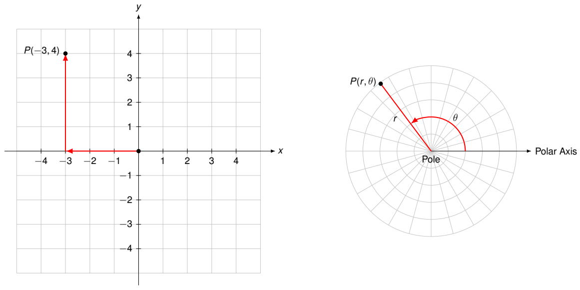
Polar Coordinates

In Section **??**, we introduced the notion of assigning ordered pairs of real numbers called 'coordinates' to points in the plane. Recall the Cartesian coordinate plane is defined using two number lines – one horizontal and one vertical – which intersect at right angles at a point called the 'origin'.

As seen below on the left, to plot a point with Cartesian coordinates, say $P(-3, 4)$, we start at the origin, travel horizontally to the left 3 units, then up 4 units. Alternatively, we could start at the origin, travel up 4 units, then to the left 3 units and arrive at the same location.

For the most part, the 'motions' of the Cartesian system (over and up) describe a rectangle, and most points can be thought of as the corner diagonally across the rectangle from the origin.[1] For this reason, the Cartesian coordinates of a point are often called 'rectangular' coordinates.



In this section, we introduce a new system for assigning coordinates to points in the plane – **polar coordinates** as diagrammed above on the right. We start with an origin point, called the **pole**, and a ray called the **polar axis**.

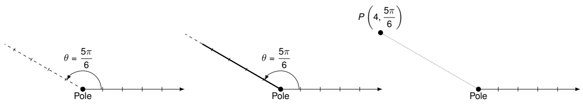
We locate a point P using two coordinates, (r, θ), where r represents a *directed* distance from the pole[2] and θ is a measure of counter-clockwise rotation from the polar axis.

Roughly speaking, the polar coordinates (r, θ) of a point measure 'how far out' the point is from the pole (that's r), and 'how far to rotate' from the polar axis, (that's θ).

For example, if we wished to plot the point P with polar coordinates $\left(4, \frac{5\pi}{6}\right)$, we'd start at the pole, move out along the polar axis 4 units, then rotate $\frac{5\pi}{6}$ radians counter-clockwise. The GeoGebra interactive below allows us to do just that by first adjusting the slider for the radius and then the slider for the angle.

GeoGebra link: https://www.geogebra.org/m/ftghtgsc

We may also visualize this process by thinking of the rotation first.[3] To plot $P\left(4, \frac{5\pi}{6}\right)$ this way, we rotate $\frac{5\pi}{6}$ counter-clockwise from the polar axis, then move outwards from the pole 4 units. Essentially we are locating a point on the terminal side of $\frac{5\pi}{6}$ which is 4 units away from the pole. The sequence of diagrams below show this process.



If $r < 0$, we begin by moving in the opposite direction on the polar axis from the pole. For example, to plot the point with polar coordinates $Q\left(-3.5, \frac{\pi}{4}\right)$ we would start from the pole, move to the left 3.5 units, then

[1] Excluding, of course, the points in which one or both coordinates are 0.

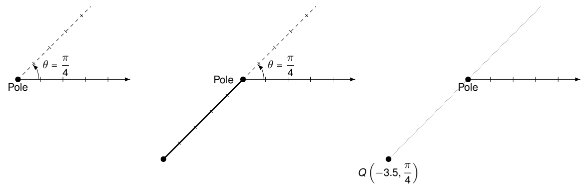
[2] We will explain more about this momentarily.

[3] As with anything in Mathematics, the more ways you have to look at something, the better. The authors encourage the reader to take time to think about both approaches to plotting points given in polar coordinates.

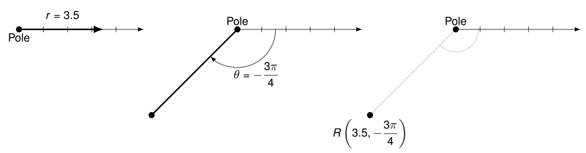
rotate counter-clockwise $\dfrac{\pi}{4}$ radians. Once again, we are fortunate to have a GeoGebra interactive to help us visualize this process.

GeoGebra link: `https://www.geogebra.org/m/cpcceeus`

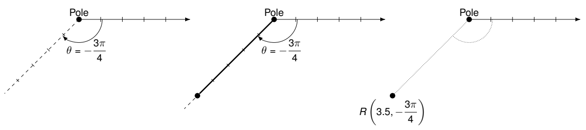
If we interpret the angle first, we rotate $\dfrac{\pi}{4}$ radians, then move back through the pole 3.5 units. Here we are locating a point 3.5 units away from the pole on the terminal side of $\dfrac{5\pi}{4}$, not $\dfrac{\pi}{4}$. This process is diagrammed below.



As you may have guessed, $\theta < 0$ means the rotation away from the polar axis is clockwise instead of counter-clockwise. Hence, to plot $R\left(3.5, -\dfrac{3\pi}{4}\right)$ we have the following.



From an 'angles first' approach, we rotate $-\dfrac{3\pi}{4}$ then move out 3.5 units from the pole. We see that R is the point on the terminal side of $\theta = -\dfrac{3\pi}{4}$ which is 3.5 units from the pole.



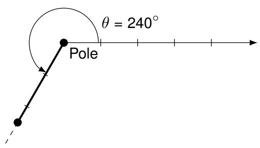
The points Q and R above are, in fact, the same point despite the fact that their polar coordinate representations are different. Unlike Cartesian coordinates where (a, b) and (c, d) represent the same point if and only if $a = c$ and $b = d$, a point can be represented by infinitely many polar coordinate pairs.

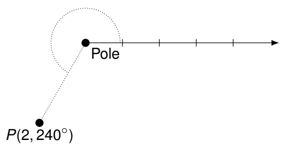
We explore this notion more in the following example.

Example 1. For each point in polar coordinates given below plot the point and then give two additional expressions for the point, one of which has $r > 0$ and the other with $r < 0$.

1. $P\left(2, 240^\circ\right)$
2. $P\left(-4, \dfrac{7\pi}{6}\right)$
3. $P\left(117, -\dfrac{5\pi}{2}\right)$
4. $P\left(-3, -\dfrac{\pi}{4}\right)$

Explanation. 1. Whether we move 2 units along the polar axis and then rotate 240° or rotate 240° then move out 2 units from the pole, we plot $P\left(2, 240^\circ\right)$ below.

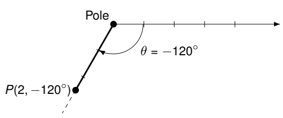


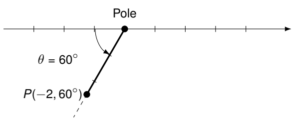


We now set about finding alternate descriptions (r, θ) for the point P. Since P is 2 units from the pole, $r = \pm 2$. Next, we choose angles θ for each of the r values.

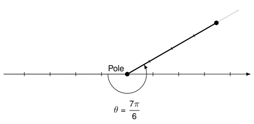
The given representation for P is $\left(2, 240^\circ\right)$ so the angle θ we choose for the $r = 2$ case must be coterminal with 240°. (Can you see why?) We choose $\theta = -120^\circ$, so one answer is $\left(2, -120^\circ\right)$.

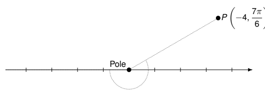
For the case $r = -2$, we visualize our rotation starting 2 units to the left of the pole. From this position, we need only to rotate $\theta = 60^\circ$ to arrive at location coterminal with 240°. Hence, our answer here is $\left(-2, 60^\circ\right)$. We check our answers by plotting them.





2. We plot $\left(-4, \dfrac{7\pi}{6}\right)$ by first moving 4 units to the left of the pole and then rotating $\dfrac{7\pi}{6}$ radians. Since $r = -4 < 0$, we find our point lies 4 units from the pole on the terminal side of $\dfrac{\pi}{6}$.

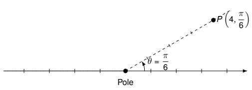


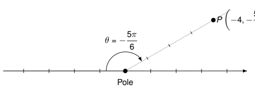


To find alternate descriptions for P, we note that the distance from P to the pole is 4 units, so any representation (r, θ) for P must have $r = \pm 4$.

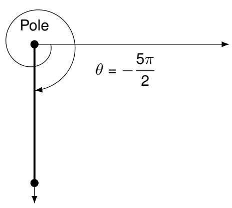
As noted above, P lies on the terminal side of $\dfrac{\pi}{6}$, so this, coupled with $r = 4$, gives us $\left(4, \dfrac{\pi}{6}\right)$ as one of our answers as seen below on the left.

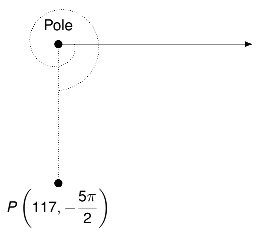
To find a different representation for P with $r = -4$, we may choose any angle coterminal with the angle $\theta = \dfrac{7\pi}{6}$. We pick $-\dfrac{5\pi}{6}$ and get $\left(-4, -\dfrac{5\pi}{6}\right)$ as our second answer, sketched below on the right.





3. To plot $P\left(117, -\dfrac{5\pi}{2}\right)$, we move along the polar axis 117 units from the pole and rotate *clockwise* $\dfrac{5\pi}{2}$ radians as illustrated below.

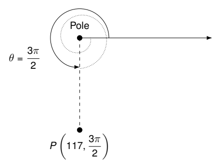


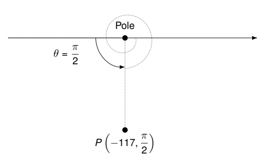


Since P is 117 units from the pole, any representation (r, θ) for P satisfies $r = \pm 117$.

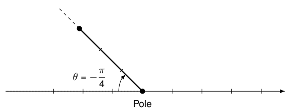
For the $r = 117$ case, we can take θ to be any angle coterminal with $-\dfrac{5\pi}{2}$. In this case, we choose $\theta = \dfrac{3\pi}{2}$, and get $\left(117, \dfrac{3\pi}{2}\right)$ as one answer as seen below on the left.

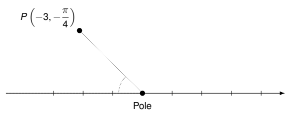
For the $r = -117$ case, we visualize moving left 117 units from the pole and then rotating through an angle θ to reach P. We find that $\theta = \dfrac{\pi}{2}$ works here, so our second answer is $\left(-117, \dfrac{\pi}{2}\right)$, sketched below on the right.





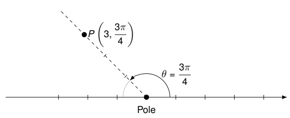
4. We move three units to the left of the pole and follow up with a clockwise rotation of $\dfrac{\pi}{4}$ radians to plot $P\left(-3, -\dfrac{\pi}{4}\right)$. We see that P lies on the terminal side of $\dfrac{3\pi}{4}$.

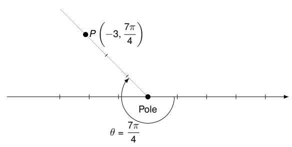




Since P lies on the terminal side of $\dfrac{3\pi}{4}$, one alternative representation for P is $\left(3, \dfrac{3\pi}{4}\right)$, sketched below on the left.

To find a different representation for P with $r = -3$, we may choose any angle coterminal with $-\dfrac{\pi}{4}$. We choose $\theta = \dfrac{7\pi}{4}$ for our final answer $\left(-3, \dfrac{7\pi}{4}\right)$, as seen below on the right.





In light of our work in Example **??**, it should come as no surprise that any given point expressed in polar coordinates has infinitely many other representations in polar coordinates.

The following result characterizes when two sets of polar coordinates determine the same point in the plane. It could be considered as a definition or a theorem, depending on your point of view. We choose to state it as a property of the polar coordinate system.

Equivalent Representations of Points in Polar Coordinates

Suppose (r, θ) and $\left(r', \theta'\right)$ are polar coordinates where $r \neq 0$, $r' \neq 0$ and the angles are measured in radians. Then (r, θ) and $\left(r', \theta'\right)$ determine the same point P if and only if one of the following is true:

- $r' = r$ and $\theta' = \theta + 2\pi k$ for some integer k
- $r' = -r$ and $\theta' = \theta + (2k + 1)\pi$ for some integer k

All polar coordinates of the form $(0, \theta)$ represent the pole regardless of the value of θ.

The key to understanding this result, and indeed the whole polar coordinate system, is to keep in mind that (r, θ) means (directed distance from pole, angle of rotation).

If $r = 0$, then no matter how much rotation is performed, the point never leaves the pole. Thus $(0, \theta)$ is the pole for all values of θ.

Now let's assume that neither r nor r' is zero. If (r, θ) and $\left(r', \theta'\right)$ determine the same point P then the (non-zero) distance from P to the pole in each case must be the same. Since this distance is controlled by the first coordinate, we have that either $r' = r$ or $r' = -r$.

If $r' = r$, then when plotting (r, θ) and $\left(r', \theta'\right)$, the angles θ and θ' have the same initial side. Hence, if (r, θ) and $\left(r', \theta'\right)$ determine the same point, we must have that θ' is coterminal with θ. We know that this means $\theta' = \theta + 2\pi k$ for some integer k, as required.

If, on the other hand, $r' = -r$, then when plotting (r, θ) and $\left(r', \theta'\right)$, the initial side of θ' is rotated π radians away from the initial side of θ. In this case, θ' must be coterminal with $\pi + \theta$. Hence, $\theta' = \pi + \theta + 2\pi k$ which we rewrite as $\theta' = \theta + (2k + 1)\pi$ for some integer k.

Conversely, if $r' = r$ and $\theta' = \theta + 2\pi k$ for some integer k, then the points $P(r, \theta)$ and $P'\left(r', \theta'\right)$ lie the same (directed) distance from the pole on the terminal sides of coterminal angles, and hence are the same point.

Now suppose $r' = -r$ and $\theta' = \theta + (2k + 1)\pi$ for some integer k. To plot P, we first move a directed distance r from the pole; to plot P', our first step is to move the same distance from the pole as P, but in the opposite direction. At this intermediate stage, we have two points equidistant from the pole rotated exactly π radians apart. Since $\theta' = \theta + (2k + 1)\pi = (\theta + \pi) + 2\pi k$ for some integer k, we see that θ' is coterminal to $(\theta + \pi)$ and it is this extra π radians of rotation which aligns the points P and P'.

Next, we marry the polar coordinate system with the Cartesian (rectangular) coordinate system. To do so, we identify the pole and polar axis in the polar system to the origin and positive x-axis, respectively, in the rectangular system. We get the following result.

Theorem 1. Conversion Between Rectangular and Polar Coordinates:

Suppose P is represented in rectangular coordinates as (x, y) and in polar coordinates as (r, θ). Then

- $x = r\cos(\theta)$ and $y = r\sin(\theta)$
- $x^2 + y^2 = r^2$ and $\tan(\theta) = \dfrac{y}{x}$ (provided $x \neq 0$)

In the case $r > 0$, Theorem **??** is an immediate consequence of Theorems **??** and **??**.

If $r < 0$, then we know an alternate representation for (r, θ) is $(-r, \theta + \pi)$. Since in this case, $-r > 0$, we know the theorem as stated is true for the representation $(-r, \theta + \pi)$ so we apply it here.

Applying Theorem **??** to $(-r, \theta + \pi)$ gives[4] $x = (-r)\cos(\theta + \pi) = (-r)(-\cos(\theta)) = r\cos(\theta)$ as well as $y = (-r)\sin(\theta + \pi) = (-r)(-\sin(\theta)) = r\sin(\theta)$.

Moreover, $x^2 + y^2 = (-r)^2 = r^2$, and $\dfrac{y}{x} = \tan(\theta + \pi) = \tan(\theta)$, so the theorem is true in this case, too.

The remaining case is $r = 0$, in which case $(r, \theta) = (0, \theta)$ is the pole. Since the pole is identified with the origin $(0, 0)$ in rectangular coordinates, the theorem in this case amounts to checking '$0 = 0$.'

Since we have argued that Theorem **??** is true in all cases, we put it to good use in the following example.

Example 2. Convert each point in rectangular coordinates given below into polar coordinates with $r \geq 0$ and $0 \leq \theta < 2\pi$. Use exact values if possible and round any approximate values to two decimal places. Check your answer by converting them back to rectangular coordinates.

1. $P\left(2, -2\sqrt{3}\right)$
2. $Q(-3, -3)$

[4] Well, Theorem **??** along with the identities $\cos(\theta + \pi) = -\cos(\theta)$ and $\sin(\theta + \pi) = -\sin(\theta)$ ...

3. $R(0, -3)$
4. $S(-3, 4)$

Explanation. Even though we are not explicitly told to do so, we can avoid many common mistakes by taking the time to plot the points before we do any calculations.

1. Plotting $P\left(2, -2\sqrt{3}\right)$, we find P lies in Quadrant IV. With $x = 2$ and $y = -2\sqrt{3}$, we calculate $r^2 = x^2 + y^2 = (2)^2 + \left(-2\sqrt{3}\right)^2 = 4 + 12 = 16$ so $r = \pm 4$. To satisfy $r \geq 0$, we choose $r = 4$.

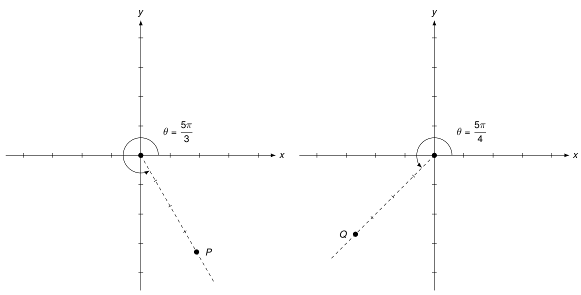
   To find θ, know $\tan(\theta) = \dfrac{y}{x} = \dfrac{-2\sqrt{3}}{2} = -\sqrt{3}$. This tells us θ has a reference angle of $\dfrac{\pi}{3}$, and since P lies in Quadrant IV, we know θ is a Quadrant IV angle. To satisfy the stipulation that $0 \leq \theta < 2\pi$, we choose $\theta = \dfrac{5\pi}{3}$. Hence, our answer is $\left(4, \dfrac{5\pi}{3}\right)$, which we sketch below on the left.

   To check, we convert the polar representation $(r, \theta) = \left(4, \dfrac{5\pi}{3}\right)$ back to rectangular coordinates. We find $x = r\cos(\theta) = 4\cos\left(\dfrac{5\pi}{3}\right) = 4\left(\dfrac{1}{2}\right) = 2$ and $y = r\sin(\theta) = 4\sin\left(\dfrac{5\pi}{3}\right) = 4\left(-\dfrac{\sqrt{3}}{2}\right) = -2\sqrt{3}$.

2. The point $Q(-3, -3)$ lies in Quadrant III. Using $x = y = -3$, we get $r^2 = (-3)^2 + (-3)^2 = 18$ so $r = \pm\sqrt{18} = \pm 3\sqrt{2}$. To satisfy$r \geq 0$, we choose $r = 3\sqrt{2}$.

   We find $\tan(\theta) = \dfrac{-3}{-3} = 1$, which means θ has a reference angle of $\dfrac{\pi}{4}$. Since Q lies in Quadrant III, we choose $\theta = \dfrac{5\pi}{4}$, to satisfy the requirement that $0 \leq \theta < 2\pi$. Our final answer is $\left(3\sqrt{2}, \dfrac{5\pi}{4}\right)$, which we sketch below on the right.

   Checking our answer, we find that $x = r\cos(\theta) = (3\sqrt{2})\cos\left(\dfrac{5\pi}{4}\right) = (3\sqrt{2})\left(-\dfrac{\sqrt{2}}{2}\right) = -3$ and compute $y = r\sin(\theta) = (3\sqrt{2})\sin\left(\dfrac{5\pi}{4}\right) = (3\sqrt{2})\left(-\dfrac{\sqrt{2}}{2}\right) = -3$, so we are done.



3. The point $R(0, -3)$ lies along the negative y-axis. While we could go through the usual computations[5] to find the polar form of R, in this case it is much more efficient to find the polar coordinates of R using the definition.

   Since the pole is identified with the origin, we see the point R is 3 units from the pole. Hence in the polar representation (r, θ) of R we know $r = \pm 3$. To satisfy $r \geq 0$, we choose $r = 3$.

   Concerning θ, we once again find more or less 'by inspection' that $\theta = \dfrac{3\pi}{2}$ satisfies $0 \leq \theta < 2\pi$ with its terminal side along the negative y-axis. Hence, our answer is $\left(3, \dfrac{3\pi}{2}\right)$, as seen below on the left.

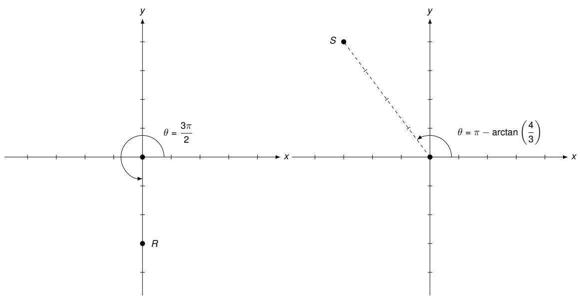
[5] Since $x = 0$, we would have to determine θ geometrically.

To check, we note $x = r\cos(\theta) = 3\cos\left(\frac{3\pi}{2}\right) = (3)(0) = 0$ and $y = r\sin(\theta) = 3\sin\left(\frac{3\pi}{2}\right) = 3(-1) = -3$.

4. The point $S(-3, 4)$ lies in Quadrant II. With $x = -3$ and $y = 4$, we get $r^2 = (-3)^2 + (4)^2 = 25$ so $r = \pm 5$. As usual, we choose $r = 5 \geq 0$ and proceed to determine θ.

   We have $\tan(\theta) = \frac{y}{x} = \frac{4}{-3} = -\frac{4}{3}$, and since this isn't the tangent of one the common angles, we resort to using the arctangent function. Since θ lies in Quadrant II and must satisfy $0 \leq \theta < 2\pi$, we choose $\theta = \pi - \arctan\left(\frac{4}{3}\right)$ radians. Hence, our answer is $(r, \theta) = \left(5, \pi - \arctan\left(\frac{4}{3}\right)\right) \approx (5, 2.21)$, sketched below on the right.

   To check our answers requires a bit of tenacity since we need to simplify expressions of the form: $\cos\left(\pi - \arctan\left(\frac{4}{3}\right)\right)$ and $\sin\left(\pi - \arctan\left(\frac{4}{3}\right)\right)$. These are good review exercises (see Section **??**) and are hence left to the reader. We find $\cos\left(\pi - \arctan\left(\frac{4}{3}\right)\right) = -\frac{3}{5}$ and $\sin\left(\pi - \arctan\left(\frac{4}{3}\right)\right) = \frac{4}{5}$, so that $x = r\cos(\theta) = (5)\left(-\frac{3}{5}\right) = -3$ and $y = r\sin(\theta) = (5)\left(\frac{4}{5}\right) = 4$ which confirms our answer.



Now that we've had practice converting representations of *points* between the rectangular and polar coordinate systems, we now set about converting *equations* from one system to another.

Just as we've used equations in x and y to represent relations in rectangular coordinates (see Section **??**), equations in the variables r and θ represent relations in polar coordinates. We convert equations between the two systems using Theorem **??** as the next example illustrates.

Example 3.

1. Convert each equation in rectangular coordinates into an equation in polar coordinates.
   (i) $(x-3)^2 + y^2 = 9$
   (ii) $y = -x$
   (iii) $y = x^2$
2. Convert each equation in polar coordinates into an equation in rectangular coordinates.
   (i) $r = -3$
   (ii) $\theta = \frac{4\pi}{3}$
   (iii) $r = 1 - \cos(\theta)$

Explanation. 1. One strategy to convert an equation from rectangular to polar coordinates is to replace every occurrence of x with $r\cos(\theta)$ and every occurrence of y with $r\sin(\theta)$ and use identities to simplify. This is the technique we employ below.

(i) We start by substituting $x = r\cos(\theta)$ and $y = \sin(\theta)$ into $(x-3)^2 + y^2 = 9$ and simplifying. With no real direction in which to proceed, we follow our mathematical instincts.[6]

$$\begin{array}{rcll} (r\cos(\theta)-3)^2 + (r\sin(\theta))^2 & = & 9 & \\ r^2\cos^2(\theta) - 6r\cos(\theta) + 9 + r^2\sin^2(\theta) & = & 9 & \\ r^2\left(\cos^2(\theta)+\sin^2(\theta)\right) - 6r\cos(\theta) & = & 0 & \text{Subtract 9 from both sides.} \\ r^2 - 6r\cos(\theta) & = & 0 & \text{Since } \cos^2(\theta)+\sin^2(\theta)=1 \\ r(r-6\cos(\theta)) & = & 0 & \text{Factor.} \end{array}$$

We get $r = 0$ or $r = 6\cos(\theta)$. From Section **??** we know the equation $(x-3)^2 + y^2 = 9$ describes a circle, and since $r = 0$ describes just a point (namely the pole/origin), we choose $r = 6\cos(\theta)$ for our final answer.[7]

(ii) Substituting $x = r\cos(\theta)$ and $y = r\sin(\theta)$ into $y = -x$ gives $r\sin(\theta) = -r\cos(\theta)$. Rearranging, we get $r\cos(\theta) + r\sin(\theta) = 0$ or $r(\cos(\theta) + \sin(\theta)) = 0$. This gives $r = 0$ or $\cos(\theta) + \sin(\theta) = 0$. Solving the latter equation for θ, we get $\theta = -\frac{\pi}{4} + \pi k$ for integers k.

As we did in the previous example, we take a step back and think geometrically. We know $y = -x$ describes a line through the origin. As before, $r = 0$ describes the origin, but nothing else. Consider the equation $\theta = -\frac{\pi}{4}$. In this equation, the variable r is free,[8] meaning it can assume any and all values including $r = 0$.

If we imagine plotting points $(r, -\frac{\pi}{4})$ for all conceivable values of r (positive, negative and zero), we are essentially drawing the line containing the terminal side of $\theta = -\frac{\pi}{4}$ which is none other than $y = -x$. Hence, we can take as our final answer $\theta = -\frac{\pi}{4}$ here.[9]

(iii) Substituting $x = r\cos(\theta)$ and $y = r\sin(\theta)$ into $y = x^2$ gives $r\sin(\theta) = (r\cos(\theta))^2$, or $r^2\cos^2(\theta) - r\sin(\theta) = 0$. Factoring, we get $r(r\cos^2(\theta) - \sin(\theta)) = 0$ so either $r = 0$ or $r\cos^2(\theta) = \sin(\theta)$.

We can solve $r\cos^2(\theta) = \sin(\theta)$ for r by dividing both sides of the equation by $\cos^2(\theta)$, but as a general rule, we never divide through by a quantity that may be 0.

In this particular case, we are safe since if $\cos^2(\theta) = 0$, then $\cos(\theta) = 0$, and for the equation $r\cos^2(\theta) = \sin(\theta)$ to hold, then $\sin(\theta)$ would also have to be 0. Since there are no angles with both $\cos(\theta) = 0$ and $\sin(\theta) = 0$, we are not losing any information by dividing both sides of $r\cos^2(\theta) = \sin(\theta)$ by $\cos^2(\theta)$.

Solving $r\cos^2(\theta) = \sin(\theta)$ gives $r = \frac{\sin(\theta)}{\cos^2(\theta)}$, or $r = \sec(\theta)\tan(\theta)$. As before, the $r = 0$ case is recovered in the solution $r = \sec(\theta)\tan(\theta)$ (let $\theta = 0$), so $r = \sec(\theta)\tan(\theta)$ is our final answer.

2. As a general rule, converting equations from polar to rectangular coordinates isn't as straight forward as the reverse process. We could solve $r^2 = x^2 + y^2$ for r to get $r = \pm\sqrt{x^2+y^2}$ and solving $\tan(\theta) = \frac{y}{x}$ requires the arctangent function to get $\theta = \arctan\left(\frac{y}{x}\right) + \pi k$ for integers k.

   Since neither of these expressions for r and θ are especially user-friendly, so we opt for a second strategy – rearrange the given polar equation so that the expressions $r^2 = x^2 + y^2$, $r\cos(\theta) = x$, $r\sin(\theta) = y$ and/or $\tan(\theta) = \frac{y}{x}$ present themselves.

   (i) Starting with $r = -3$, we can square both sides to get $r^2 = (-3)^2$ or $r^2 = 9$. We may now substitute $r^2 = x^2 + y^2$ to get the equation $x^2 + y^2 = 9$.

[6] Experience is the mother of all instinct, and necessity is the mother of invention. Study this example and see what techniques are employed, then try your best to get your answers in the homework to match Jeff's.

[7] Note when we substitute $\theta = \frac{\pi}{2}$ into the equation $r = 6\cos(\theta)$, we recover the point $r = 0$.

[8] See Section **??**.

[9] We could take it to be *any* of $\theta = -\frac{\pi}{4} + \pi k$ for integers k.

As we have seen,[10] squaring an equation does not, in general, produce an equivalent equation. The concern here is that the equation $r^2 = 9$ might be satisfied by more points than $r = -3$.
On the surface, this certainly appears to be the case since $r^2 = 9$ is equivalent to $r = \pm 3$, not just $r = -3$. That being said, any point with polar coordinates $(3, \theta)$ can be represented as $(-3, \theta + \pi)$, which means any point (r, θ) whose polar coordinates satisfy the relation $r = \pm 3$ has an equivalent[11] representation which satisfies $r = -3$.

(ii) We take the tangent of both sides the equation $\theta = \frac{4\pi}{3}$ to get $\tan(\theta) = \tan\left(\frac{4\pi}{3}\right) = \sqrt{3}$. Since $\tan(\theta) = \frac{y}{x}$, we get $\frac{y}{x} = \sqrt{3}$ or $y = x\sqrt{3}$. Of course, we pause a moment to wonder if, geometrically, the equations $\theta = \frac{4\pi}{3}$ and $y = x\sqrt{3}$ generate the same set of points.[12] The same argument presented in number **??** applies equally well here so we are done.

(iii) Once again, we need to manipulate $r = 1 - \cos(\theta)$ a bit before using the conversion formulas given in Theorem **??**. We could square both sides of this equation like we did in part **??** above to obtain an r^2 on the left hand side, but that does nothing helpful for the right hand side.
Instead, we multiply both sides by r to obtain $r^2 = r - r\cos(\theta)$. We now have an r^2 and an $r\cos(\theta)$ in the equation, which we can easily handle, but we also have another r to deal with. Rewriting the equation as $r = r^2 + r\cos(\theta)$ and squaring both sides yields $r^2 = \left(r^2 + r\cos(\theta)\right)^2$. Substituting $r^2 = x^2 + y^2$ and $r\cos(\theta) = x$ gives $x^2 + y^2 = \left(x^2 + y^2 + x\right)^2$.
Once again, we have performed some algebraic maneuvers which may have altered the set of points described by the original equation. First, we multiplied both sides by r. This means that now $r = 0$ is a viable solution to the equation. In the original equation, $r = 1 - \cos(\theta)$, we see that $\theta = 0$ gives $r = 0$, so the multiplication by r doesn't introduce any new points.
The squaring of both sides of this equation is also a reason to pause. That is, are there points with coordinates (r, θ) which satisfy $r^2 = \left(r^2 + r\cos(\theta)\right)^2$ but do not satisfy $r = r^2 + r\cos(\theta)$?
Suppose $\left(r', \theta'\right)$ satisfies $r^2 = \left(r^2 + r\cos(\theta)\right)^2$. Then $r' = \pm\left((r')^2 + r'\cos(\theta')\right)$. If it turns out that $r' = (r')^2 + r'\cos(\theta')$, then we are done.
If $r' = -\left((r')^2 + r'\cos(\theta')\right) = -(r')^2 - r'\cos(\theta')$, we claim that the coordinates $(-r', \theta' + \pi)$, which determine the same point as (r', θ'), satisfy $r = r^2 + r\cos(\theta)$. To show this, we substitute $r = -r'$ and $\theta = \theta' + \pi$ into the equation $r = r^2 + r\cos(\theta)$:

$$\begin{array}{rcll} -r' & \stackrel{?}{=} & \left(-r'\right)^2 + \left(-r'\cos(\theta' + \pi)\right) & \\ -\left(-(r')^2 - r'\cos(\theta')\right) & \stackrel{?}{=} & (r')^2 - r'\cos(\theta' + \pi) & \text{Since } r' = -(r')^2 - r'\cos(\theta') \\ (r')^2 + r'\cos(\theta') & \stackrel{?}{=} & (r')^2 - r'(-\cos(\theta')) & \text{Since } \cos(\theta' + \pi) = -\cos(\theta') \\ (r')^2 + r'\cos(\theta') & \stackrel{\text{check}}{=} & (r')^2 + r'\cos(\theta') & \end{array}$$

Since both sides worked out to be equal, $(-r', \theta' + \pi)$ satisfies $r = r^2 + r\cos(\theta)$.
Hence, any point (r, θ) which satisfies $r^2 = \left(r^2 + r\cos(\theta)\right)^2$ has a representation which satisfies $r = r^2 + r\cos(\theta)$, so a rectangular representation of $r = 1 - \cos(\theta)$ is $x^2 + y^2 = \left(x^2 + y^2 + x\right)^2$.

In practice, much of the pedantic verification of the equivalence of equations in Example **??** is left unsaid. Indeed, in most textbooks, squaring equations like $r = -3$ to arrive at $r^2 = 9$ happens without a second thought. Your instructor will ultimately decide how much, if any, justification is warranted.

[10] See Exercises **??** - **??** in Section **??**, for instance ...

[11] As *ordered pairs*, $(3, 0)$ and $(-3, \pi)$ are different, but since they correspond to the same point in the plane, we consider them 'equivalent' in this context. Technically speaking, the equations $r^2 = 9$ and $r = -3$ represent different relations per Definition **??** in Section **??** since they generate different sets of ordered pairs. Since polar coordinates were defined geometrically to describe the location of points in the plane, however, we concern ourselves only with ensuring that the sets of *points* in the plane generated by two equations are the same.

[12] In addition to taking the tangent of both sides of an equation (There are infinitely many solutions to $\tan(\theta) = \sqrt{3}$, and $\theta = \frac{4\pi}{3}$ is only one of them!), we also went from $\frac{y}{x} = \sqrt{3}$, in which x cannot be 0, to $y = x\sqrt{3}$ in which we assume x can be 0.

If you take anything away from Example **??**, it should be that relatively simple equations in rectangular coordinates, such as $y = x^2$, can become quite complicated in polar coordinates, and vice-versa.

In the next section, we devote our attention to graphing equations like the ones given in Example **??** number **??** on the Cartesian coordinate plane without converting back to rectangular coordinates. If nothing else, number **??** above shows the price we pay if we insist on always converting to back to the more familiar rectangular coordinate system.