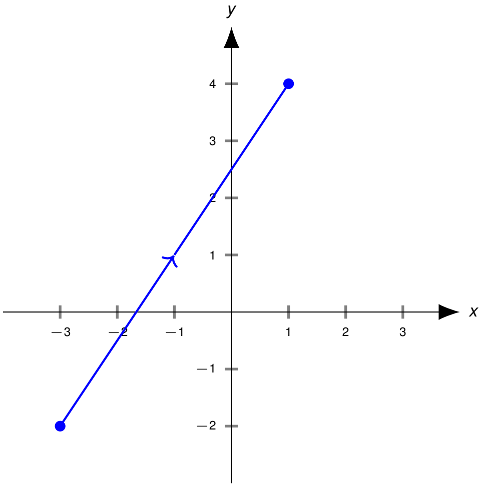
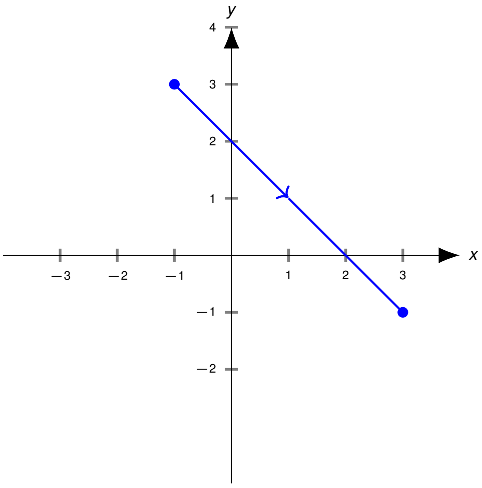
Exercises for Parametric Equations

Question 1 *In Exercises **??** - **??**, plot the set of parametric equations by hand. Be sure to indicate the orientation imparted on the curve by the parametrization.*

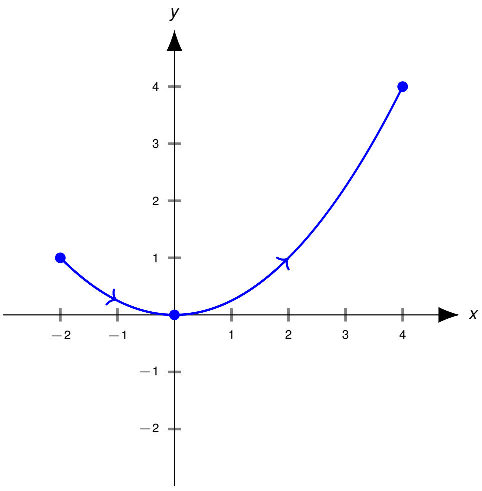
Problem 1.1 $\begin{cases} x = 4t - 3 \\ y = 6t - 2 \end{cases}$ *for* $0 \leq t \leq 1$



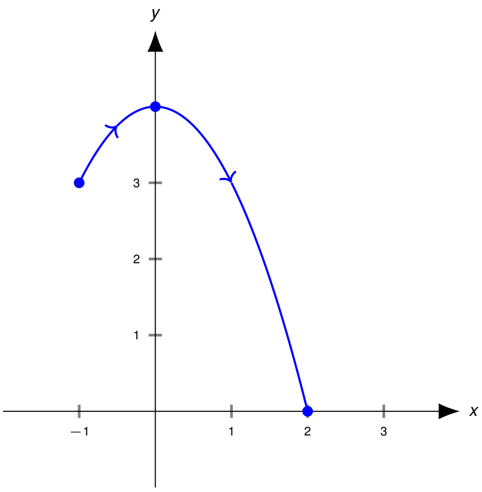
Problem 1.2 $\begin{cases} x = 4t - 1 \\ y = 3 - 4t \end{cases}$ *for* $0 \leq t \leq 1$



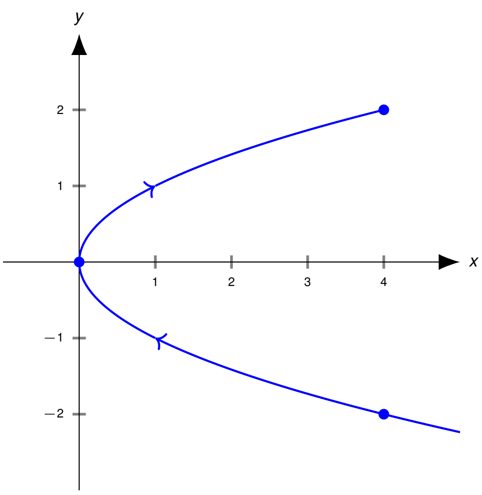
Problem 1.3 $\begin{cases} x = 2t \\ y = t^2 \end{cases}$ *for* $-1 \leq t \leq 2$



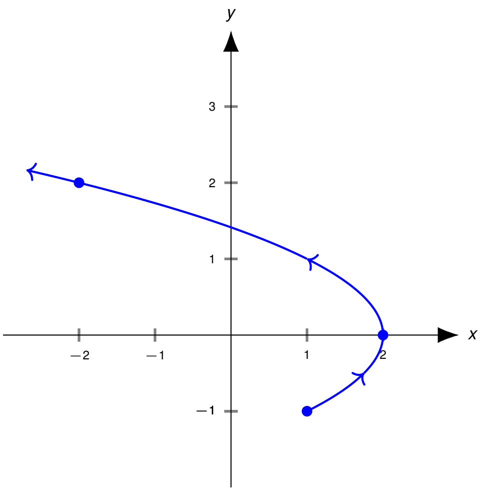
Problem 1.4 $\begin{cases} x = t - 1 \\ y = 3 + 2t - t^2 \end{cases}$ *for* $0 \leq t \leq 3$



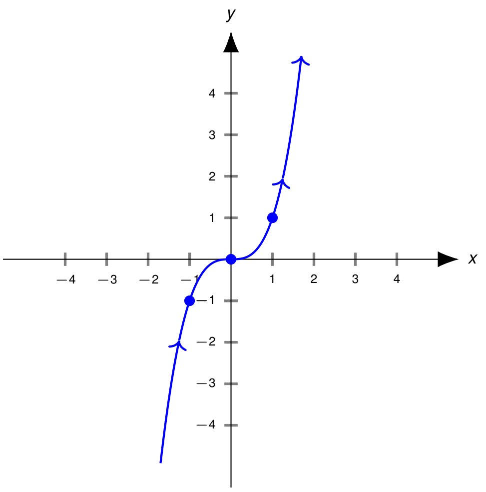
Problem 1.5 $\begin{cases} x = t^2 + 2t + 1 \\ y = t + 1 \end{cases}$ *for* $t \leq 1$



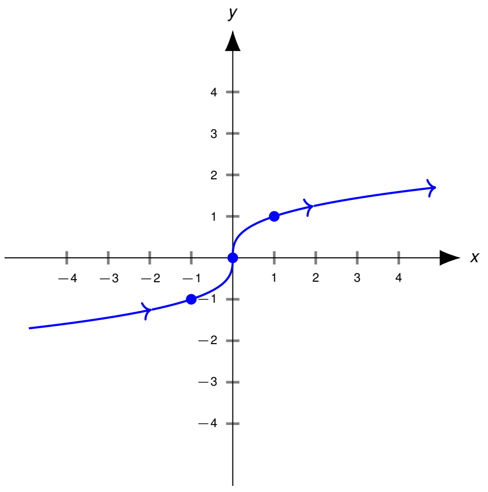
Problem 1.6 $\begin{cases} x = \frac{1}{9}\left(18 - t^2\right) \\ y = \frac{1}{3}t \end{cases}$ *for* $t \geq -3$



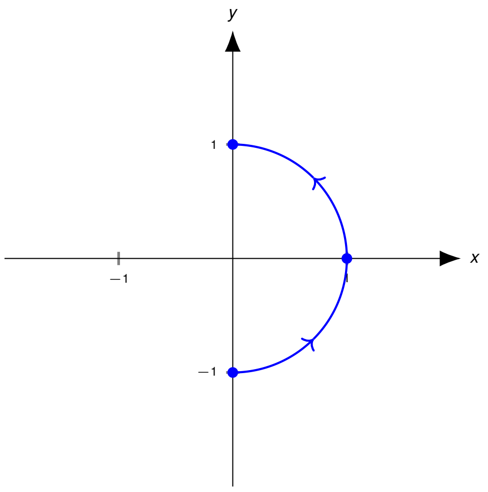
Problem 1.7 $\begin{cases} x = t \\ y = t^3 \end{cases}$ *for* $-\infty < t < \infty$



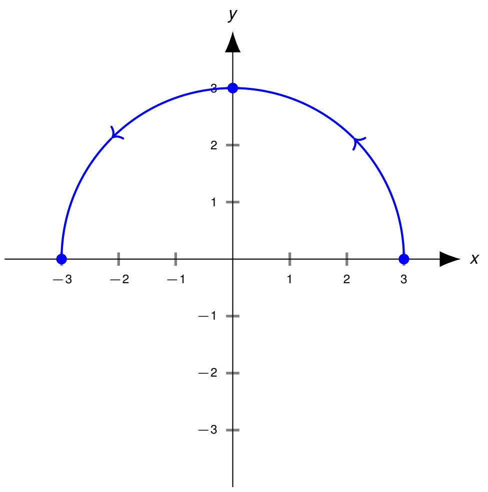
Problem 1.8 $\begin{cases} x = t^3 \\ y = t \end{cases}$ *for* $-\infty < t < \infty$



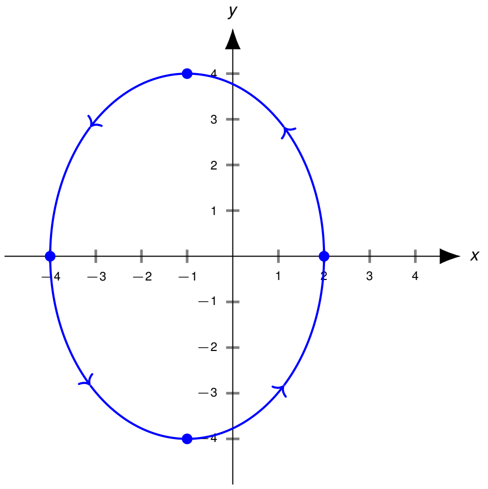
Problem 1.9 $\begin{cases} x = \cos(t) \\ y = \sin(t) \end{cases}$ *for* $-\dfrac{\pi}{2} \leq t \leq \dfrac{\pi}{2}$



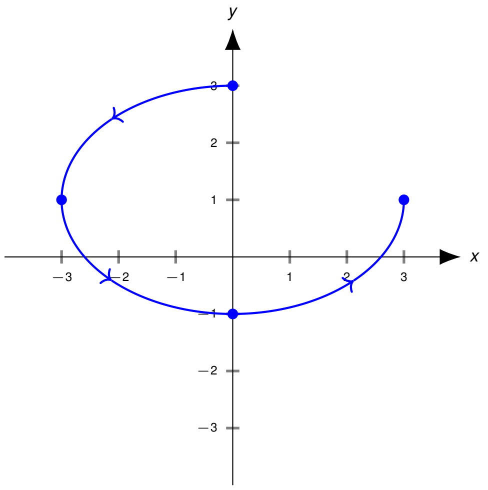
Problem 1.10 $\begin{cases} x = 3\cos(t) \\ y = 3\sin(t) \end{cases}$ *for* $0 \leq t \leq \pi$



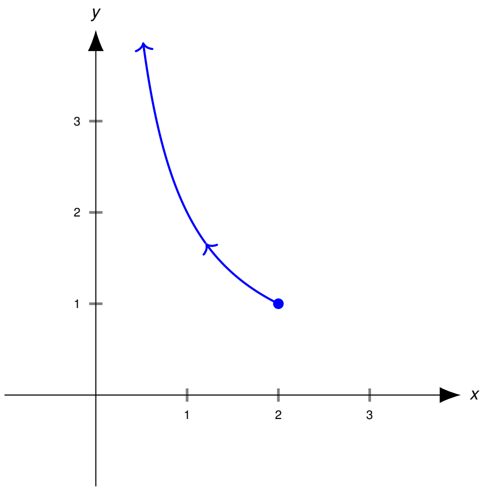
Problem 1.11 $\begin{cases} x = -1 + 3\cos(t) \\ y = 4\sin(t) \end{cases}$ *for* $0 \le t \le 2\pi$



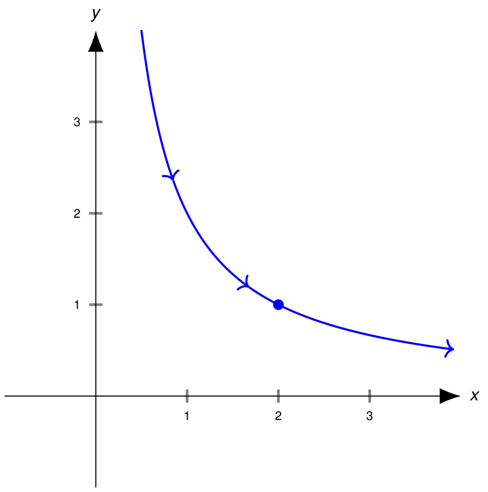
Problem 1.12 $\begin{cases} x = 3\cos(t) \\ y = 2\sin(t) + 1 \end{cases}$ *for* $\frac{\pi}{2} \le t \le 2\pi$



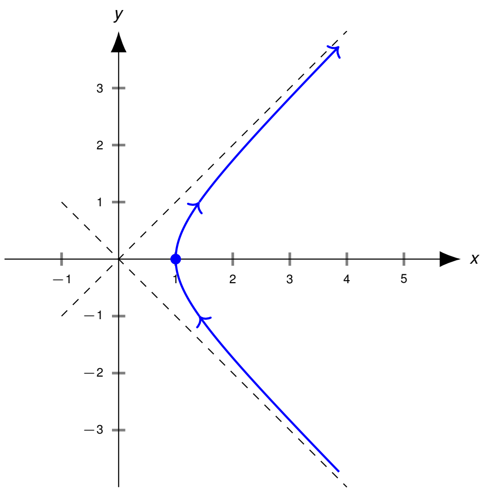
Problem 1.13 $\begin{cases} x = 2\cos(t) \\ y = \sec(t) \end{cases}$ *for* $0 \le t < \frac{\pi}{2}$



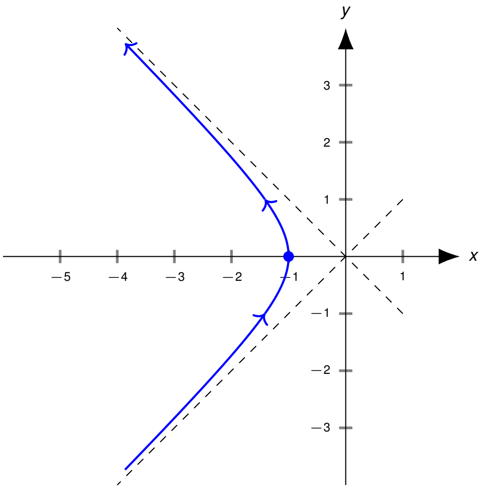
Problem 1.14 $\begin{cases} x = 2\tan(t) \\ y = \cot(t) \end{cases}$ *for* $0 < t < \dfrac{\pi}{2}$



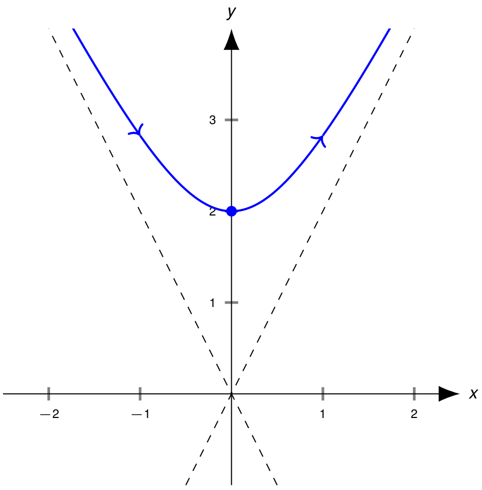
Problem 1.15 $\begin{cases} x = \sec(t) \\ y = \tan(t) \end{cases}$ *for* $-\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$



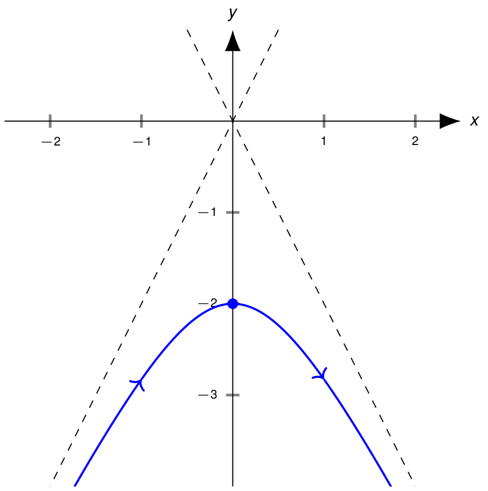
Problem 1.16 $\begin{cases} x = \sec(t) \\ y = \tan(t) \end{cases}$ *for* $\frac{\pi}{2} < t < \frac{3\pi}{2}$



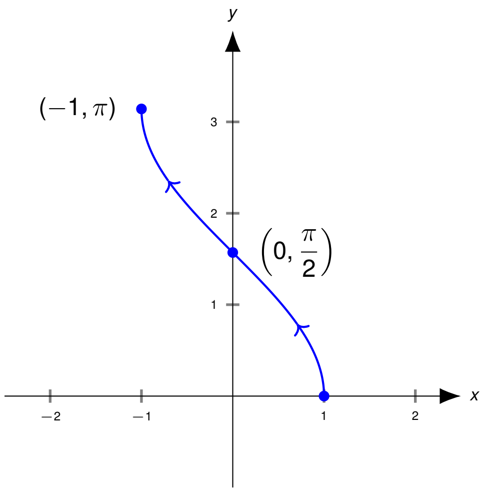
Problem 1.17 $\begin{cases} x = \tan(t) \\ y = 2\sec(t) \end{cases}$ *for* $-\frac{\pi}{2} < t < \frac{\pi}{2}$



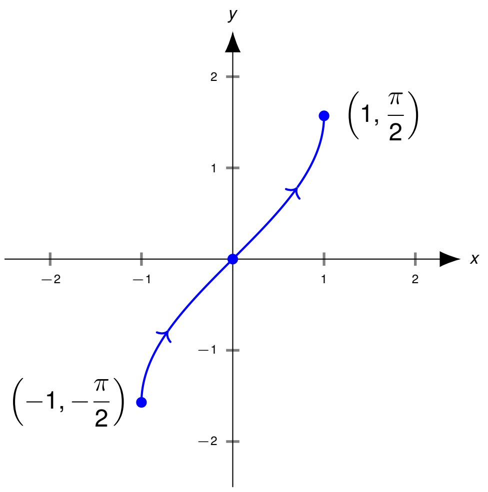
Problem 1.18 $\begin{cases} x = \tan(t) \\ y = 2\sec(t) \end{cases}$ *for* $\frac{\pi}{2} < t < \frac{3\pi}{2}$



Problem 1.19 $\begin{cases} x = \cos(t) \\ y = t \end{cases}$ *for* $0 \leq t \leq \pi$

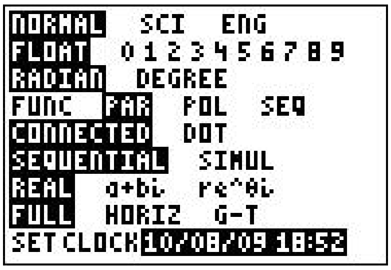


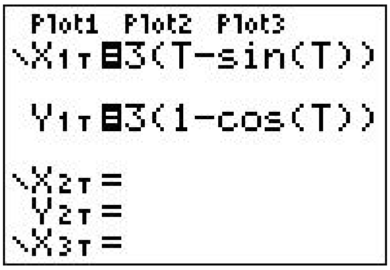
Problem 1.20 $\begin{cases} x = \sin(t) \\ y = t \end{cases}$ *for* $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$



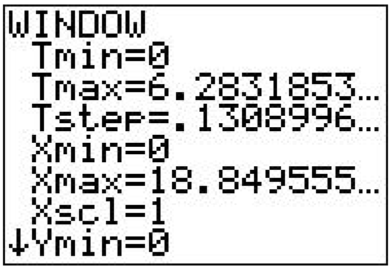
In the same way (and for the same reason) we took the time on page **??** in Section **??** to show how to graph polar equations using a graphing calculator, we take a few moments here to explain how to graph a system of parametric equations using a calculator. Our task is to graph the cycloid from Example **??**, $\{x = 3(t - \sin(t)),\ y = 3(1 - \cos(t))$ for $t \geq 0$ using a graphing calculator.

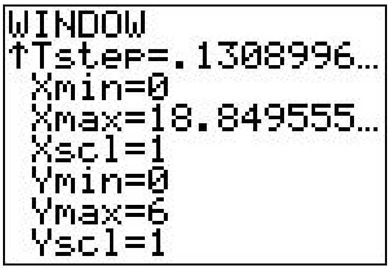
We first must ensure that the calculator is in 'Parametric Mode' and 'radian mode' when we enter the equations and advance to the 'Window' screen.



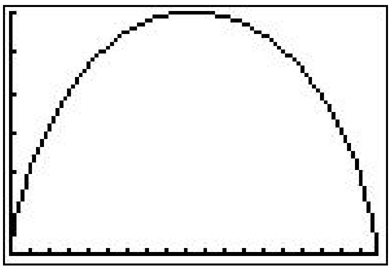


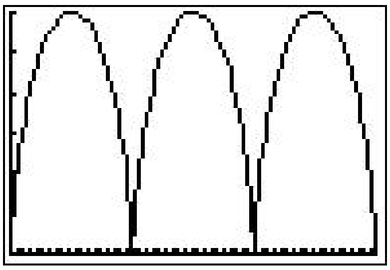
Our next step is to find appropriate bounds on the parameter, t, as well as for x and y. We know that one full revolution of the circle occurs over the interval $0 \leq t < 2\pi$, so it seems reasonable to keep these as our bounds on t. The 'Tstep' seems reasonably small – too large a value here can lead to incorrect graphs.[1] We know from our derivation of the equations of the cycloid that the center of the generating circle has coordinates $(r\theta, r) = (3t, 3)$. Since t ranges between 0 and 2π, we set x to range between 0 and 6π. The values of y go from the bottom of the circle to the top, so y ranges between 0 and 6.





Below we graph the cycloid with these settings, and then extend t to range from 0 to 6π which forces x to range from 0 to 18π yielding three arches of the cycloid.[2]



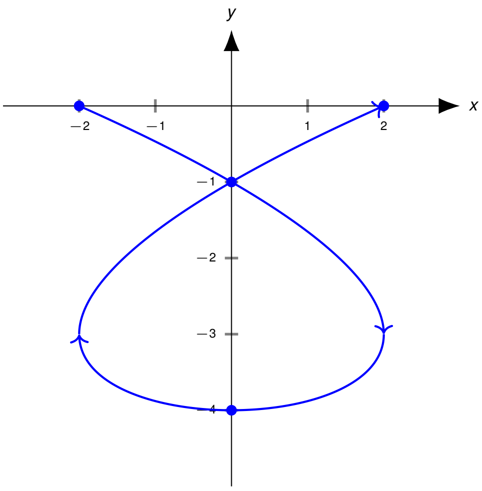


Question 2 *In Exercises* **??** *-* **??***, plot the set of parametric equations with the help of a graphing utility. Be sure to indicate the orientation imparted on the curve by the parametrization.*

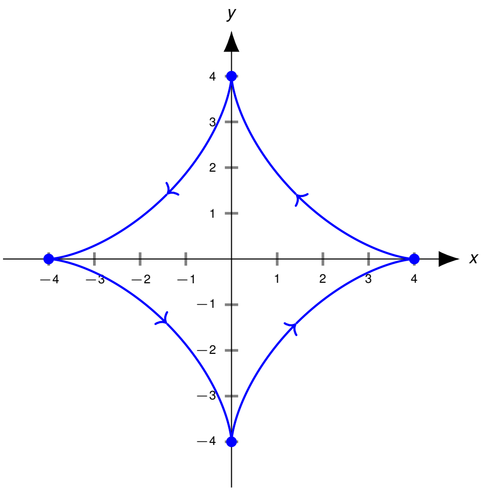
Problem 2.1 $\left\{ \begin{array}{l} x = t^3 - 3t \\ y = t^2 - 4 \end{array} \right.$ *for* $-2 \leq t \leq 2$

[1] Again, see page **??** in Section **??**.

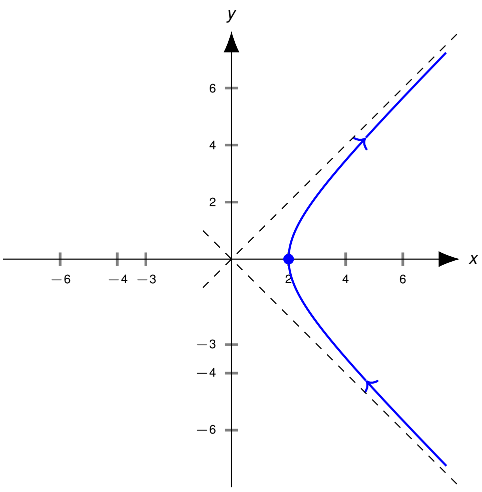
[2] It is instructive to note that keeping the y settings between 0 and 6 skews the aspect ratio of the cycloid. Using the 'Zoom Square' feature on the graphing calculator gives a true geometric perspective of the three arches.



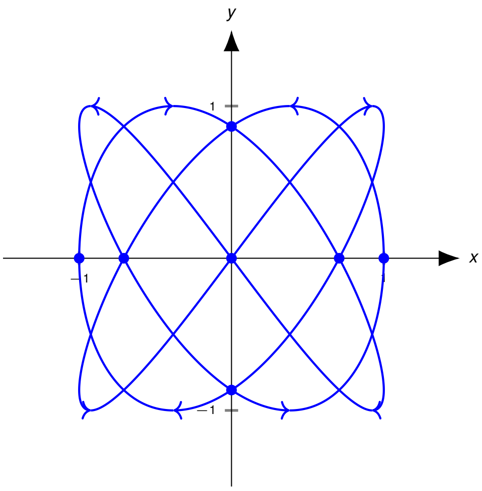
Problem 2.2 $\begin{cases} x = 4\cos^3(t) \\ y = 4\sin^3(t) \end{cases}$ *for* $0 \leq t \leq 2\pi$



Problem 2.3 $\begin{cases} x = e^t + e^{-t} \\ y = e^t - e^{-t} \end{cases}$ *for* $-2 \leq t \leq 2$



Problem 2.4 $\begin{cases} x = \cos(3t) \\ y = \sin(4t) \end{cases}$ *for* $0 \le t \le 2\pi$



Question 3 *In Exercises* **??** - **??**, *find a parametric description for the given oriented curve.*

Problem 3.1 *the directed line segment from* $(3, -5)$ *to* $(-2, 2)$

$\begin{cases} x = 3 - 5t \\ y = -5 + 7t \end{cases}$ *for* $0 \le t \le 1$

Problem 3.2 *the directed line segment from* $(-2, -1)$ *to* $(3, -4)$

$\begin{cases} x = 5t - 2 \\ y = -1 - 3t \end{cases}$ *for* $0 \le t \le 1$

Problem 3.3 *the curve* $y = 4 - x^2$ *from* $(-2, 0)$ *to* $(2, 0)$

$\begin{cases} x = t \\ y = 4 - t^2 \end{cases}$ *for* $-2 \le t \le 2$

Problem 3.4 *the curve* $y = 4 - x^2$ *from* $(-2, 0)$ *to* $(2, 0)$ *(Shift the parameter so* $t = 0$ *corresponds to* $(-2, 0)$*.)*

$\begin{cases} x = t - 2 \\ y = 4t - t^2 \end{cases}$ *for* $0 \le t \le 4$

Problem 3.5 *the curve* $x = y^2 - 9$ *from* $(-5, -2)$ *to* $(0, 3)$

$\begin{cases} x = t^2 - 9 \\ y = t \end{cases}$ *for* $-2 \le t \le 3$

Problem 3.6 *the curve* $x = y^2 - 9$ *from* $(0, 3)$ *to* $(-5, -2)$*. (Shift the parameter so* $t = 0$ *corresponds to* $(0, 3)$*.)*

$\begin{cases} x = t^2 - 6t \\ y = 3 - t \end{cases}$ *for* $0 \le t \le 5$

Problem 3.7 *the circle* $x^2 + y^2 = 25$*, oriented counter-clockwise*

$$\begin{cases} x = 5\cos(t) \\ y = 5\sin(t) \end{cases} \quad \textit{for } 0 \le t < 2\pi$$

Problem 3.8 *the circle* $(x-1)^2 + y^2 = 4$*, oriented counter-clockwise*

$$\begin{cases} x = 1 + 2\cos(t) \\ y = 2\sin(t) \end{cases} \quad \textit{for } 0 \le t < 2\pi$$

Problem 3.9 *the circle* $x^2 + y^2 - 6y = 0$*, oriented counter-clockwise*

$$\begin{cases} x = 3\cos(t) \\ y = 3 + 3\sin(t) \end{cases} \quad \textit{for } 0 \le t < 2\pi$$

Problem 3.10 *the circle* $x^2 + y^2 - 6y = 0$*, oriented* clockwise *(Shift the parameter so* t *begins at* 0*.)*

$$\begin{cases} x = 3\cos(t) \\ y = 3 - 3\sin(t) \end{cases} \quad \textit{for } 0 \le t < 2\pi$$

Problem 3.11 *the circle* $(x-3)^2 + (y+1)^2 = 117$*, oriented counter-clockwise*

$$\begin{cases} x = 3 + \sqrt{117}\cos(t) \\ y = -1 + \sqrt{117}\sin(t) \end{cases} \quad \textit{for } 0 \le t < 2\pi$$

Problem 3.12 *the ellipse* $(x-1)^2 + 9y^2 = 9$*, oriented counter-clockwise*

$$\begin{cases} x = 1 + 3\cos(t) \\ y = \sin(t) \end{cases} \quad \textit{for } 0 \le t < 2\pi$$

Problem 3.13 *the ellipse* $9x^2 + 4y^2 + 24y = 0$*, oriented counter-clockwise*

$$\begin{cases} x = 2\cos(t) \\ y = 3\sin(t) - 3 \end{cases} \quad \textit{for } 0 \le t < 2\pi$$

Problem 3.14 *the ellipse* $9x^2 + 4y^2 + 24y = 0$*, oriented clockwise (Shift the parameter so* $t = 0$ *corresponds to* $(0,0)$*.)*

$$\begin{cases} x = 2\cos\left(t - \frac{\pi}{2}\right) = 2\sin(t) \\ y = -3 - 3\sin\left(t - \frac{\pi}{2}\right) = -3 + 3\cos(t) \end{cases} \quad \textit{for } 0 \le t < 2\pi$$

Problem 3.15 *the triangle with vertices* $(0,0)$*,* $(3,0)$*,* $(0,4)$*, oriented counter-clockwise (Shift the parameter so* $t = 0$ *corresponds to* $(0,0)$*.)*

$\{x(t),\ y(t)$ *where:*

$$x(t) = \begin{cases} 3t, & 0 \le t \le 1 \\ 6 - 3t, & 1 \le t \le 2 \\ 0, & 2 \le t \le 3 \end{cases} \qquad y(t) = \begin{cases} 0, & 0 \le t \le 1 \\ 4t - 4, & 1 \le t \le 2 \\ 12 - 4t, & 2 \le t \le 3 \end{cases}$$

Problem 4 *Use parametric equations and a graphing utility to graph the inverse of* $f(x) = x^3 + 3x - 4$*.*

The parametric equations for the inverse are $\begin{cases} x = t^3 + 3t - 4 \\ y = t \end{cases}$ *for* $-\infty < t < \infty$

Problem 5 *Every polar curve $r = f(\theta)$ can be translated to a system of parametric equations with parameter θ by $\{x = r\cos(\theta) = f(\theta)\cos(\theta),\ y = r\sin(\theta) = f(\theta)\sin(\theta)\}$. Convert $r = 6\cos(2\theta)$ to a system of parametric equations. Check your answer by graphing $r = 6\cos(2\theta)$ by hand using the techniques presented in Section* **??** *and then graphing the parametric equations you found using a graphing utility.*

$r = 6\cos(2\theta)$ *translates to*

$$\begin{cases} x = 6\cos(2\theta)\cos(\theta) \\ y = 6\cos(2\theta)\sin(\theta) \end{cases} \quad \text{for } 0 \le \theta < 2\pi.$$

Problem 6 *Use your results from Exercises* **??** *and* **??** *in Section* **??** *to find the parametric equations which model a passenger's position as they ride the London Eye.*

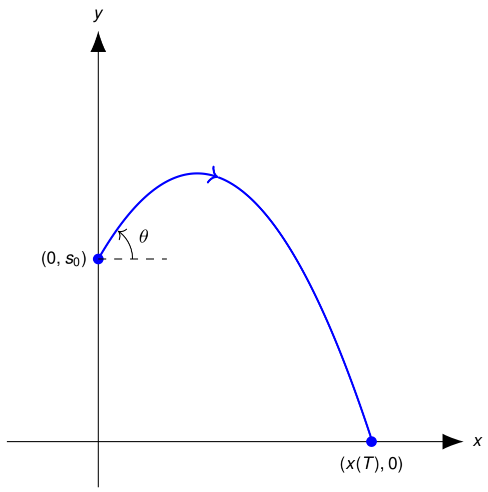
The parametric equations which describe the locations of passengers on the London Eye are

$$\begin{cases} x = 67.5\cos\left(\frac{\pi}{15}t - \frac{\pi}{2}\right) = 67.5\sin\left(\frac{\pi}{15}t\right) \\ y = 67.5\sin\left(\frac{\pi}{15}t - \frac{\pi}{2}\right) + 67.5 = 67.5 - 67.5\cos\left(\frac{\pi}{15}t\right) \end{cases} \quad \text{for } -\infty < t < \infty$$

Question 7 *Suppose an object, called a projectile, is launched into the air. Ignoring everything except the force gravity, the path of the projectile is given by*[3]

$$\begin{cases} x = v_0\cos(\theta)\,t \\ y = -\frac{1}{2}gt^2 + v_0\sin(\theta)\,t + s_0 \end{cases} \quad \text{for } 0 \le t \le T$$

where v_0 is the initial speed of the object, θ is the angle from the horizontal at which the projectile is launched,[4] *g is the acceleration due to gravity, s_0 is the initial height of the projectile above the ground and T is the time when the object returns to the ground. (See the figure below.)*



Problem 7.1 *Carl's friend Jason competes in Highland Games Competitions across the country. In one event, the 'hammer throw', he throws a 56 pound weight for distance. If the weight is released 6 feet above the ground at an angle of 42° with respect to the horizontal with an initial speed of 33 feet per second, find*

[3] A nice mix of vectors and Calculus are needed to derive this.

[4] We've seen this before. It's the angle of elevation which was defined on page **??**.

the parametric equations for the flight of the hammer. (Here, use $g = 32\frac{ft.}{s^2}$.) When will the hammer hit the ground? How far away will it hit the ground? Check your answer using a graphing utility.

The parametric equations for the hammer throw are $\begin{cases} x = 33\cos(42^\circ)t \\ y = -16t^2 + 33\sin(42^\circ)t + 6 \end{cases}$ *for* $t \geq 0$.

To find when the hammer hits the ground, we solve $y(t) = 0$ *and get* $t \approx -0.23$ *or* 1.61. *Since* $t \geq 0$, *the hammer hits the ground after approximately* $t = 1.61$ *seconds after it was launched into the air.*

To find how far away the hammer hits the ground, we find $x(1.61) \approx 39.48$ *feet from where it was thrown into the air.*

Problem 7.2 *Eliminate the parameter in the equations for projectile motion to show that the path of the projectile follows the curve*

$$y = -\frac{g\sec^2(\theta)}{2v_0^2}x^2 + \tan(\theta)x + s_0$$

*Use the vertex formula (Equation **??**) to show the maximum height of the projectile is*

$$y = \frac{v_0^2\sin^2(\theta)}{2g} + s_0 \quad \textit{when} \quad x = \frac{v_0^2\sin(2\theta)}{2g}$$

Problem 7.3 *In another event, the 'sheaf toss', Jason throws a 20 pound weight for height. If the weight is released 5 feet above the ground at an angle of* 85° *with respect to the horizontal and the sheaf reaches a maximum height of 31.5 feet, use your results from part **??** to determine how fast the sheaf was launched into the air. (Once again, use $g = 32\frac{ft.}{s^2}$.)*

We solve $y = \frac{v_0^2\sin^2(\theta)}{2g} + s_0 = \frac{v_0^2\sin^2(85^\circ)}{2(32)} + 5 = 31.5$ *to get* $v_0 = \pm 41.34$.

The initial speed of the sheaf was approximately 41.34 *feet per second.*

Problem 7.4 *Suppose* $\theta = \frac{\pi}{2}$. *(The projectile was launched vertically.) Simplify the general parametric formula given for* $y(t)$ *above using* $g = 9.8\,\frac{m}{s^2}$ *and compare that to the formula for* $s(t)$ *given in Exercise **??** in Section **??**. What is* $x(t)$ *in this case?*

Problem 8 *If f and g are functions, explain why the function* $\mathbf{r}(t) = \langle f(t), g(t)\rangle$ *is a function. The function* $\mathbf{r}$ *is called a **vector-valued function** since it matches real number inputs, t, with vector outputs,* $\mathbf{r}(t)$. *Explain why when the vectors* $\mathbf{r}(t)$ *are plotted in standard position, their terminal points trace out the curve described parametrically by the system of equations:* $\{x = f(t)\ y = g(t)$ *(In Calculus, you will see systems of parametric equations 'packaged' together using vectors.)*

Question 9 *In Exercises **??** - **??**, we explore the **hyperbolic cosine** function, denoted* $\cosh(t)$, *and the **hyperbolic sine** function, denoted* $\sinh(t)$, *defined below:*

$$\cosh(t) = \frac{e^t + e^{-t}}{2} \quad \textit{and} \quad \sinh(t) = \frac{e^t - e^{-t}}{2}$$

Problem 9.1 *Using a graphing utility as needed, verify the following:*

1. *the domain of* $\cosh(t)$ *is* $(-\infty, \infty)$ *and the range of* $\cosh(t)$ *is* $[1, \infty)$.
2. *the domain and range of* $\sinh(t)$ *are both* $(-\infty, \infty)$.

Problem 9.2 *Show that* $\{x(t) = \cosh(t),\ y(t) = \sinh(t)$ *parametrize the right half of the 'unit' hyperbola* $x^2 - y^2 = 1$. *(Hence the use of the adjective 'hyperbolic.')*

Problem 9.3 *Compare and contrast the definitions of* $\cosh(t)$ *and* $\sinh(t)$ *to the formulas for* $\cos(t)$ *and* $\sin(t)$ *given in Exercise* **??** *in Section* **??**.

Problem 9.4 *Four other hyperbolic functions are waiting to be defined: the hyperbolic secant sech*(t)*, the hyperbolic cosecant csch*(t)*, the hyperbolic tangent* $\tanh(t)$ *and the hyperbolic cotangent* $\coth(t)$*. Define these functions in terms of* $\cosh(t)$ *and* $\sinh(t)$*, then convert them to formulas involving* e^t *and* e^{-t}*. Consult a suitable reference (a Calculus book, or this entry on the hyperbolic functions) and spend some time reliving the thrills of trigonometry with these 'hyperbolic' functions.*

Problem 9.5 *If these functions look familiar, they should. Enjoy some nostalgia and revisit Exercise* **??** *in Section* **??***, Exercise* **??** *in Section* **??** *and the answer to Exercise* **??** *in Section* **??**.