{aligned}$$

   Given that profit = (revenue) $-$ (cost), if profit = 0, then revenue = cost. Hence, the zeros of P are called the 'break-even' points - where just enough product is sold to recover the cost spent to make the product. Also, x represents a number of game systems, which is a whole number, so instead of using the exact values of the zeros, or even their approximations, we consider $x = 0$ and $x = 1$ along with $x = 112$ and $x = 113$. We find $P(0) = -150$, $P(1) = 18.5$, $P(112) = 74$ and $P(113) = -93.5$. These data suggest that, in order to be profitable, at least 1 but not more than 112 systems should be produced and sold, as borne out in the graph below.

4. Knowing the zeros of P, we have two x-intercepts: $\left(\frac{170 - 20\sqrt{70}}{3}, 0\right) \approx (0.89, 0)$ and $\left(\frac{170 + 20\sqrt{70}}{3}, 0\right) \approx (112.44, 0)$. Since $P(0) = -150$, we get the y-intercept is $(0, -150)$. To find the vertex, we appeal to Equation **??**. Substituting $a = -1.5$ and $b = 170$, we get $x = -\frac{170}{2(-1.5)} = \frac{170}{3} = 56.\overline{6}$. To find the y-coordinate of the vertex, we compute $P\left(\frac{170}{3}\right) = \frac{14000}{3} = 4666.\overline{6}$. Hence, the vertex is $(56.\overline{6}, 4666.\overline{6})$. The domain is restricted $0 \leq x \leq 166$ and we find $P(166) = -13264$. Attempting to plot all of these points on the same graph to any sort of scale is challenging. Instead, we present a portion of the graph for $0 \leq x \leq 113$ with the help of Desmos. Even with this, the intercepts near the origin are crowded.

   Desmos link: `https://www.desmos.com/calculator/9qarfctjwf`

5. From the vertex, we see that the maximum of P is $4666.\overline{6}$ when $x = 56.\overline{6}$. As before, x represents the number of PortaBoy systems produced and sold, so we cannot produce and sell $56.\overline{6}$ systems. Hence, by comparing $P(56) = 4666$ and $P(57) = 4666.5$, we conclude that we will make a maximum profit of \$4666.50 if we sell 57 game systems.

6. We've determined that we need to sell 57 PortaBoys to maximize profit, so we substitute $x = 57$ into the price-demand function to get $p(57) = -1.5(57) + 250 = 164.5$. In other words, to sell 57 systems, and thereby maximize the profit, we should set the price at \$164.50 per system.

7. To find the average rate of change of P over $[0, 57]$, we compute

$$\frac{\Delta[P(x)]}{\Delta x} = \frac{P(57) - P(0)}{57 - 0} = \frac{4666.5 - (-150)}{57} = 84.5.$$

   This means that as the number of systems produced and sold ranges from 0 to 57, the average profit per system is increasing at a rate of \$84.50. In other words, for each additional system produced and sold, the profit, on average, increased by \$84.50. ■

We hope Example **??** shows the value of using a continuous model to describe a discrete situation. True, we could have 'run the numbers' and computed $P(1), P(2), \ldots, P(166)$ to eventually determine the maximum profit, but the vertex formula made much quicker work of the problem.

Along these same lines, in our next example we revisit Skippy's temperature data from Example **??** in Section **??**. We found a piecewise-linear model in Section **??** to model the temperature over the course the day and now we seek a quadratic function to do the job. The methodology used here is similar to that of the least squares regression line discussed in Section **??** but instead of finding the line closest to the data points, we want the **parabola** closest to them that comes from a function of the form $f(x) = ax^2+bx+c$. The Mathematics required to find the desired quadratic function is beyond the scope of this text, but most graphing utilities can do these quickly. In the quadratic case, the machine will return a value of R^2 such that $0 \leq R^2 \leq 1$. The closer R^2 is to 1, the better the fit. (Again, how R^2 is computed is beyond this text.)

Example 5.

1. Use a graphing utility to fit a quadratic model to the time and temperature data in Example **??**. Comment on the goodness of fit.
2. Use your model to predict the temperature at 7 AM and 3 PM. Round your answers to one decimal place. How do your results compare with those from Example **??**?
3. According to the model, what was the warmest temperature of the day? When did that occur? Round your answers to one decimal place.

Explanation.

1. Entering the data in Desmos we find $T = F(t) = -0.1905t^2 + 3.9643t + 62.405$ with an R^2 value of 0.97, indicating a pretty strong fit.

   Desmos link: `https://www.desmos.com/calculator/svtsequege`

2. Since 7 AM corresponds to $t = 1$, we find $T = F(1) \approx 66.18$. Hence our quadratic model predicts a temperature of 66.2° F at 7 AM - identical (when rounded) to the 66.2° F predicted in Example **??**. Similarly, 3 PM corresponds to $t = 9$, so we find $T = F(9) \approx 82.65$. Thus the model predicts an outdoor temperature of 82.6° F which is very close to the 82.9°F prediction from Example **??**.

3. The model is quadratic with $a < 0$ so the maximum (warmest) temperature can be determined by finding the vertex. We get

$$t = -\frac{b}{2a} = -\frac{3.9643}{2(-0.1905)} \approx -10.40, \quad T = F(-10.40) \approx 83.03,$$

   or, in other words, the warmest temperature is 83.0° F at 4:24 PM (10.40 hours after 6 AM.)

■

It is interesting how close the predictions from Examples **??** and **??** despite one using linear models and one using a quadratic model. Which model is the 'better' model? We leave that discussion to the reader and their classmates.

Our next example is classic application of optimizing a quadratic function.

Example 6. Much to Donnie's surprise and delight, he inherits a large parcel of land in Ashtabula County from one of his (e)strange(d) relatives so the time is right for him to pursue his dream of raising alpaca. He wishes to build a rectangular pasture and estimates that he has enough money for 200 linear feet of fencing material. If he makes the pasture adjacent to a river (so that no fencing is required on that side), what are the dimensions of the pasture which maximize the area? What is the maximum area? If an average alpaca needs 25 square feet of grazing area, how many alpaca can Donnie keep in his pasture?

Explanation. We are asked to find the dimensions of the pasture which would give a maximum area, so we begin by assigning variables. We let w denote the width of the pasture and ℓ denote the length of the pasture, each with units of feet. The area of the pasture, which we'll call A, is related to w and ℓ by the equation $A = w\ell$. Since w and ℓ are both measured in feet, A has units of feet^2, or square feet. We are also told that the total amount of fencing available is 200 feet, which means $w + \ell + w = 200$, or, $\ell + 2w = 200$.

Before we dive right into algebraic machinations, let's take a moment to visualize the problem situation. In the GeoGebra interactive below, we see a representation of the fenced-in pasture on the left. As we move the slider for the width, w, we can see the corresponding shape of the pasture subject to the condition (constraint) that the total fencing available is 200 feet. Below this diagram is a calculator which gives the maximum number of alpaca supported by the pasture. On the right is the graph of the area of the pasture as a function of w which is traced as we move the slider.

GeoGebra link: `https://www.geogebra.org/m/mxxvbanv`

The interactive *suggests* a width of 50 feet will maximize the pasture area allowing Donnie to keep 200 alpaca. It's now time to **prove** this using the algebraic tools we've developed in this section.[7]

We now have two equations, $A = w\ell$ and $\ell + 2w = 200$. In order to use the tools given to us in this section to **maximize** A, we need to use the information given to write A as a function of just **one** variable, either w or ℓ. This is where we use the equation $\ell + 2w = 200$. Solving for ℓ, we find $\ell = 200 - 2w$, and we substitute this into our equation for A. We get $A = w\ell = w(200 - 2w) = 200w - 2w^2$. We now have A as a function of w, $A = f(w) = 200w - 2w^2 = -2w^2 + 200w$.

Before we go any further, we need to find the applied domain of f so that we know what values of w make sense in this situation.[8] Given that w represents the width of the pasture we need $w > 0$. Likewise, ℓ represents the length of the pasture, so $\ell = 200 - 2w > 0$. Solving this latter inequality yields $w < 100$. Hence, the function we wish to maximize is $f(w) = -2w^2 + 200w$ for $0 < w < 100$. We know two things about the quadratic function f: the graph of $A = f(w)$ is a parabola and (since the coefficient of w^2 is -2) the parabola opens downwards.

This means that there is a maximum value to be found, and we know it occurs at the vertex. Using the vertex formula, we find $w = -\dfrac{200}{2(-2)} = 50$, and $A = f(50) = -2(50)^2 + 200(50) = 5000$. Since $w = 50$ lies in the applied domain, $0 < w < 100$, we have that the area of the pasture is maximized when the width is 50 feet. To find the length, we use $\ell = 200 - 2w$ and find $\ell = 200 - 2(50) = 100$, so the length of the pasture is 100 feet. The maximum area is $A = f(50) = 5000$, or 5000 square feet. If an average alpaca requires 25 square feet of pasture, Donnie can raise $\dfrac{5000}{25} = 200$ average alpaca. This solution tracks with what we learned from the GeoGebra interactive earlier. ■

The function f in Example **??** is called the **objective function** for this problem - it's the function we're trying to optimize. In the case above, we were trying to maximize f. The equation $\ell + 2w = 200$ along with the inequalities $w > 0$ and $\ell > 0$ are called the **constraints**. As we saw in this example, and as we'll see again and again, the constraint equation is used to rewrite the objective function in terms of just one of the variables where constraint inequalities, if any, help determine the applied domain.

2 Inequalities involving Quadratic Functions

We now turn our attention to solving inequalities involving quadratic functions. Consider the inequality $x^2 \leq 6$. We could use the fact that the square root is increasing[9] to get: $\sqrt{x^2} \leq \sqrt{6}$, or $|x| \leq \sqrt{6}$. This reduces to $-\sqrt{6} \leq x \leq \sqrt{6}$ or, using interval notation, $[-\sqrt{6}, \sqrt{6}]$. If, however, we had to solve $x^2 \leq x + 6$, things are

[7] Mitch Korcheck used to always say that technology *suggests* solutions; algebra *proves* them. Rest in peace, Mitch.

[8] Donnie would be very upset if, for example, we told him the width of the pasture needs to be -50 feet.

[9] That is, if $a < b$, then $\sqrt{a} < \sqrt{b}$.

more complicated. One approach is to complete the square:

$$\begin{aligned} x^2 &\leq x+6 \\ x^2 - x &\leq 6 \\ x^2 - x + \frac{1}{4} &\leq 6 + \frac{1}{4} \\ \left(x - \frac{1}{2}\right)^2 &\leq \frac{25}{4} \\ \sqrt{\left(x - \frac{1}{2}\right)^2} &\leq \sqrt{\frac{25}{4}} \\ \left|x - \frac{1}{2}\right| &\leq \frac{5}{2} \\ -\frac{5}{2} \leq x - \frac{1}{2} &\leq \frac{5}{2} \\ -2 \leq x &\leq 3 \end{aligned}$$

We get the solution $[-2, 3]$. While there is nothing wrong with this approach, we seek methods here that will generalize to higher degree polynomials such as those we'll see in Chapter **??**.

To that end, we look at the inequality $x^2 \leq x+6$ graphically. Identifying $f(x) = x^2$ and $g(x) = x+6$, we graph f and g on the same set of axes below and look for where the graph of f (the parabola) meets or is below the graph of g (the line).

Desmos link: `https://www.desmos.com/calculator/vduhlovrae`

There are two points of intersection which we determine by solving $f(x) = g(x)$ or $x^2 = x+6$. As usual, we rewrite this equation as $x^2 - x - 6 = 0$ in order to use the primary tools we've developed to handle these types[10] of quadratic equations: factoring, or failing that, the Quadratic Formula. We find $x^2 - x - 6 = (x+2)(x-3)$ so we get two solutions to $(x+2)(x-3) = 0$, namely $x = -2$ and $x = 3$. Putting these together with the graph, we obtain the same solution: $[-2, 3]$.

Yet a third way to attack $x^2 \leq x+6$ is to rewrite the inequality as $x^2 - x - 6 \leq 0$. Here, we graph $f(x) = x^2 - x - 6$ to look for where the graph meets or is below the graph of $g(x) = 0$, a.k.a. the x-axis.

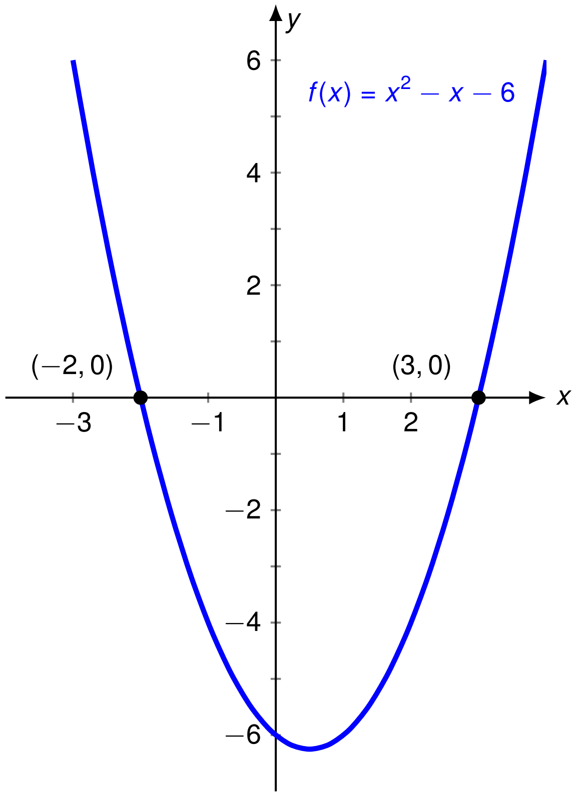
Desmos link: `https://www.desmos.com/calculator/7kboi4ddjw`

To find where the graph of f meets the x-axis, we find the zeros of f by solving $f(x) = x^2 - x - 6 = 0$. We obtain $x = -2$ and $x = 3$ as before, and find the same solution, $[-2, 3]$.

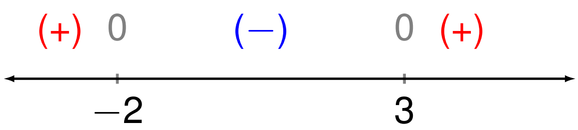
One advantage to using this last approach is that we are essentially concerned with one function and its **zeros**. This approach can be generalized to all functions - not just quadratics, so we take the time to develop this method more thoroughly now.

Consider the graph of $f(x) = x^2 - x - 6$ below.

[10] Namely ones with a nonzero coefficient of 'x'.



The zeros of f are $x = -2$ and $x = 3$ and they divide the domain (the x-axis) into three intervals: $(-\infty, -2)$, $(-2, 3)$ and $(3, \infty)$. For every number in $(-\infty, -2)$, the graph of f is above the x-axis; in other words, $f(x) > 0$ for all x in $(-\infty, -2)$. Similarly, $f(x) < 0$ for all x in $(-2, 3)$, and $f(x) > 0$ for all x in $(3, \infty)$. We represent this schematically with the **sign diagram** below.



The $(+)$ above a portion of the number line indicates $f(x) > 0$ for those values of x and the $(-)$ indicates $f(x) < 0$ there. The numbers labeled on the number line are the zeros of f, so we place 0 above them. For the inequality $f(x) = x^2 - x - 6 \leq 0$, we read from the sign diagram that the solution is $[-2, 3]$.

Our next goal is to establish a procedure by which we can generate the sign diagram without graphing the function. While parabolas aren't that bad to graph knowing what we know, our sights are set on more general functions whose graphs are more complicated.

An important property of parabolas is that a parabola can't be above the x-axis at one point and below the x-axis at another point without crossing the x-axis at some point in between. Said differently, if the function is positive at one point and negative at another, the function must have at least one zero in between. This property is a consequence of quadratic functions being **continuous**. A precise definition of 'continuous' requires the language of Calculus, but it suffices for us to know that the graph of a continuous function has no gaps or holes. This allows us to determine the sign of **all** of the function values on a given interval by testing the function at just **one** value in the interval.

The result below applies to all continuous functions defined on an interval of real numbers, but we restrict our attention to quadratic functions for the time being,

Steps for Creating A Sign Diagram for A Quadratic Function

Suppose f is a quadratic function.

1. Find the zeros of f and place them on the number line with the number 0 above them.
2. Choose a real number, called a **test value**, in each of the intervals determined in step 1.
3. Determine and record the sign of $f(x)$ for each test value in step 2.

To use a sign diagram to solve an inequality, we must always remember to compare the function to 0.

Solving Inequalities using Sign Diagrams

To solve an inequality using a sign diagram:

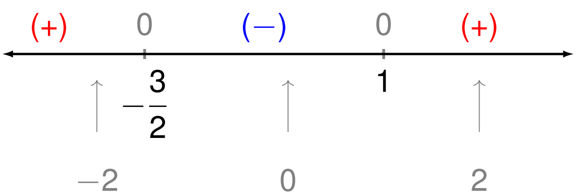
1. Rewrite the inequality so some function $f(x)$ is being compared to '0.'
2. Make a sign diagram for f.
3. Record the solution.

We practice this approach in the following example.

Example 7. Solve the following inequalities analytically[11] and check your solutions graphically.

1. $2x^2 \leq 3 - x$
2. $t^2 - 2t > 1$
3. $x^2 + 1 \leq 2x$
4. $2t - t^2 \geq |t - 1| - 1$

Explanation. 1. To solve $2x^2 \leq 3-x$, we rewrite it as $2x^2+x-3 \leq 0$. We find the zeros of $f(x) = 2x^2+x-3$ by solving $2x^2 + x - 3 = 0$. Factoring gives $(2x + 3)(x - 1) = 0$, so $x = -\frac{3}{2}$ or $x = 1$. We place these values on the number line with 0 above them and choose test values in the intervals $\left(-\infty, -\frac{3}{2}\right)$, $\left(-\frac{3}{2}, 1\right)$ and $(1, \infty)$. For the interval $\left(-\infty, -\frac{3}{2}\right)$, we choose[12] $x = -2$; for $\left(-\frac{3}{2}, 1\right)$, we pick $x = 0$; and for $(1, \infty)$, $x = 2$. Evaluating the function at the three test values gives us $f(-2) = 3 > 0$, so we place $(+)$ above $\left(-\infty, -\frac{3}{2}\right)$; $f(0) = -3 < 0$, so $(-)$ goes above the interval $\left(-\frac{3}{2}, 1\right)$; and, $f(2) = 7$, which means $(+)$ is placed above $(1, \infty)$.



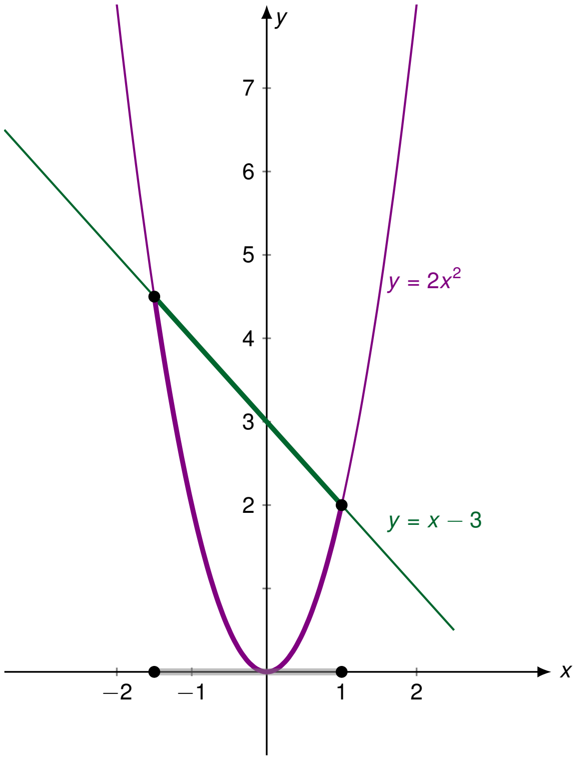
We are solving $2x^2 + x - 3 \leq 0$ so we need solutions to $2x^2 + x - 3 < 0$ as well as solutions for $2x^2 + x - 3 = 0$. For $2x^2 + x - 3 < 0$, we need the intervals which we have a $(-)$ above them. The sign

[11] By 'solve analytically' we mean 'algebraically' using a sign diagram.

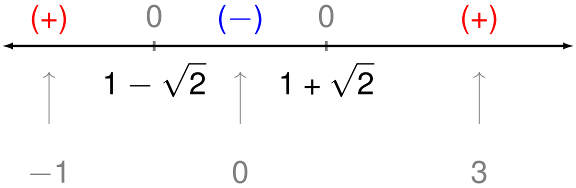
[12] We have to choose **something** in each interval. If you don't like our choices, please feel free to choose different numbers. You'll get the same sign chart.

diagram shows only one: $\left(-\frac{3}{2}, 1\right)$. Also, we know $2x^2 + x - 3 = 0$ when $x = -\frac{3}{2}$ and $x = 1$, so our final answer is $\left[-\frac{3}{2}, 1\right]$.

To verify our solution graphically, we refer to the original inequality, $2x^2 \leq 3 - x$. We let $g(x) = 2x^2$ and $h(x) = 3 - x$. We are looking for the x values where the graph of g is below that of h (the solution to $g(x) < h(x)$) as well as the points of intersection (the solutions to $g(x) = h(x)$). The graph below checks our solution is indeed $\left[-\frac{3}{2}, 1\right]$.

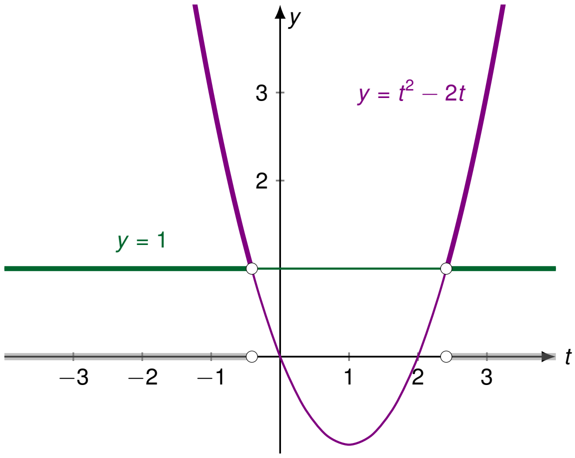


2. Once again, we re-write $t^2 - 2t > 1$ as $t^2 - 2t - 1 > 0$ and we identify $f(t) = t^2 - 2t - 1$. When we go to find the zeros of f, we find, to our chagrin, that the quadratic $t^2 - 2t - 1$ doesn't factor nicely. Hence, we resort to the Quadratic Formula and find $t = 1 \pm \sqrt{2}$. As before, these zeros divide the number line into three pieces. To help us decide on test values, we approximate $1 - \sqrt{2} \approx -0.4$ and $1 + \sqrt{2} \approx 2.4$. We choose $t = -1$, $t = 0$ and $t = 3$ as our test values and find $f(-1) = 2$, which is $(+)$; $f(0) = -1$ which is $(-)$; and $f(3) = 2$ which is $(+)$ again.

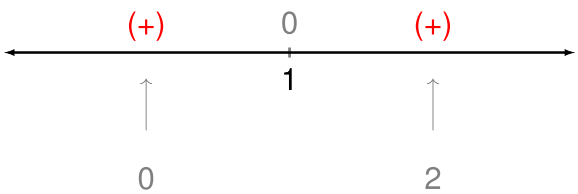


Our solution to $t^2 - 2t - 1 > 0$ is where we have $(+)$, so, in interval notation $\left(-\infty, 1 - \sqrt{2}\right) \cup \left(1 + \sqrt{2}, \infty\right)$.

To check the inequality $t^2 - 2t > 1$ graphically, we set $g(t) = t^2 - 2t$ and $h(t) = 1$. We are looking for the t values where the graph of g is above the graph of h. The graph below checks our solution is $\left(-\infty, 1-\sqrt{2}\right) \cup \left(1+\sqrt{2}, \infty\right)$.



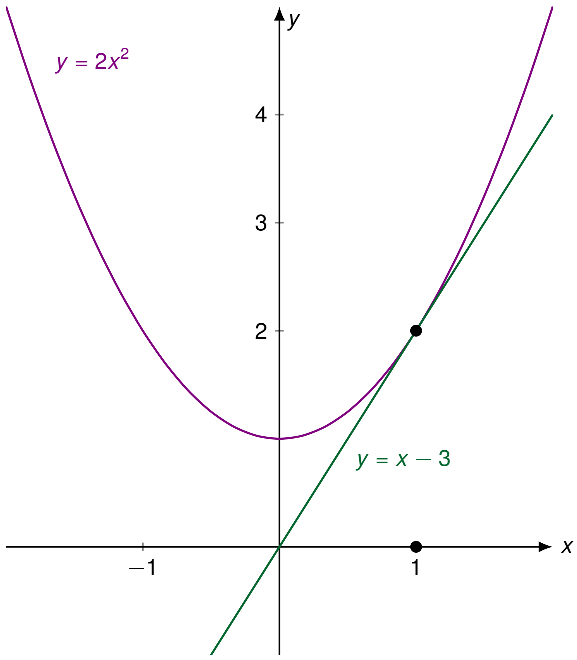
3. To solve $x^2 + 1 \leq 2x$, as before, we solve $x^2 - 2x + 1 \leq 0$. Setting $f(x) = x^2 - 2x + 1 = 0$, we find only one zero of f: $x = 1$. This one x value divides the number line into two intervals, from which we choose $x = 0$ and $x = 2$ as test values. We find $f(0) = 1 > 0$ and $f(2) = 1 > 0$.



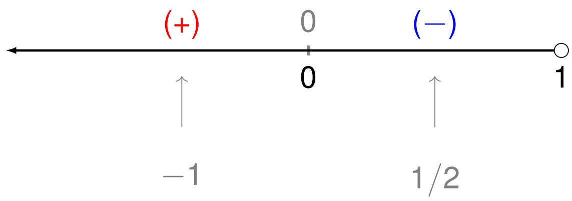
Since we are looking for solutions to $x^2 - 2x + 1 \leq 0$, we are looking for x values where $x^2 - 2x + 1 < 0$ as well as where $x^2 - 2x + 1 = 0$. Looking at our sign diagram, there are no places where $x^2 - 2x + 1 < 0$ (there are no $(-)$), so our solution is only $x = 1$ (where $x^2 - 2x + 1 = 0$). We write this as $\{1\}$.

Graphically, we solve $x^2 + 1 \leq 2x$ by graphing $g(x) = x^2 + 1$ and $h(x) = 2x$. We are looking for the x values where the graph of g is below the graph of h (for $x^2 + 1 < 2x$) and where the two graphs intersect ($x^2 + 1 = 2x$). Notice that the line and the parabola touch at $(1, 2)$, but the parabola is always above the line otherwise.[13]

[13] In this case, we say the line $y = 2x$ is **tangent** to $y = x^2 + 1$ at $(1, 2)$. Finding tangent lines to arbitrary functions is a fundamental problem solved, in general, with Calculus.

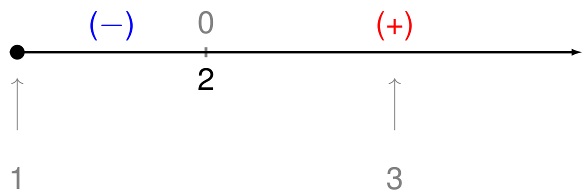


4. To solve $2t - t^2 \geq |t-1| - 1$ analytically we first rewrite the absolute value using cases. For $t < 1$, $|t-1| = -(t-1) = -t+1$, so we get $2t - t^2 \geq (-t+1) - 1$ which simplifies to $t^2 - 3t \leq 0$. Finding the zeros of $f(t) = t^2 - 3t$, we get $t = 0$ and $t = 3$. However, we are concerned only with the portion of the number line where $t < 1$, so the only zero that we deal with is $t = 0$. This divides the interval $t < 1$ into two intervals: $(-\infty, 0)$ and $(0, 1)$. We choose $t = -1$ and $t = \frac{1}{2}$ as our test values. We find $f(-1) = 4$ and $f\left(\frac{1}{2}\right) = -\frac{5}{4}$. Hence, our solution to $t^2 - 3t \leq 0$ for $t < 1$ is $[0, 1)$.



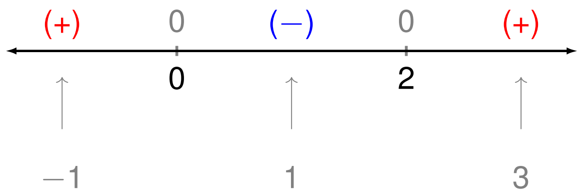
Solving $2t - t^2 \geq |t-1| - 1$ for $t < 1$.

Next, we turn our attention to the case $t \geq 1$. Here, $|t-1| = t-1$, so our original inequality becomes $2t - t^2 \geq (t-1) - 1$, or $t^2 - t - 2 \leq 0$. Setting $g(t) = t^2 - t - 2$, we find the zeros of g to be $t = -1$ and $t = 2$. Of these, only $t = 2$ lies in the region $t \geq 1$, so we ignore $t = -1$. Our test intervals are now $[1, 2)$ and $(2, \infty)$. We choose $t = 1$ and $t = 3$ as our test values and find $g(1) = -2$ and $g(3) = 4$. Hence, our solution to $g(t) = t^2 - t - 2 \leq 0$, in this region is $[1, 2]$.

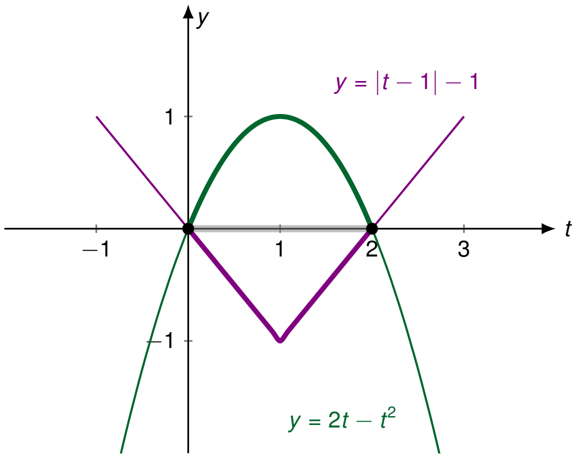


Solving $2t - t^2 \geq |t-1| - 1$ for $t \geq 1$.

Combining these into one sign diagram, we have that our solution is $[0,2]$.



Graphically, to check $2t - t^2 \geq |t-1| - 1$, we set $h(t) = 2t - t^2$ and $i(t) = |t-1| - 1$ and look for the t values where the graph of h intersects or is above the the graph of i. Our graph below checks that our answer is $[0,2]$.



■

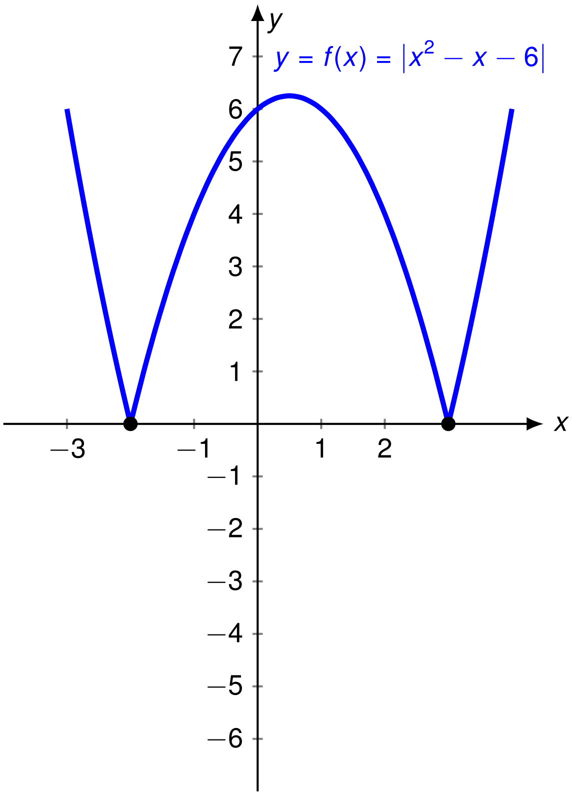
We end this section with an example that combines quadratic inequalities with piecewise functions.

Example 8. Rewrite $g(x) = |x^2 - x - 6|$ as a piecewise-defined function and graph. Compare the graph of g to the graph of $f(x) = x^2 - x - 6$. What do you notice?

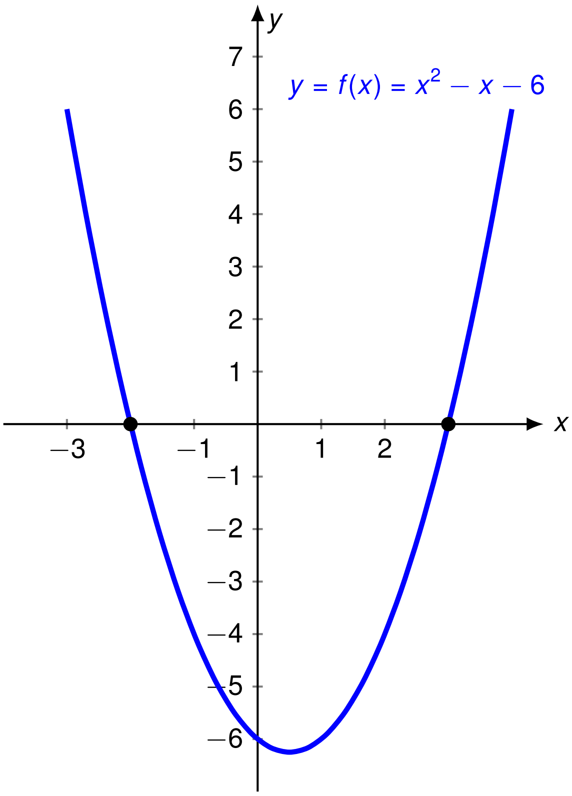
Explanation. Using the definition of absolute value, Definition **??** and the sign diagram we constructed for $f(x) = x^2 - x - 6$ near the beginning of the subsection, we get:

$$g(x) = |x^2 - x - 6| = \begin{cases} -(x^2 - x - 6) & \text{if } (x^2 - x - 6) < 0, \\ (x^2 - x - 6) & \text{if } (x^2 - x - 6) \geq 0. \end{cases} \longrightarrow \quad g(x) = \begin{cases} -x^2 + x + 6 & \text{if } -2 < x < 3, \\ x^2 - x - 6 & \text{if } x \leq -2 \text{ or } x \geq 3. \end{cases}$$

Going through the usual machinations results in the graph below.



Next, we graph $f(x) = x^2 - x - 6$.



If we take a step back and look at the graphs of f and g, we notice that to obtain the graph of g from the graph of f, we reflect the **portion** of the graph of f which lies below the x-axis to above the x-axis and leave the portion of the graph of f which lies above the x-axis alone.

In general, if $g(x) = |f(x)|$, then:

$$g(x) = |f(x)| = \begin{cases} -f(x) & \text{if } f(x) < 0, \\ f(x) & \text{if } f(x) \geq 0. \end{cases}$$

The function g is defined so that when $f(x)$ is negative (i.e., when its graph is below the x-axis), the graph of g is the refection of the graph of f across the x-axis. This is a general method to graph functions of the form $g(x) = |f(x)|$. Indeed, the graph of $g(x) = |x|$ can be obtained by reflection the portion of the line $f(x) = x$ which is below the x-axis back above the x-axis creating the characteristic '$\vee$' shape.[14] ■

[14] See Exercise **??** in Section **??**.