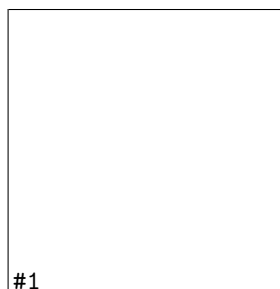


PARAMETRIC EQUATIONS

As we have seen in Exercises ?? - ?? in Section ??, Chapter ?? and most recently in Section ??, there are scores of interesting curves which, when plotted in the xy -plane, neither represent y as a function of x nor x as a function of y .

In this section, we present a new concept which allows us to use functions to study these kinds of curves. To motivate the idea, we imagine a bug crawling across a table top starting at the point O and tracing out a curve C in the plane, as shown below.



The curve C does not represent y as a function of x because it fails the Vertical Line Test and it does not represent x as a function of y because it fails the Horizontal Line Test.

However, since the bug can be in only one *place* $P(x, y)$ at any given time t , we can define the x -coordinate of P as a function of t and the y -coordinate of P as a (usually, but not necessarily) different function of t . Traditionally, $f(t)$ is used for x and $g(t)$ is used for y .

The independent variable t in this case is called a **parameter** and the system of equations

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

is called a **system of parametric equations** or a **parametrization** of the curve C .¹

The parametrization of C endows it with an *orientation* and the arrows on C indicate motion in the direction of increasing values of t .

In this case, our bug starts at the point O , travels upwards to the left, then loops back around to cross its path² at the point Q and finally heads off into the first quadrant.

It is important to note that the curve itself is a set of points and as such is devoid of any orientation. The parametrization determines the orientation and as we shall see, different parametrizations can determine different orientations.

If all of this seems hauntingly familiar, it should. By definition, the system of equations $\{x = \cos(t), y = \sin(t)\}$ parametrizes the Unit Circle, giving it a counter-clockwise orientation.

More generally, the equations of circular motion $\{x = r \cos(\omega t), y = r \sin(\omega t)\}$ developed on page ?? in Section ?? are parametric equations which trace out a circle of radius r centered at the origin.

If $\omega > 0$, the orientation is counter-clockwise; if $\omega < 0$, the orientation is clockwise. The angular frequency ω determines 'how fast' the object moves around the circle.

In particular, the equations $\left\{x = 2960 \cos\left(\frac{\pi}{12}t\right), y = 2960 \sin\left(\frac{\pi}{12}t\right)\right\}$ that model the motion of Lakeland Community College as the earth rotates (see Example ?? in Section ??) parameterize a circle of radius 2960

¹Note the use of the indefinite article 'a'. As we shall see, there are infinitely many different parametric representations for any given curve.

²Here, the bug reaches the point Q at two different times. While this does not contradict our claim that $f(t)$ and $g(t)$ are *functions* of t , it shows that neither f nor g can be one-to-one. (Think about this before reading on.)

PARAMETRIC EQUATIONS

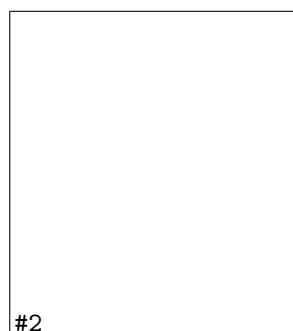
with a counter-clockwise rotation which completes one revolution as t runs through the interval $[0, 24)$. It is time for another example.

Example 1. Sketch the curve described by $\begin{cases} x = t^2 - 3 \\ y = 2t - 1 \end{cases}$ for $t \geq -2$.

Solution. We follow the same procedure here as we have time and time again when asked to graph anything new – choose values for t , then plot and connect the corresponding points.

Since we are told $t \geq -2$, we start there and as we plot successive points, we draw an arrow to indicate the direction of the path for increasing values of t .

t	$x(t)$	$y(t)$	$(x(t), y(t))$
-2	1	-5	(1, -5)
-1	-2	-3	(-2, -3)
0	-3	-1	(-3, -1)
1	-2	1	(-2, 1)
2	1	3	(1, 3)
3	6	5	(6, 5)



The curve sketched out in Example ?? certainly looks like a parabola, and the presence of the t^2 term in the equation $x = t^2 - 3$ reinforces this hunch.

Since the parametric equations $\{x = t^2 - 3, y = 2t - 1\}$ given to describe this curve are a *system* of equations, we can use the technique of substitution as described in Section ?? to eliminate the parameter t and get an equation involving just x and y .

To do so, we choose to solve the equation $y = 2t - 1$ for t to get $t = \frac{y+1}{2}$. Substituting this into the equation $x = t^2 - 3$ yields $x = \left(\frac{y+1}{2}\right)^2 - 3$ or, after some rearrangement, $(y+1)^2 = 4(x+3)$.

Thinking back to Section ??, we see that the graph of this equation is a parabola with vertex $(-3, -1)$ which opens to the right, as required.

Technically speaking, the equation $(y+1)^2 = 4(x+3)$ describes the *entire* parabola, while the parametric equations $\{x = t^2 - 3, y = 2t - 1 \text{ for } t \geq -2\}$ describe only a *portion* of the parabola.

In this case,³ we can remedy this situation by restricting the bounds on y . Since the portion of the parabola we want is exactly the part where $y \geq -5$, the equation $(y+1)^2 = 4(x+3)$ coupled with the restriction $y \geq -5$ describes the same curve as the given parametric equations. The one piece of information we can never recover after eliminating the parameter, however, is the orientation of the curve.

Eliminating the parameter and obtaining an equation in terms of x and y , whenever possible, can be a great help in graphing curves determined by parametric equations.

If the system of parametric equations contains algebraic functions, as was the case in Example ??, then the usual techniques of substitution and elimination as learned in Section ?? can be applied to the system $\{x = f(t), y = g(t)\}$ to eliminate the parameter.

³We will have an example shortly where no matter how we restrict x and y , we can never accurately describe the curve once we've eliminated the parameter.

PARAMETRIC EQUATIONS

If, on the other hand, the parametrization involves the trigonometric functions, the strategy changes slightly. In this case, it is often best to solve for the trigonometric functions and relate them using an identity.

We demonstrate these techniques in the following example.

Example 2. Sketch the curves described by the following parametric equations.

$$1. \begin{cases} x = t^3 \\ y = 2t^2 \end{cases} \text{ for } -1 \leq t \leq 1$$

$$3. \begin{cases} x = \sin(t) \\ y = \csc(t) \end{cases} \text{ for } 0 < t < \pi$$

$$2. \begin{cases} x = e^{-t} \\ y = e^{-2t} \end{cases} \text{ for } t \geq 0$$

$$4. \begin{cases} x = 1 + 3\cos(t) \\ y = 2\sin(t) \end{cases} \text{ for } 0 \leq t \leq \frac{3\pi}{2}$$

Solution.

1. To get a feel for the curve described by the system $\{x = t^3, y = 2t^2\}$ we first sketch the graphs of $x = t^3$ and $y = 2t^2$ over the interval $[-1, 1]$ below on the left in the middle, respectively.

We note that as t takes on values in the interval $[-1, 1]$, $x = t^3$ ranges between -1 and 1 , and $y = 2t^2$ ranges between 0 and 2 . This means that all of the action is happening on a portion of the plane, namely $\{(x, y) \mid -1 \leq x \leq 1, 0 \leq y \leq 2\}$.

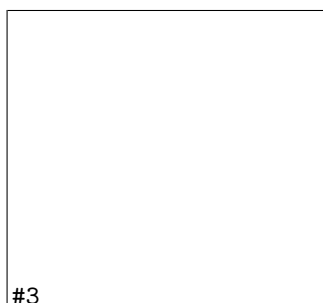
Next, we plot a few points to get a sense of the position and orientation of the curve. Certainly, $t = -1$ and $t = 1$ are good values to pick since these are the extreme values of t . We also choose $t = 0$, since that corresponds to a (local) minimum⁴ on the graph of $y = 2t^2$. Plugging in $t = -1$ gives the point $(-1, 2)$, $t = 0$ gives $(0, 0)$ and $t = 1$ gives $(1, 2)$.

More generally, we see that $x = t^3$ is *increasing* over the entire interval $[-1, 1]$ whereas $y = 2t^2$ is *decreasing* over the interval $[-1, 0]$ and then *increasing* over $[0, 1]$.

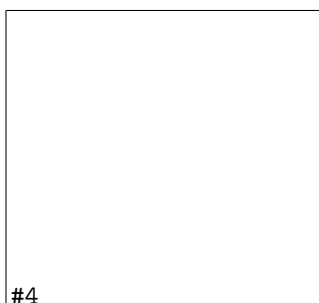
Geometrically, this means that in order to trace out the path described by the parametric equations, we start at $(-1, 2)$ (where $t = -1$), then move to the right (since x is increasing) and down (since y is decreasing) to $(0, 0)$ (where $t = 0$).

We continue to move to the right (since x is still increasing) but now move upwards (since y is now increasing) until we reach $(1, 2)$ (where $t = 1$).

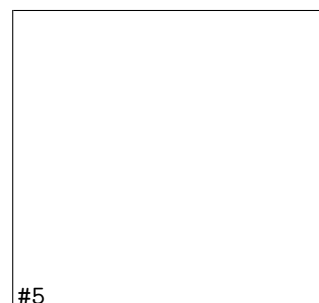
Finally, to get a good sense of the shape of the curve, we eliminate the parameter. Solving $x = t^3$ for t , we get $t = \sqrt[3]{x}$. Substituting this into $y = 2t^2$ gives $y = 2(\sqrt[3]{x})^2 = 2x^{2/3}$. Our experience in Section ?? yields the graph of our final answer below on the right.



$$x = t^3, -1 \leq t \leq 1$$



$$y = 2t^2, -1 \leq t \leq 1$$



$$\{x = t^3, y = 2t^2, -1 \leq t \leq 1\}$$

2. For the system $\{x = 2e^{-t}, y = e^{-2t}\}$ for $t \geq 0$, we proceed as in the previous example and graph $x = 2e^{-t}$ and $y = e^{-2t}$ over the interval $[0, \infty)$ below on the left and in the middle, respectively.

⁴You should review Definitions ?? and ?? if you've forgotten what 'increasing', 'decreasing' and 'local minimum' mean.

PARAMETRIC EQUATIONS

We find that the range of x in this case is $(0, 2]$ and the range of y is $(0, 1]$, so our graph will reside in a portion of Quadrant I: $\{(x, y) \mid 0 < x \leq 2, 0 < y \leq 1\}$.

Next, we plug in some friendly values of t to get a sense of the orientation of the curve. Since t lies in the exponent here, 'friendly' values of t involve natural logarithms. Starting with $t = \ln(1) = 0$ we get⁵ $(2, 1)$, for $t = \ln(2)$ we get $\left(1, \frac{1}{4}\right)$ and for $t = \ln(3)$ we get $\left(\frac{2}{3}, \frac{1}{9}\right)$.

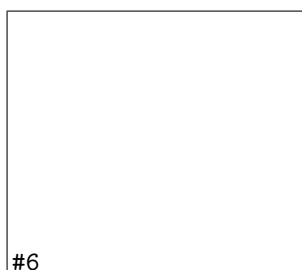
Since t is ranging over the unbounded interval $[0, \infty)$, we take the time to analyze the end behavior of both x and y . We find $\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} 2e^{-t} = 0$ and $\lim_{t \rightarrow \infty} y(t) = \lim_{t \rightarrow \infty} e^{-2t} = 0$. This means the graph of $\{x = 2e^{-t}, y = e^{-2t}\}$ approaches the point $(0, 0)$.

Since both $x = 2e^{-t}$ and $y = e^{-2t}$ are always decreasing for $t \geq 0$, we know that our final graph will start at $(2, 1)$ (where $t = 0$), and move consistently to the left (since x is decreasing) and down (since y is decreasing) to approach the origin.

To eliminate the parameter, one way to proceed is to solve $x = 2e^{-t}$ for t to get $t = -\ln\left(\frac{x}{2}\right)$. Substituting this for t in $y = e^{-2t}$ gives $y = e^{-2(-\ln(x/2))} = e^{2\ln(x/2)} = e^{\ln(x/2)^2} = \left(\frac{x}{2}\right)^2 = \frac{x^2}{4}$.

Alternatively, we could recognize that $y = e^{-2t} = (e^{-t})^2$, and since $x = 2e^{-t}$ means $e^{-t} = \frac{x}{2}$, we get $y = \left(\frac{x}{2}\right)^2 = \frac{x^2}{4}$ this way as well.

Either way, the graph of $\{x = 2e^{-t}, y = e^{-2t} \text{ for } t \geq 0\}$ is a portion of the parabola $y = \frac{x^2}{4}$ which starts at the point $(2, 1)$ and heads towards, but never reaches,⁶ $(0, 0)$ as seen below on the right.



$$x = 2e^{-t}, t \geq 0$$



$$y = e^{-2t}, t \geq 0$$



$$\{x = 2e^{-t}, y = e^{-2t}, t \geq 0\}$$

3. For the system $\{x = \sin(t), y = \csc(t) \text{ for } 0 < t < \pi\}$, we start by graphing $x = \sin(t)$ and $y = \csc(t)$ over the interval $(0, \pi)$ below on the left and in the middle, respectively.

We find that the range of x is $(0, 1]$ while the range of y is $[1, \infty)$ which means our graph will lie in the first quadrant.

Plotting a few friendly points, we see that $t = \frac{\pi}{6}$ gives the point $\left(\frac{1}{2}, 2\right)$, $t = \frac{\pi}{2}$ gives $(1, 1)$ and $t = \frac{5\pi}{6}$ returns us to $\left(\frac{1}{2}, 2\right)$.

Since $t = 0$ and $t = \pi$ aren't included in the domain for t , (because $y = \csc(t)$ is undefined at these t -values), we analyze the behavior of the system as t approaches 0 and π .

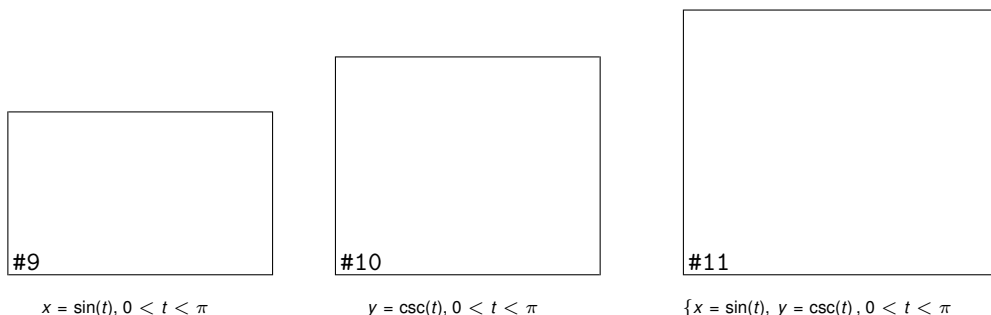
We have $\lim_{t \rightarrow 0^+} x(t) = \lim_{t \rightarrow 0^+} \sin(t) = 0$ and $\lim_{t \rightarrow \pi^-} x(t) = \lim_{t \rightarrow \pi^-} \sin(t) = 0$. Also, $\lim_{t \rightarrow 0^+} y(t) = \lim_{t \rightarrow 0^+} \csc(t) = \infty$ and $\lim_{t \rightarrow \pi^-} y(t) = \lim_{t \rightarrow \pi^-} \csc(t) = \infty$. Piecing all of this information together, we get that for t near 0, we have points with very small positive x -values, but very large positive y -values.

⁵The reader is encouraged to review Sections ?? and ?? as needed.

⁶Note the open circle at the origin. See our discussion about holes in graphs in Example ?? in Section ??.

PARAMETRIC EQUATIONS

As t ranges through the interval $\left(0, \frac{\pi}{2}\right]$, $x = \sin(t)$ is increasing and $y = \csc(t)$ is decreasing. This means that we are moving to the right and downwards, through $\left(\frac{1}{2}, 2\right)$ when $t = \frac{\pi}{6}$ to $(1, 1)$ when $t = \frac{\pi}{2}$. Once $t = \frac{\pi}{2}$, the orientation reverses, and we start to head to the left, since $x = \sin(t)$ is now decreasing, and up, since $y = \csc(t)$ is now increasing. We pass back through $\left(\frac{1}{2}, 2\right)$ when $t = \frac{5\pi}{6}$ back to the points with small positive x -coordinates and large positive y -coordinates.



To better explain this behavior, we eliminate the parameter. Using a reciprocal identity, we write $y = \csc(t) = \frac{1}{\sin(t)}$. Since $x = \sin(t)$, the curve traced out by this parametrization is a portion of the graph of $y = \frac{1}{x}$. We now can explain the unusual behavior as $t \rightarrow 0^+$ and $t \rightarrow \pi^-$ – for these values of t , we are hugging the vertical asymptote $x = 0$ of the graph of $y = \frac{1}{x}$.

We see that the parametrization given above traces out the portion of $y = \frac{1}{x}$ for $0 < x \leq 1$ twice as t runs through the interval $(0, \pi)$ as indicated above on the right.

4. Proceeding as above, we set about graphing $\{x = 1 + 3\cos(t), y = 2\sin(t) \text{ for } 0 \leq t \leq \frac{3\pi}{2}\}$ by first graphing $x = 1 + 3\cos(t)$ and $y = 2\sin(t)$ on the interval $\left[0, \frac{3\pi}{2}\right]$ below on the left and middle, respectively.

We see that x ranges from -2 to 4 and y ranges from -2 to 2 . Hence our graph will reside in the region $\{(x, y) \mid -2 \leq x \leq 4, -2 \leq y \leq 2\}$.

Plugging in $t = 0, \frac{\pi}{2}, \pi$ and $\frac{3\pi}{2}$ gives the points $(4, 0), (1, 2), (-2, 0)$ and $(1, -2)$, respectively.

As t ranges from 0 to $\frac{\pi}{2}$, $x = 1 + 3\cos(t)$ is decreasing, while $y = 2\sin(t)$ is increasing. This means that we start tracing out our answer at $(4, 0)$ and continue moving to the left and upwards towards $(1, 2)$. For $\frac{\pi}{2} \leq t \leq \pi$, x is decreasing, as is y , so the motion is still right to left, but now is downwards from $(1, 2)$ to $(-2, 0)$. On the interval $\left[\pi, \frac{3\pi}{2}\right]$, x begins to increase, while y continues to decrease. Hence, the motion becomes left to right but continues downwards, connecting $(-2, 0)$ to $(1, -2)$.

To eliminate the parameter here, we note that the trigonometric functions involved, namely $\cos(t)$ and $\sin(t)$, are related by the Pythagorean Identity $\cos^2(t) + \sin^2(t) = 1$. Hence, we solve $x = 1 + 3\cos(t)$ for $\cos(t)$ to get $\cos(t) = \frac{x-1}{3}$, and we solve $y = 2\sin(t)$ for $\sin(t)$ to get $\sin(t) = \frac{y}{2}$.

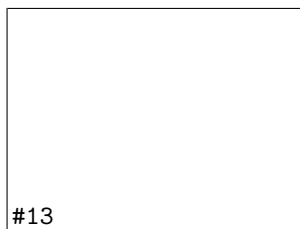
Substituting these expressions into $\cos^2(t) + \sin^2(t) = 1$ gives $\left(\frac{x-1}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$, or $\frac{(x-1)^2}{9} + \frac{y^2}{4} = 1$.

PARAMETRIC EQUATIONS

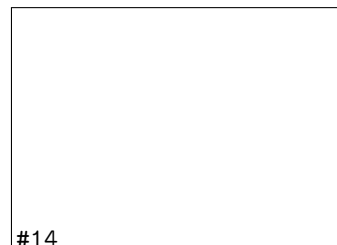
From Section ??, we know that the graph of this equation is an ellipse centered at $(1, 0)$ with vertices at $(-2, 0)$ and $(4, 0)$ with a minor axis of length 4. Our parametric equations here are tracing out three-quarters of this ellipse, in a counter-clockwise direction.



$$x = 1 + 3 \cos(t), 0 \leq t \leq \frac{3\pi}{2}$$



$$y = 2 \sin(t), 0 \leq t \leq \frac{3\pi}{2}$$



$$\{x = 1 + 3 \cos(t), y = 2 \sin(t), 0 \leq t \leq \frac{3\pi}{2}\}$$

Now that we have had some good practice sketching the graphs of parametric equations, we turn to the problem of finding parametric representations of curves. We start with the following.

Parametrizations of Common Curves

- The graph of $y = f(x)$ as x runs through some interval I is parametrized by:
 $\{x = t, y = f(t) \text{ as } t \text{ runs through } I.$
- The graph of $x = g(y)$ as y runs through some interval I is parametrized by:
 $\{x = g(t), y = t \text{ as } t \text{ runs through } I.$
- The graph of a directed line segment from (x_0, y_0) to (x_1, y_1) is parametrized by:
 $\{x = x_0 + (x_1 - x_0)t, y = y_0 + (y_1 - y_0)t \text{ for } 0 \leq t \leq 1.$
- The graph of a circle or ellipse $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$ where $a, b > 0$ is parametrized by:
 $\{x = h + a \cos(t), y = k + b \sin(t) \text{ for } 0 \leq t < 2\pi.$

NOTE: This will impart a *counter-clockwise* orientation.

The reader is encouraged to verify the above formulas by eliminating the parameter and, when indicated, checking the orientation. We put these formulas to good use in the following example.

Example 3. Find a parametrization for each of the following curves and check your answers.

1. $y = x^2$ from $x = -3$ to $x = 2$
2. $y = f^{-1}(x)$ where $f(x) = x^5 + 2x + 1$
3. The line segment which starts at $(2, -3)$ and ends at $(1, 5)$
4. The circle $x^2 + 2x + y^2 - 4y = 4$
5. The left half of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$

Solution.

1. Since $y = x^2$ is written in the form $y = f(x)$, we let $x = t$ and $y = f(t) = t^2$. Since $x = t$, the bounds on t match precisely the bounds on x we get $\{x = t, y = t^2 \text{ for } -3 \leq t \leq 2.$

The check is almost trivial; with $x = t$ we have $y = t^2 = x^2$ as $t = x$ runs from -3 to 2 .

PARAMETRIC EQUATIONS

2. We are told to parametrize $y = f^{-1}(x)$ for $f(x) = x^5 + 2x + 1$ so it is safe to assume that f is one-to-one. (Otherwise, f^{-1} would not exist.) To find a formula $y = f^{-1}(x)$, we follow the procedure outlined on page ?? – we start with the equation $y = f(x)$, interchange x and y and solve for y .

Doing so gives us the equation $x = y^5 + 2y + 1$. While we could attempt to solve this equation for y to get an *explicit* formula for $f^{-1}(x)$, we don't need to. We can parametrize the *implicit* function⁷ $x = f(y) = y^5 + 2y + 1$ by setting $y = t$ so that $x = t^5 + 2t + 1$.

We know from Section ?? that since $f(x) = x^5 + 2x + 1$ is an odd-degree polynomial, the range of $y = f(x) = x^5 + 2x + 1$ is $(-\infty, \infty)$. Hence, in order to trace out the entire graph of $x = f(y) = y^5 + 2y + 1$, we need to let y run through all real numbers.

Hence, our final answer to this problem is $\{x = t^5 + 2t + 1, y = t \text{ for } -\infty < t < \infty\}$. As in the previous problem, our solution is trivial to check.⁸

3. To parametrize line segment which starts at $(2, -3)$ and ends at $(1, 5)$, we make use of the formulas $x = x_0 + (x_1 - x_0)t$ and $y = y_0 + (y_1 - y_0)t$ for $0 \leq t \leq 1$. While these equations at first glance are quite a handful,⁹ they can be summarized as 'starting point + (displacement) t '.

To find the equation for x , we have that the line segment *starts* at $x = 2$ and *ends* at $x = 1$. This means the *displacement* in the x -direction is $\Delta x = (1 - 2) = -1$. Hence, the equation for x is $x = 2 + (-1)t = 2 - t$.

Similarly for y , we note that the line segment starts at $y = -3$ and ends at $y = 5$. Hence, the displacement in the y -direction is $\Delta y = (5 - (-3)) = 8$, so we get $y = -3 + 8t$.

Putting together our answers for x and y , we get $\{x = 2 - t, y = -3 + 8t \text{ for } 0 \leq t \leq 1\}$.

To check, we can solve $x = 2 - t$ for t to get $t = 2 - x$. Substituting this into $y = -3 + 8t$ gives $y = -3 + 8t = -3 + 8(2 - x)$, or $y = -8x + 13$. We know this is the graph of a line, so all we need to check is that it starts and stops at the correct points.

When $t = 0$, $x = 2 - t = 2$, and when $t = 1$, $x = 2 - t = 1$. Plugging in $x = 2$ gives $y = -8(2) + 13 = -3$, for an initial point of $(2, -3)$. When $x = 1$, $y = -8(1) + 13 = 5$ for an ending point of $(1, 5)$, as required.

4. In order to use the formulas above to parametrize the circle $x^2 + 2x + y^2 - 4y = 4$, we first need to put the equation into the correct form.

After completing the squares, we get $(x + 1)^2 + (y - 2)^2 = 9$, or $\frac{(x + 1)^2}{9} + \frac{(y - 2)^2}{9} = 1$.

Once again, the formulas $x = h + a \cos(t)$ and $y = k + b \sin(t)$ can be a challenge to memorize, but they come from the Pythagorean Identity $\cos^2(t) + \sin^2(t) = 1$, so we can always use the identity to get our parametrization instead of relying on memorizing a formula.

In the equation $\frac{(x + 1)^2}{9} + \frac{(y - 2)^2}{9} = 1$, we identify $\cos(t) = \frac{x + 1}{3}$ and $\sin(t) = \frac{y - 2}{3}$. Rearranging these last two equations, we get $x = -1 + 3 \cos(t)$ and $y = 2 + 3 \sin(t)$.

In order to complete one revolution around the circle, we let t range through the interval $[0, 2\pi)$, so our final answer $\{x = -1 + 3 \cos(t), y = 2 + 3 \sin(t) \text{ for } 0 \leq t < 2\pi\}$.

To check our answer, we could eliminate the parameter by solving $x = -1 + 3 \cos(t)$ for $\cos(t)$ and $y = 2 + 3 \sin(t)$ for $\sin(t)$, invoking a Pythagorean Identity, and then manipulating the resulting equation in x and y into the original equation $x^2 + 2x + y^2 - 4y = 4$.

Instead, we opt for a more direct approach. We substitute $x = -1 + 3 \cos(t)$ and $y = 2 + 3 \sin(t)$ into the equation $x^2 + 2x + y^2 - 4y = 4$ and show that the latter is satisfied for all t such that $0 \leq t < 2\pi$.

⁷See the discussion preceding Example ?? in Section ?? for a review of this concept.

⁸Provided you followed the inverse function theory, of course.

⁹Compare and contrast this with Exercise ?? in Section ??.

PARAMETRIC EQUATIONS

$$\begin{aligned}
 x^2 + 2x + y^2 - 4y &= 4 \\
 (-1 + 3 \cos(t))^2 + 2(-1 + 3 \cos(t)) + (2 + 3 \sin(t))^2 - 4(2 + 3 \sin(t)) &\stackrel{?}{=} 4 \\
 1 - 6 \cos(t) + 9 \cos^2(t) - 2 + 6 \cos(t) + 4 + 12 \sin(t) + 9 \sin^2(t) - 8 - 12 \sin(t) &\stackrel{?}{=} 4 \\
 9 \cos^2(t) + 9 \sin^2(t) - 5 &\stackrel{?}{=} 4 \\
 9 (\cos^2(t) + \sin^2(t)) - 5 &\stackrel{?}{=} 4 \\
 9(1) - 5 &\stackrel{?}{=} 4 \\
 4 &= 4 \checkmark
 \end{aligned}$$

Now that we know the parametric equations give us points on the circle, we can go through the usual analysis as demonstrated in Example ?? to show that the entire circle is covered as t ranges through the interval $[0, 2\pi)$.

5. In the equation $\frac{x^2}{4} + \frac{y^2}{9} = 1$, we can either use the formulas above or think back to the Pythagorean Identity to get $x = 2 \cos(t)$ and $y = 3 \sin(t)$.

The normal range on the parameter in this case is $0 \leq t < 2\pi$, but since we are interested in only the left half of the ellipse, we restrict t to the values which correspond to Quadrant II and Quadrant III angles, namely $\frac{\pi}{2} \leq t \leq \frac{3\pi}{2}$. Hence, our final answer is $\{x = 2 \cos(t), y = 3 \sin(t) \text{ for } \frac{\pi}{2} \leq t \leq \frac{3\pi}{2}\}$.

Substituting $x = 2 \cos(t)$ and $y = 3 \sin(t)$ into $\frac{x^2}{4} + \frac{y^2}{9} = 1$ gives $\frac{4 \cos^2(t)}{4} + \frac{9 \sin^2(t)}{9} = 1$, which reduces to the Pythagorean Identity $\cos^2(t) + \sin^2(t) = 1$. This proves the points generated by the parametric equations $\{x = 2 \cos(t), y = 3 \sin(t)\}$ lie on the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

Employing the techniques demonstrated in Example ??, we find that the restriction $\frac{\pi}{2} \leq t \leq \frac{3\pi}{2}$ generates the left half of the ellipse, as required. ■

We note that the formulas given on page ?? offer only *one* of literally *infinitely* many ways to parametrize the common curves listed there. At times, the formulas offered there need to be altered to suit the situation.

Adjusting Parametric Equations

- **Reversing Orientation:**

Replacing every occurrence of t with $-t$ in a parametric description for a curve (including any inequalities which describe the bounds on t) reverses the orientation of the curve.

- **Shift of Parameter:**

Replacing every occurrence of t with $(t - c)$ in a parametric description for a curve (including any inequalities which describe the bounds on t) shifts the start of the parameter t ahead by c units.

We demonstrate these techniques in the following example.

Example 4. Find a parametrization for the following curves.

1. The curve which starts at $(2, 4)$ and follows the parabola $y = x^2$ to end at $(-1, 1)$. Shift the parameter so that the path starts at $t = 0$.
2. The two part path which starts at $(0, 0)$, travels along a line to $(3, 4)$, then travels along a line to $(5, 0)$.
3. The Unit Circle, oriented clockwise, with $t = 0$ corresponding to $(0, -1)$.

PARAMETRIC EQUATIONS

Solution.

1. We can parametrize $y = x^2$ from $x = -1$ to $x = 2$ using the formula given on Page ?? as $\{x = t, y = t^2$ for $-1 \leq t \leq 2$. This parametrization, however, starts at $(-1, 1)$ and ends at $(2, 4)$. Hence, we need to reverse the orientation.

To this end, we replace every occurrence of t with $-t$: $\{x = -t, y = (-t)^2$ for $-1 \leq -t \leq 2$. After simplifying, we get $\{x = -t, y = t^2$ for $-2 \leq t \leq 1$.

We would like t to begin at $t = 0$ instead of $t = -2$. The problem here is that the parametrization we have starts 2 units 'too soon', so we need to introduce a 'time delay' of 2.

Replacing every occurrence of t with $(t - 2)$ gives $\{x = -(t - 2), y = (t - 2)^2$ for $-2 \leq t - 2 \leq 1$. Simplifying yields $\{x = 2 - t, y = t^2 - 4t + 4$ for $0 \leq t \leq 3$.

We leave it to the reader to verify this system traces $y = x^2$ starting with $(2, 4)$ and ending at $(-1, 1)$.

2. Again, when parameterizing line segments, we think: 'starting point + (displacement) t '. For the first part of the path, we get $\{x = 3t, y = 4t$ for $0 \leq t \leq 1$, and for the second part we get $\{x = 3 + 2t, y = 4 - 4t$ for $0 \leq t \leq 1$.

Since the first parametrization leaves off at $t = 1$, we shift the parameter in the second part so it starts at $t = 1$. Our current description of the second part starts at $t = 0$, so we need to introduce a 'time delay' of 1 unit to the second set of parametric equations.

Replacing t with $(t - 1)$ in the second set of equations gives $\{x = 3 + 2(t - 1), y = 4 - 4(t - 1)$ for $0 \leq t - 1 \leq 1$. Simplifying yields $\{x = 1 + 2t, y = 8 - 4t$ for $1 \leq t \leq 2$. Hence, we may parametrize the path as $\{x = f(t), y = g(t)$ for $0 \leq t \leq 2$ where

$$f(t) = \begin{cases} 3t, & \text{for } 0 \leq t \leq 1 \\ 1 + 2t, & \text{for } 1 \leq t \leq 2 \end{cases} \quad \text{and} \quad g(t) = \begin{cases} 4t, & \text{for } 0 \leq t \leq 1 \\ 8 - 4t, & \text{for } 1 \leq t \leq 2 \end{cases}$$

Again, we encourage the reader to check our solution.

3. We know that $\{x = \cos(t), y = \sin(t)$ for $0 \leq t < 2\pi$ gives a *counter-clockwise* parametrization of the Unit Circle with $t = 0$ corresponding to $(1, 0)$, so our first task is to reverse orientation.

Replacing t with $-t$ gives $\{x = \cos(-t), y = \sin(-t)$ for $0 \leq -t < 2\pi$. Using the Even/Odd Identities, we simplify: $\{x = \cos(t), y = -\sin(t)$ for $-2\pi < t \leq 0$. This parametrization gives a clockwise orientation, but $t = 0$ still corresponds to the point $(1, 0)$; the point $(0, -1)$ is reached when $t = -\frac{3\pi}{2}$.

Our strategy is to first get the parametrization to 'start' at the point $(0, -1)$ and then shift the parameter accordingly so the 'start' coincides with $t = 0$.

We know that any interval of length 2π will parametrize the entire circle, so we keep the equations $\{x = \cos(t), y = -\sin(t)$, but start the parameter t at $-\frac{3\pi}{2}$, and find the upper bound by adding 2π so $-\frac{3\pi}{2} \leq t < \frac{\pi}{2}$. We leave it to the reader to verify that $\{x = \cos(t), y = -\sin(t)$ for $-\frac{3\pi}{2} \leq t < \frac{\pi}{2}$ traces out the Unit Circle clockwise starting at the point $(0, -1)$.

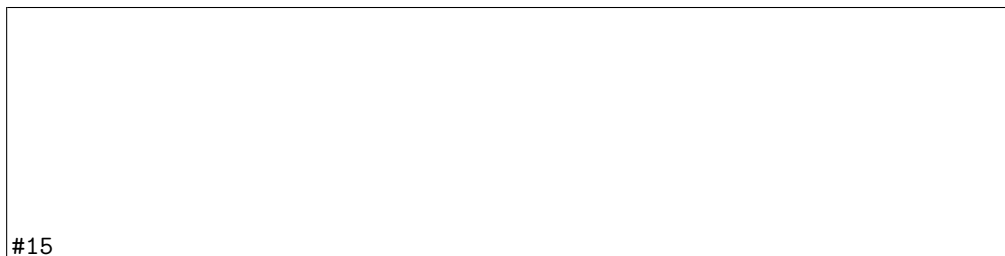
We now shift the parameter by introducing a 'time delay' of $\frac{3\pi}{2}$ units by replacing every occurrence of t with $\left(t - \frac{3\pi}{2}\right)$. We get $\left\{x = \cos\left(t - \frac{3\pi}{2}\right), y = -\sin\left(t - \frac{3\pi}{2}\right)\right.$ for $-\frac{3\pi}{2} \leq t - \frac{3\pi}{2} < \frac{\pi}{2}$. This simplifies courtesy of the Sum/Difference Formulas to $\{x = -\sin(t), y = -\cos(t)$ for $0 \leq t < 2\pi$.

We leave the check of our solution to the reader. ■

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We put our answer to Example ?? number ?? to good use to derive the equation of a [cycloid](#).

Suppose a circle of radius r rolls along the positive x -axis at a constant velocity v as pictured below. Let θ be the angle in radians which measures the amount of clockwise rotation experienced by the radius highlighted in the figure.



Our goal is to find parametric equations for the coordinates of the point $P(x, y)$ in terms of θ . From our work in Example ?? number ??, we know that clockwise motion along the Unit Circle starting at the point $(0, -1)$ can be modeled by the equations $\{x = -\sin(\theta), y = -\cos(\theta)\}$ for $0 \leq \theta < 2\pi$. (We have renamed the parameter ' θ ' to match the context of this problem.)

To model this motion on a circle of radius r , all we need to do¹⁰ is multiply both x and y by the factor r which yields $\{x = -r \sin(\theta), y = -r \cos(\theta)\}$.

Next, we adjust for the fact that the circle isn't stationary with center $(0, 0)$, but rather, is rolling along the positive x -axis. Since the velocity v is constant, we know that at time t , the center of the circle has traveled a distance vt down the positive x -axis. Furthermore, since the radius of the circle is r and the circle isn't moving vertically, we know that the center of the circle is always r units above the x -axis. Putting these two facts together, we have that at time t , the center of the circle is at the point (vt, r) .

From Section ??, we know $v = \frac{r\theta}{t}$, or $vt = r\theta$. Hence, the center of the circle, in terms of the parameter θ , is $(r\theta, r)$. As a result, we need to modify the equations $\{x = -r \sin(\theta), y = -r \cos(\theta)\}$ by shifting the x -coordinate to the right $r\theta$ units (by adding $r\theta$ to the expression for x) and the y -coordinate up r units¹¹ (by adding r to the expression for y).

We get $\{x = -r \sin(\theta) + r\theta, y = -r \cos(\theta) + r\}$, which can be written as $\{x = r(\theta - \sin(\theta)), y = r(1 - \cos(\theta))\}$. Since the motion starts at $\theta = 0$ and proceeds indefinitely, we set $\theta \geq 0$.

We end the section by using technology to graph a cycloid.

Example 5. Find the parametric equations of a cycloid which results from a circle of radius 3 rolling down the positive x -axis as described above. Graph your answer using a graphing utility.

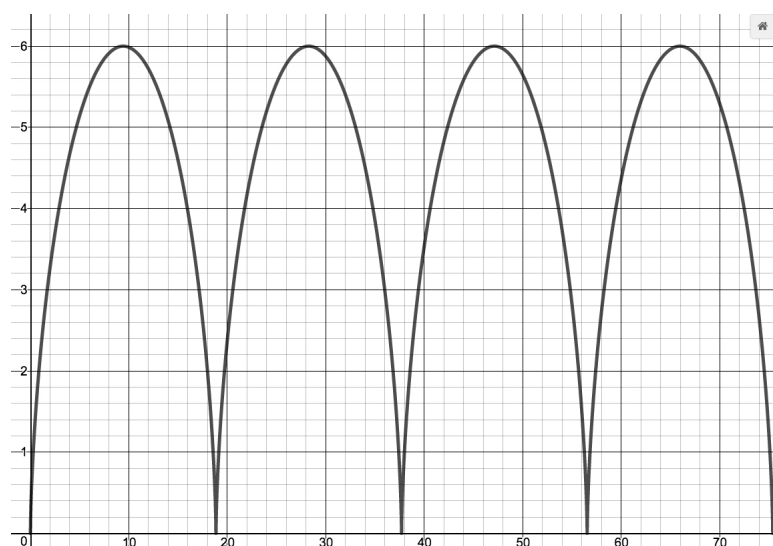
Solution. We have $r = 3$ which gives the equations $\{x = 3(t - \sin(t)), y = 3(1 - \cos(t))\}$ for $t \geq 0$. (Here we have returned to the convention of using t as the parameter.)

Sketching the cycloid by hand is a wonderful exercise in Calculus, but for the purposes of this book, we use a graphing utility. Below is the graph of this cycloid created by [desmos](#).

¹⁰If we replace x with $\frac{x}{r}$ and y with $\frac{y}{r}$ in the equation for the Unit Circle $x^2 + y^2 = 1$, we obtain $\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$ which reduces to $x^2 + y^2 = r^2$. In the language of Section ??, we are stretching the graph by a factor of r in both the x - and y -directions. Hence, we multiply both the x - and y -coordinates of points on the graph by r .

¹¹Does this seem familiar? See Example ?? in Section ??.

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We see the equations create a series of ‘arches’ and can (partially) verify the reasonableness the graph by finding the x -intercepts. To do this, we set $y = 3(1 - \cos(t)) = 0$, which amounts to solving $\cos(t) = 1$.

We get $t = 2\pi k$ and since $t \geq 0$, k can be any *nonnegative* integer. Substituting a few of these values for t , $t = 0$, $t = 2\pi$, $t = 4\pi$, and $t = 6\pi$ into the equations $x = 3(t - \sin(t))$ and $y = 3(1 - \cos(t))$ we obtain the points $(0, 0)$, $(6\pi, 0) \approx (18.85, 0)$, $(12\pi, 0) \approx (37.70, 0)$ and $(18\pi, 0) \approx (56.55, 0)$, which match the graph. In general, the x -intercepts are $(6\pi k, 0)$ for nonnegative integers k . We leave the details to the reader.

We note it is also possible to analytically determine the (local) maximums of the graph using the techniques demonstrated in Example ?? by analyzing $y = 3(1 - \cos(t))$. The maximums occur when $t = (2k + 1)\pi$ where k is a nonnegative integer, which isn’t too surprising just looking at the problem from a symmetry perspective. Substituting these values for t into our equations for x and y produce points of the form $(3(2k + 1)\pi, 6)$. We leave the details to the reader. ■