Exercises for Polar Complex

Question 1 *In Exercises ?? - ??, find a polar representation for the complex number z. Identify $Re(z)$, $Im(z)$, $|z|$, $arg(z)$ and $Arg(z)$.*

Problem 1.1 $z = 9 + 9i$

$z = 9\sqrt{2}cis\left(\frac{\pi}{4}\right)$

$Re(z) = 9$

$Im(z) = 9$

$|z| = 9\sqrt{2}$

$arg(z) = \left\{\frac{\pi}{4} + 2\pi k \mid k \text{ is an integer}\right\}$

$Arg(z) = \frac{\pi}{4}$

Problem 1.2 $z = 5 + 5i\sqrt{3}$

$z = 10cis\left(\frac{\pi}{3}\right)$ *(Choose θ so $0 \leq \theta \leq 2\pi$.)*

$Re(z) = 5$

$Im(z) = 5\sqrt{3}$

$|z| = 10$

$arg(z) = \left\{\frac{\pi}{3} + 2\pi k \mid k \text{ is an integer}\right\}$ *(Choose θ so $0 \leq \theta \leq 2\pi$.)*

$Arg(z) = \frac{\pi}{3}$

Problem 1.3 $z = 6i$

$z = 6cis\left(\frac{\pi}{2}\right)$

$Re(z) = 0$

$Im(z) = 6$

$|z| = 6$

$arg(z) = \left\{\frac{\pi}{2} + 2\pi k \mid k \text{ is an integer}\right\}$

$Arg(z) = \frac{\pi}{2}$

Problem 1.4 $z = -3\sqrt{2} + 3i\sqrt{2}$

$z = 6cis\left(\frac{3\pi}{4}\right)$ *(Choose θ so $0 \leq \theta \leq 2\pi$.)*

$Re(z) = -3\sqrt{2}$

$Im(z) = 3\sqrt{2}$

$|z| = 6$

$arg(z) = \left\{\frac{3\pi}{4} + 2\pi k \mid k \text{ is an integer}\right\}$ *(Choose θ so $0 \leq \theta \leq 2\pi$.)*

$Arg(z) = \frac{3\pi}{4}$

Problem 1.5 $z = -6\sqrt{3} + 6i$

$z = 12cis\left(\frac{5\pi}{6}\right)$

$Re(z) = -6\sqrt{3}$

$Im(z) = 6$

$|z| = 12$

$arg(z) = \left\{\frac{5\pi}{6} + 2\pi k \mid k \text{ is an integer}\right\}$

$Arg(z) = \frac{5\pi}{6}$

Problem 1.6 $z = -2$

$z = 2cis(\pi)$ (Choose θ so $0 \leq \theta \leq 2\pi$.)

$Re(z) = -2$

$Im(z) = 0$

$|z| = 2$

$arg(z) = \{\pi + 2\pi k = (2k+1)\pi \mid k \text{ is an integer}\}$ (Choose θ so $0 \leq \theta \leq 2\pi$.)

$Arg(z) = \pi$

Problem 1.7 $z = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$

$z = cis\left(\frac{7\pi}{6}\right)$

$Re(z) = -\frac{\sqrt{3}}{2}$

$Im(z) = -\frac{1}{2}$

$|z| = 1$

$arg(z) = \left\{\frac{7\pi}{6} + 2\pi k \mid k \text{ is an integer}\right\}$

$Arg(z) = -\frac{5\pi}{6}$

Problem 1.8 $z = -3 - 3i$

$z = 3\sqrt{2}cis\left(\frac{5\pi}{4}\right)$ (Choose θ so $0 \leq \theta \leq 2\pi$.)

$Re(z) = -3$

$Im(z) = -3$

$|z| = 3\sqrt{2}$

$arg(z) = \left\{\frac{5\pi}{4} + 2\pi k \mid k \text{ is an integer}\right\}$ (Choose θ so $0 \leq \theta \leq 2\pi$.)

$Arg(z) = -\frac{3\pi}{4}$

Problem 1.9 $z = -5i$

$z = 5cis\left(\frac{3\pi}{2}\right)$

$Re(z) = 0$

$Im(z) = -5$

$|z| = 5$

$arg(z) = \left\{ \frac{3\pi}{2} + 2\pi k \mid k \text{ is an integer} \right\}$

$Arg(z) = -\frac{\pi}{2}$

Problem 1.10 $z = 2\sqrt{2} - 2i\sqrt{2}$

$z = 4cis\left(\frac{7\pi}{4}\right)$ *(Choose θ so $0 \leq \theta \leq 2\pi$.)*

$Re(z) = 2\sqrt{2}$

$Im(z) = -2\sqrt{2}$

$|z| = 4$

$arg(z) = \left\{ \frac{7\pi}{4} + 2\pi k \mid k \text{ is an integer} \right\}$ *(Choose θ so $0 \leq \theta \leq 2\pi$.)*

$Arg(z) = -\frac{\pi}{4}$

Problem 1.11 $z = 6$

$z = 6cis(0)$

$Re(z) = 6$

$Im(z) = 0$

$|z| = 6$

$arg(z) = \{0 + 2\pi k = 2\pi k \mid k \text{ is an integer}\}$

$Arg(z) = 0$

Problem 1.12 $z = i\sqrt[3]{7}$

$z = \sqrt[3]{7}cis\left(\frac{\pi}{2}\right)$ *(Choose θ so $0 \leq \theta \leq 2\pi$.)*

$Re(z) = 0$

$Im(z) = \sqrt[3]{7}$

$|z| = \sqrt[3]{7}$

$arg(z) = \left\{ \frac{\pi}{2} + 2\pi k \mid k \text{ is an integer} \right\}$ *(Choose θ so $0 \leq \theta \leq 2\pi$.)*

$Arg(z) = \frac{\pi}{2}$

Problem 1.13 $z = 3 + 4i$

$z = 5cis\left(\arctan\left(\frac{4}{3}\right)\right)$

$Re(z) = 3$

$Im(z) = 4$

$|z| = 5$

$arg(z) = \left\{ \arctan\left(\frac{4}{3}\right) + 2\pi k \mid k \text{ is an integer} \right\}$

$Arg(z) = \arctan\left(\frac{4}{3}\right)$

Problem 1.14 $z = \sqrt{2} + i$

$z = \sqrt{3} cis\left(\arctan\left(\frac{\sqrt{2}}{2}\right)\right)$ *(Choose θ so $0 \leq \theta \leq 2\pi$.)*

$Re(z) = \sqrt{2}$

$Im(z) = 1$

$|z| = \sqrt{3}$

$arg(z) = \left\{\arctan\left(\frac{\sqrt{2}}{2}\right) + 2\pi k \mid k \text{ is an integer}\right\}$ *(Choose θ so $0 \leq \theta \leq 2\pi$.)*

$Arg(z) = \arctan\left(\frac{\sqrt{2}}{2}\right)$

Problem 1.15 $z = -7 + 24i$

$z = 25cis\left(\pi - \arctan\left(\frac{24}{7}\right)\right)$

$Re(z) = -7$

$Im(z) = 24$

$|z| = 25$

$arg(z) = \left\{\pi - \arctan\left(\frac{24}{7}\right) + 2\pi k \mid k \text{ is an integer}\right\}$

$Arg(z) = \pi - \arctan\left(\frac{24}{7}\right)$

Problem 1.16 $z = -2 + 6i$

$z = 2\sqrt{10} cis\left(\pi - \arctan(3)\right)$ *(Choose θ so $0 \leq \theta \leq 2\pi$.)*

$Re(z) = -2$

$Im(z) = 6$

$|z| = 2\sqrt{10}$

$arg(z) = \{\pi - \arctan(3) + 2\pi k \mid k \text{ is an integer}\}$ *(Choose θ so $0 \leq \theta \leq 2\pi$.)*

$Arg(z) = \pi - \arctan(3)$.

Problem 1.17 $z = -12 - 5i$

$z = 13cis\left(\pi + \arctan\left(\frac{5}{12}\right)\right)$

$Re(z) = -12$

$Im(z) = -5$

$|z| = 13$

$arg(z) = \left\{\pi + \arctan\left(\frac{5}{12}\right) + 2\pi k \mid k \text{ is an integer}\right\}$

$Arg(z) = \arctan\left(\frac{5}{12}\right) - \pi$.

Problem 1.18 $z = -5 - 2i$

$z = \sqrt{29}cis\left(\pi + \arctan\left(\frac{2}{5}\right)\right)$ *(Choose θ so $0 \leq \theta \leq 2\pi$.)*

$Re(z) = -5$

$Im(z) = -2$

$|z| = \sqrt{29}$

$arg(z) = \left\{\pi + \arctan\left(\frac{2}{5}\right) + 2\pi k \mid k \text{ is an integer}\right\}$

$Arg(z) = \arctan\left(\frac{2}{5}\right) - \pi.$

Problem 1.19 $z = 4 - 2i$

$z = 2\sqrt{5}cis\left(\arctan\left(-\frac{1}{2}\right)\right)$

$Re(z) = 4$

$Im(z) = -2$

$|z| = 2\sqrt{5}$

$arg(z) = \left\{\arctan\left(-\frac{1}{2}\right) + 2\pi k \mid k \text{ is an integer}\right\}$

$Arg(z) = \arctan\left(-\frac{1}{2}\right) = -\arctan\left(\frac{1}{2}\right).$

Problem 1.20 $z = 1 - 3i$

$z = \sqrt{10}cis\left(\arctan\left(-3\right)\right)$ *(Choose θ so $-\pi < \theta \leq \pi$.)*

$Re(z) = 1$

$Im(z) = -3$

$|z| = \sqrt{10}$

$arg(z) = \left\{\arctan\left(-3\right) + 2\pi k \mid k \text{ is an integer}\right\}$ *(Choose θ so $-\pi < \theta \leq \pi$.)*

$Arg(z) = \arctan\left(-3\right).$

Question 2 *In Exercises **??** - **??**, find the rectangular form of the given complex number. Use whatever identities are necessary to find the exact values.*

Problem 2.1 $z = 6cis(0)$

$z = 6$

Problem 2.2 $z = 2cis\left(\frac{\pi}{6}\right)$

$z = \sqrt{3} + 1i$

Problem 2.3 $z = 7\sqrt{2}cis\left(\frac{\pi}{4}\right)$

$z = 7 + 7i$

Problem 2.4 $z = 3cis\left(\frac{\pi}{2}\right)$

$z = 0 + 3i$

Problem 2.5 $z = 4cis\left(\frac{2\pi}{3}\right)$

$z = -2 + 2i\sqrt{3}$

Problem 2.6 $z = \sqrt{6}cis\left(\frac{3\pi}{4}\right)$

$z = -\sqrt{3} + \sqrt{3}\,i$

Problem 2.7 $z = 9cis\left(\pi\right)$

$z = -9$

Problem 2.8 $z = 3cis\left(\frac{4\pi}{3}\right)$

$z = -\frac{3}{2} + -\frac{3\sqrt{3}}{2}i$

Problem 2.9 $z = 7cis\left(-\frac{3\pi}{4}\right)$

$z = -\frac{7\sqrt{2}}{2} - \frac{7\sqrt{2}}{2}i$

Problem 2.10 $z = \sqrt{13}cis\left(\frac{3\pi}{2}\right)$

$z = -\sqrt{13}\,i$

Problem 2.11 $z = \frac{1}{2}cis\left(\frac{7\pi}{4}\right)$

$z = \frac{\sqrt{2}}{4} - i\frac{\sqrt{2}}{4}$

Problem 2.12 $z = 12cis\left(-\frac{\pi}{3}\right)$

$z = 6 + -6\sqrt{3}\,i$

Problem 2.13 $z = 8cis\left(\frac{\pi}{12}\right)$

$z = 4\sqrt{2 + \sqrt{3}} + 4i\sqrt{2 - \sqrt{3}}$

Problem 2.14 $z = 2cis\left(\frac{7\pi}{8}\right)$

$z = -\sqrt{2 + \sqrt{2}} + \sqrt{2 - \sqrt{2}}\,i$

Problem 2.15 $z = 5cis\left(\arctan\left(\frac{4}{3}\right)\right)$

$z = 3 + 4i$

Problem 2.16 $z = \sqrt{10}cis\left(\arctan\left(\frac{1}{3}\right)\right)$

$z = 3 + 1\,i$

Problem 2.17 $z = 15cis\,(\arctan(-2))$

$z = 3\sqrt{5} - 6i\sqrt{5}$

Problem 2.18 $z = \sqrt{3}cis\left(\arctan\left(-\sqrt{2}\right)\right)$

$z = 1 + -\sqrt{2}\,i$

Problem 2.19 $z = 50cis\left(\pi - \arctan\left(\frac{7}{24}\right)\right)$

$z = -48 + 14i$

Problem 2.20 $z = \frac{1}{2}cis\left(\pi + \arctan\left(\frac{5}{12}\right)\right)$

$z = -\frac{6}{13} + -\frac{5}{26}i$

Question 3 *For Exercises **??** - **??**, use* $z = -\frac{3\sqrt{3}}{2} + \frac{3}{2}i$ *and* $w = 3\sqrt{2} - 3i\sqrt{2}$ *to compute the quantity. Express your answers in polar form using the principal argument.*

Problem 3.1 zw

$zw = 18cis\left(\frac{7\pi}{12}\right)$

Problem 3.2 $\frac{z}{w}$

$\frac{z}{w} = \frac{1}{2}cis\left(-\frac{11\pi}{12}\right)$

Problem 3.3 $\frac{w}{z}$

$\frac{w}{z} = 2cis\left(\frac{11\pi}{12}\right)$

Problem 3.4 z^4

$z^4 = 81cis\left(-\frac{2\pi}{3}\right)$

Problem 3.5 w^3

$w^3 = 216cis\left(-\frac{3\pi}{4}\right)$

Problem 3.6 z^5w^2

$z^5w^2 = 8748cis\left(-\frac{\pi}{3}\right)$

Problem 3.7 z^3w^2

$z^3w^2 = 972cis(0)$

Problem 3.8 $\frac{z^2}{w}$

$\frac{z^2}{w} = \frac{3}{2}cis\left(-\frac{\pi}{12}\right)$

Problem 3.9 $\frac{w}{z^2}$

$\frac{w}{z^2} = \frac{2}{3}cis\left(\frac{\pi}{12}\right)$

Problem 3.10 $\frac{z^3}{w^2}$

$\frac{z^3}{w^2} = \frac{3}{4}cis(\pi)$

Problem 3.11 $\frac{w^2}{z^3}$

$\frac{w^2}{z^3} = \frac{4}{3}cis(\pi)$

Problem 3.12 $\left(\frac{w}{z}\right)^6$

$\left(\frac{w}{z}\right)^6 = 64cis\left(-\frac{\pi}{2}\right)$

Question 4 *In Exercises* **??** - **??***, use DeMoivre's Theorem to find the indicated power of the given complex number. Express your final answers in rectangular form.*

Problem 4.1 $\left(-2+2i\sqrt{3}\right)^3$

$\left(-2+2i\sqrt{3}\right)^3 = 64$

Problem 4.2 $(-\sqrt{3}-i)^3$

$(-\sqrt{3}-i)^3 = 0 + -8\,i$

Problem 4.3 $(-3+3i)^4$

$(-3+3i)^4 = -324$

Problem 4.4 $(\sqrt{3}+i)^4$

$(\sqrt{3}+i)^4 = -8+8\sqrt{3}\,i$

Problem 4.5 $\left(\frac{5}{2}+\frac{5}{2}i\right)^3$

$\left(\frac{5}{2}+\frac{5}{2}i\right)^3 = -\frac{125}{4}+\frac{125}{4}i$

Problem 4.6 $\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^6$

$\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^6 = 1+0i$

Problem 4.7 $\left(\frac{3}{2}-\frac{3}{2}i\right)^3$

$\left(\frac{3}{2}-\frac{3}{2}i\right)^3 = -\frac{27}{4}-\frac{27}{4}i$

Problem 4.8 $\left(\frac{\sqrt{3}}{3} - \frac{1}{3}i\right)^4$

$\left(\frac{\sqrt{3}}{3} - \frac{1}{3}i\right)^4 = -\frac{8}{81} + -\frac{8\sqrt{3}}{81}i$

Problem 4.9 $\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^4$

$\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^4 = -1$

Problem 4.10 $(2+2i)^5$

$(2+2i)^5 = -128 + -128i$

Problem 4.11 $(\sqrt{3} - i)^5$

$(\sqrt{3} - i)^5 = -16\sqrt{3} - 16i$

Problem 4.12 $(1-i)^8$

$(1-i)^8 = 16 + 0i$

Question 5 *In Exercises **??** - **??**, find the indicated complex roots. Express your answers in polar form and then convert them into rectangular form.*

Problem 5.1 *the two square roots of* $z = 4i$

Since $z = 4i = 4cis\left(\frac{\pi}{2}\right)$ *we have*

$w_0 = 2cis\left(\frac{\pi}{4}\right) = \sqrt{2} + i\sqrt{2}$

$w_1 = 2cis\left(\frac{5\pi}{4}\right) = -\sqrt{2} - i\sqrt{2}$

Problem 5.2 *the two square roots of* $z = -25i$

Since $z = -25i = 25cis\left(\frac{3\pi}{2}\right)$ *we have*

$w_0 = 5cis\left(\frac{3\pi}{4}\right) = -\frac{5\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}i$

$w_1 = 5cis\left(\frac{7\pi}{4}\right) = \frac{5\sqrt{2}}{2} - \frac{5\sqrt{2}}{2}i$

Problem 5.3 *the two square roots of* $z = 1 + i\sqrt{3}$

Since $z = 1 + i\sqrt{3} = 2cis\left(\frac{\pi}{3}\right)$ *we have*

$w_0 = \sqrt{2}cis\left(\frac{\pi}{6}\right) = \frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i$

$w_1 = \sqrt{2}cis\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i$

Problem 5.4 *the two square roots of* $\frac{5}{2} - \frac{5\sqrt{3}}{2}i$

Since $z = \frac{5}{2} - \frac{5\sqrt{3}}{2}i = 5cis\left(\frac{5\pi}{3}\right)$ *we have*

$w_0 = \sqrt{5}cis\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{15}}{2} + \frac{\sqrt{5}}{2}i$

$w_1 = \sqrt{5}cis\left(\frac{11\pi}{6}\right) = \frac{\sqrt{15}}{2} - \frac{\sqrt{5}}{2}i$

Problem 5.5 *the three cube roots of* $z = 64$

Since $z = 64 = 64cis(0)$ *we have*

$w_0 = 4cis(0) = 4$

$w_1 = 4cis\left(\frac{2\pi}{3}\right) = -2 + 2i\sqrt{3}$

$w_2 = 4cis\left(\frac{4\pi}{3}\right) = -2 - 2i\sqrt{3}$

Problem 5.6 *the three cube roots of* $z = -125$

Since $z = -125 = 125cis(\pi)$ *we have*

$w_0 = 5cis\left(\frac{\pi}{3}\right) = \frac{5}{2} + \frac{5\sqrt{3}}{2}i$

$w_1 = 5cis(\pi) = -5$

$w_2 = 5cis\left(\frac{5\pi}{3}\right) = \frac{5}{2} - \frac{5\sqrt{3}}{2}i$

Problem 5.7 *the three cube roots of* $z = i$

Since $z = i = cis\left(\frac{\pi}{2}\right)$ *we have*

$w_0 = cis\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}i$

$w_1 = cis\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$

$w_2 = cis\left(\frac{3\pi}{2}\right) = -i$

Problem 5.8 *the three cube roots of* $z = -8i$

Since $z = -8i = 8cis\left(\frac{3\pi}{2}\right)$ *we have*

$w_0 = 2cis\left(\frac{\pi}{2}\right) = 2i$

$w_1 = 2cis\left(\frac{7\pi}{6}\right) = -\sqrt{3} - i$

$w_2 = cis\left(\frac{11\pi}{6}\right) = \sqrt{3} - i$

Problem 5.9 *the four fourth roots of* $z = 16$

Since $z = 16 = 16cis\,(0)$ *we have*

$w_0 = 2cis\,(0) = 2$

$w_1 = 2cis\left(\frac{\pi}{2}\right) = 2i$

$w_2 = 2cis\,(\pi) = -2$

$w_3 = 2cis\left(\frac{3\pi}{2}\right) = -2i$

Problem 5.10 *the four fourth roots of* $z = -81$

Since $z = -81 = 81cis\,(\pi)$ *we have*

$w_0 = 3cis\left(\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$

$w_1 = 3cis\left(\frac{3\pi}{4}\right) = -\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$

$w_2 = 3cis\left(\frac{5\pi}{4}\right) = -\frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}i$

$w_3 = 3cis\left(\frac{7\pi}{4}\right) = \frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}i$

Problem 5.11 *the six sixth roots of* $z = 64$

Since $z = 64 = 64cis(0)$ *we have*

$w_0 = 2cis(0) = 2$

$w_1 = 2cis\left(\frac{\pi}{3}\right) = 1 + \sqrt{3}i$

$w_2 = 2cis\left(\frac{2\pi}{3}\right) = -1 + \sqrt{3}i$

$w_3 = 2cis\,(\pi) = -2$

$w_4 = 2cis\left(-\frac{2\pi}{3}\right) = -1 - \sqrt{3}i$

$w_5 = 2cis\left(-\frac{\pi}{3}\right) = 1 - \sqrt{3}i$

Problem 5.12 *the six sixth roots of* $z = -729$

Since $z = -729 = 729cis(\pi)$ *we have*

$w_0 = 3cis\left(\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{2} + \frac{3}{2}i$

$w_1 = 3cis\left(\frac{\pi}{2}\right) = 3i$

$w_2 = 3cis\left(\frac{5\pi}{6}\right) = -\frac{3\sqrt{3}}{2} + \frac{3}{2}i$

$w_3 = 3cis\left(\frac{7\pi}{6}\right) = -\frac{3\sqrt{3}}{2} - \frac{3}{2}i$

$w_4 = 3cis\left(-\frac{3\pi}{2}\right) = -3i$

$w_5 = 3cis\left(-\frac{11\pi}{6}\right) = \frac{3\sqrt{3}}{2} - \frac{3}{2}i$

Problem 6 *Use the Sum and Difference Identities in Theorem **??** or the Half Angle Identities in Theorem **??** to convert the three cube roots of $z = \sqrt{2} + i\sqrt{2}$ we found in Example **??**, number **??** from polar form to rectangular form.*

Note: In the answers for w_0 and w_2 the first rectangular form comes from applying the appropriate Sum or Difference Identity ($\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$ and $\frac{17\pi}{12} = \frac{2\pi}{3} + \frac{3\pi}{4}$, respectively) and the second comes from using the Half-Angle Identities.

$$w_0 = \sqrt[3]{2}cis\left(\frac{\pi}{12}\right) = \sqrt[3]{2}\left(\frac{\sqrt{6}+\sqrt{2}}{4} + i\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)\right) = \sqrt[3]{2}\left(\frac{\sqrt{2+\sqrt{3}}}{2} + i\frac{\sqrt{2-\sqrt{3}}}{2}\right)$$

$$w_1 = \sqrt[3]{2}cis\left(\frac{3\pi}{4}\right) = \sqrt[3]{2}\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$$

$$w_2 = \sqrt[3]{2}cis\left(\frac{17\pi}{12}\right) = \sqrt[3]{2}\left(\frac{\sqrt{2}-\sqrt{6}}{4} + i\left(\frac{-\sqrt{2}-\sqrt{6}}{4}\right)\right) = \sqrt[3]{2}\left(\frac{\sqrt{2-\sqrt{3}}}{2} + i\frac{\sqrt{2+\sqrt{3}}}{2}\right)$$

Problem 7 *Use a calculator to approximate the rectangular form of the five fifth roots of 1 we found in Example **??**, number **??**.*

$$w_0 = cis(0) = 1$$

$$w_1 = cis\left(\frac{2\pi}{5}\right) \approx 0.309 + 0.951i$$

$$w_2 = cis\left(\frac{4\pi}{5}\right) \approx -0.809 + 0.588i$$

$$w_3 = cis\left(\frac{6\pi}{5}\right) \approx -0.809 - 0.588i$$

$$w_4 = cis\left(\frac{8\pi}{5}\right) \approx 0.309 - 0.951i$$

Problem 8 *According to Theorem **??** in Section **??**, the polynomial $p(x) = x^4 + 4$ can be factored into the product linear and irreducible quadratic factors. In Exercise **??** in Section **??**, we showed you how to factor this polynomial into the product of two irreducible quadratic factors using a system of non-linear equations. Now that we can compute the complex fourth roots of -4 directly using Theorem **??**, we can apply the Complex Factorization Theorem, Theorem **??**, to obtain the linear factorization $p(x) = (x - (1+i))(x - (1-i))(x - (-1+i))(x - (-1-i))$. By multiplying the first two factors together and then the second two factors together, thus pairing up the complex conjugate pairs of zeros Theorem **??** told us we'd get, we have that $p(x) = (x^2 - 2x + 2)(x^2 + 2x + 2)$. Use the 12 complex 12^{th} roots of 4096 to factor $p(x) = x^{12} - 4096$ into a product of linear and irreducible quadratic factors.*

$$p(x) = x^{12} - 4096 = (x-2)(x+2)(x^2+4)(x^2-2x+4)(x^2+2x+4)(x^2-2\sqrt{3}x+4)(x^2+2\sqrt{3}+4)$$

Problem 9 *Use Exercise **??** from Section **??** to show the the Triangle Inequality $|z+w| \leq |z| + |w|$ holds for all complex numbers z and w as well. Identify the complex number $z = a + bi$ with the vector $u = \langle a, b\rangle$ and identify the complex number $w = c + di$ with the vector $v = \langle c, d\rangle$ and just follow your nose!*

Problem 10 *Complete the proof of Theorem **??** by showing that if $w \neq 0$ than $\left|\frac{1}{w}\right| = \frac{1}{|w|}$.*

Problem 11 *Recall from Section **??** that given a complex number $z = a + bi$ its complex conjugate, denoted $\overline{z}$, is given by $\overline{z} = a - bi$.*

1. *Prove that $|\overline{z}| = |z|$.*

2. *Prove that* $|z| = \sqrt{z\overline{z}}$

3. *Show that* $Re(z) = \dfrac{z+\overline{z}}{2}$ *and* $Im(z) = \dfrac{z-\overline{z}}{2i}$

4. *Show that if* $\theta \in arg(z)$ *then* $-\theta \in arg(\overline{z})$*. Interpret this result geometrically.*

5. *Is it always true that* $Arg(\overline{z}) = -Arg(z)$*?*

Problem 12 *Given a natural number* $n \geq 2$*, the* n *complex* n^{th} *roots of* $z = 1$ *are called the* ***n^{th} Roots of Unity****. In the following exercises, assume that* n *is a fixed, but arbitrary, natural number such that* $n \geq 2$*.*

1. *Show that* $w = 1$ *is an* n^{th} *root of unity.*

2. *Show that if both* w_j *and* w_k *are* n^{th} *roots of unity then so is their product* $w_j w_k$*.*

3. *Show that if* w_j *is an* n^{th} *root of unity then there is an* n^{th} *root of unity* $w_{j'}$ *so that* $w_j w_{j'} = 1$*.*
   HINT: If $w_j = cis(\theta)$ *let* $w_{j'} = cis(2\pi - \theta)$*. Show* $w_{j'} = cis(2\pi - \theta)$ *is indeed an* n^{th} *root of unity.*

Problem 13 *Another way to express the polar form of a complex number is to use the exponential function. For real numbers* t*, Euler's Formula defines* $e^{it} = \cos(t) + i\sin(t)$*.*

1. *Use Theorem* **??** *to show that:*

   (i) $e^{ix}e^{iy} = e^{i(x+y)}$ *for all real numbers* x *and* y*.*

   (ii) $\left(e^{ix}\right)^n = e^{i(nx)}$ *for any real number* x *and any natural number* n*.*

   (iii) $\dfrac{e^{ix}}{e^{iy}} = e^{i(x-y)}$ *for all real numbers* x *and* y*.*

2. *If* $z = r\,cis(\theta)$ *is the polar form of* z*, show that* $z = re^{it}$ *where* $\theta = t$ *radians.*

3. *Show that* $e^{i\pi} + 1 = 0$*. (This famous equation relates the five most important constants in all of Mathematics with the three most fundamental operations in Mathematics.)*

4. *Show that* $\cos(t) = \dfrac{e^{it}+e^{-it}}{2}$ *and that* $\sin(t) = \dfrac{e^{it}-e^{-it}}{2i}$ *for all real numbers* t*.*