THE GRAPHS OF POLAR EQUATIONS

In this section, we discuss how to graph equations relating the *polar coordinate* variables r and θ on the *rectangular coordinate* plane. Since every point in the plane has infinitely many different representations in polar coordinates, in order for a point P to be on the graph of a given equation, there must be *at least one* representation of $P(r, \theta)$ that satisfies that equation.

In our first example, only one of the variables r and θ is present making the other variable free.[1] This makes these graphs easier to visualize than others.

Example 1. Graph the following polar equations in the xy-plane.

1. $r = 4$
2. $r = -3\sqrt{2}$
3. $\theta = \dfrac{5\pi}{4}$
4. $\theta = -\dfrac{3\pi}{2}$

Explanation.

1. In the equation $r = 4$, θ is free. The graph of this equation is, therefore, all points which have a polar coordinate representation $(4, \theta)$, for any choice of θ.

   We can explore this relationship using the GeoGebra interactive below. Adjusting the slider for θ while keeping r fixed at 4 sweeps out a circle of radius 4, centered at the origin.

   GeoGebra link: `https://www.geogebra.org/m/euypn4hp`

2. Once again we have θ being free in the equation $r = -3\sqrt{2}$. Using the GeoGebra interactive below, we see that plotting all of the points of the form $(-3\sqrt{2}, \theta)$ gives us a circle of radius $3\sqrt{2}$ centered at the origin.

   GeoGebra link: `https://www.geogebra.org/m/kqassh6e`

3. In the equation $\theta = \dfrac{5\pi}{4}$, r is free, so we plot all of the points with polar representation $\left(r, \dfrac{5\pi}{4}\right)$. As seen below in the GeoGebra interactive, the result is the line containing the terminal side of $\theta = \dfrac{5\pi}{4}$, when plotted in standard position.

   GeoGebra link: `https://www.geogebra.org/m/f94fkjpn`

4. As in the previous example, the variable r is free in the equation $\theta = -\dfrac{3\pi}{2}$. Plotting $\left(r, -\dfrac{3\pi}{2}\right)$ for various values of r shows us that we are tracing out the y-axis. This is clearly seen in the GeoGebra interative below.

   GeoGebra link: `https://www.geogebra.org/m/q5krpqj3`

Hopefully, our experience in Example **??** makes the following result clear.

Theorem 1. Graphs of Constant r and θ: Suppose a and α are constants, $a \neq 0$.

- The graph of the polar equation $r = a$ on the Cartesian plane is a circle centered at the origin of radius $|a|$.

[1]See the discussion in Example **??** number **??** in Section **??**.

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- The graph of the polar equation $\theta = \alpha$ on the Cartesian plane is the line containing the terminal side of α when plotted in standard position.

Suppose we wish to graph $r = 6\cos(\theta)$. A reasonable way to start is to treat θ as the independent variable, r as the dependent variable, evaluate $r = f(\theta)$ at some 'friendly' values of θ and plot the resulting points.[2]

θ	$r = 6\cos(\theta)$	(r, θ)
0	6	$(6, 0)$
$\frac{\pi}{4}$	$3\sqrt{2}$	$\left(3\sqrt{2}, \frac{\pi}{4}\right)$
$\frac{\pi}{2}$	0	$\left(0, \frac{\pi}{2}\right)$
$\frac{3\pi}{4}$	$-3\sqrt{2}$	$\left(-3\sqrt{2}, \frac{3\pi}{4}\right)$
π	-6	$(-6, \pi)$
$\frac{5\pi}{4}$	$-3\sqrt{2}$	$\left(-3\sqrt{2}, \frac{5\pi}{4}\right)$
$\frac{3\pi}{2}$	0	$\left(0, \frac{3\pi}{2}\right)$
$\frac{7\pi}{4}$	$3\sqrt{2}$	$\left(3\sqrt{2}, \frac{7\pi}{4}\right)$
2π	6	$(6, 2\pi)$

Using the GeoGebra interactive below, we can plot these points, in sequence, to get an idea of the graph.

GeoGebra link: `https://www.geogebra.org/m/rmusbpbp`

To our dismay, despite having nine ordered pairs, we get only four distinct points on the graph. For this reason, we employ a slightly different strategy.

In the interactive below, we have the graph of $r = 6\cos(\theta)$ in the xy-plane below on the left along with the graph of $r = 6\cos(\theta)$ in the θr-plane below on the right.[3]

As we adjust the slider for θ from 0 to $\frac{\pi}{2}$, r ranges from 6 to 0. In the xy-plane, this means that the curve starts 6 units from the origin on the positive x-axis ($\theta = 0$) and gradually returns to the origin by the time the curve reaches the y-axis, $\theta = \frac{\pi}{2}$. This action in the xy-plane is matched by tracing the point $(0, 6)$ in the θr plane to the point $\left(\frac{\pi}{2}, 0\right)$.

GeoGebra link: `https://www.geogebra.org/m/rjj6bump`

The arrows drawn in the interactive are meant to help you visualize this process. In the xy-plane, each of these arrows starts at the origin and is rotated through the corresponding angle θ, in accordance with how we plot polar coordinates. In the θr-plane, the arrows are drawn from the θ-axis to the curve $r = 6\cos(\theta)$. Understanding this correspondence can help us quickly (albeit less accurately) sketch the graph of a polar curve in the xy-plane by using our knowledge of sinusoids gained in Section **??**.

Next, we repeat the process as θ ranges from $\frac{\pi}{2}$ to π. Here, the r values are all negative. This means that in the xy-plane, instead of graphing in Quadrant II, we graph in Quadrant IV, with all of the angle rotations starting from the negative x-axis.

[2] For a review of these concepts and this process, see Section **??**.

[3] The graph looks exactly like $y = 6\cos(x)$ in the xy-plane, and for good reason. At this stage, we are just graphing the relationship between r and θ before we interpret them as polar coordinates (r, θ) on the xy-plane.

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GeoGebra link: `https://www.geogebra.org/m/w5pdqrfw`

As θ ranges from π to $\frac{3\pi}{2}$, the r values are still negative, which means the graph is traced out in Quadrant I instead of Quadrant III. Since the $|r|$ for these values of θ match the r values for θ in $\left[0, \frac{\pi}{2}\right]$, we have that the curve begins to retrace itself at this point.

Proceeding further, we find that when $\frac{3\pi}{2} \leq \theta \leq 2\pi$, we retrace the portion of the curve in Quadrant IV that we first traced out as $\frac{\pi}{2} \leq \theta \leq \pi$. The reader is invited to verify that plotting any range of θ outside the interval $[0, \pi]$ results in retracting some portion of the curve.[4]

GeoGebra link: `https://www.geogebra.org/m/jeyx22fe`

Example 2. Graph the following polar equations in the *xy*-plane.

1. $r = 4 - 2\sin(\theta)$
2. $r = 2 + 4\cos(\theta)$
3. $r = 5\sin(2\theta)$
4. $r^2 = 16\cos(2\theta)$

Explanation.

1. As with our motivational example above, we approach graphing $r = 4 - 2\sin(\theta)$ in the *xy*-plane first by thinking about graphing the sinusoid $r = 4 - 2\sin(\theta)$ in the θr plane. To that end, we divide up the interval $[0, 2\pi]$ into the usual four subintervals $\left[0, \frac{\pi}{2}\right]$, $\left[\frac{\pi}{2}, \pi\right]$, $\left[\pi, \frac{3\pi}{2}\right]$ and $\left[\frac{3\pi}{2}, 2\pi\right]$.

   We can adjust the slider for θ in the interactive below through each of the intervals to see how the action on the graph of $r = 4 - 2\sin(\theta)$ in the *xy*-plane is matched with that of the corresponding sinusoid in the θr plane.

   GeoGebra link: `https://www.geogebra.org/m/szhdtpxn`

   We see that as θ ranges from 0 to $\frac{\pi}{2}$, r decreases from 4 to 2. Hence, the curve in the *xy*-plane starts 4 units from the origin on the positive *x*-axis and gradually pulls in towards the origin as it moves towards the positive *y*-axis.

   Next, as θ runs from $\frac{\pi}{2}$ to π, we see that r increases from 2 to 4. Picking up where we left off, we gradually pull the graph away from the origin until we reach the negative *x*-axis.

   Over the interval $\left[\pi, \frac{3\pi}{2}\right]$, we see that r increases from 4 to 6. On the *xy*-plane, the curve sweeps out away from the origin as it travels from the negative *x*-axis to the negative *y*-axis.

   Finally, as θ takes on values from $\frac{3\pi}{2}$ to 2π, r decreases from 6 back to 4. The graph on the *xy*-plane pulls in from the negative *y*-axis to finish where we started.

   We leave it to the reader to verify that plotting points corresponding to values of θ outside the interval $[0, 2\pi]$ results in retracing portions of the curve, so we are finished.

2. When plotting the sinusoid $r = 2 + 4\cos(\theta)$ in the θr-plane, we note that the graph crosses through the θ-axis. This corresponds to the graph of the curve passing through the origin in the *xy*-plane, so our first task is to determine when this happens.

[4] The graph of $r = 6\cos(\theta)$ looks suspiciously like a circle, for good reason. See number **??** in Example **??**.

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Setting $r = 0$ we get $2 + 4\cos(\theta) = 0$, or $\cos(\theta) = -\frac{1}{2}$. Solving for θ in $[0, 2\pi]$ gives $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$. Since these values of θ are important geometrically, we break the interval $[0, 2\pi]$ into six subintervals: $\left[0, \frac{\pi}{2}\right]$, $\left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$, $\left[\frac{2\pi}{3}, \pi\right]$, $\left[\pi, \frac{4\pi}{3}\right]$, $\left[\frac{4\pi}{3}, \frac{3\pi}{2}\right]$ and $\left[\frac{3\pi}{2}, 2\pi\right]$.

As in the previous example, we can use the GeoGebra interactive below to adjust θ through the interval $[0, 2\pi]$ and observe how the graph of $r = 2 + 4\cos(\theta)$ is traced out both in the xy- and θr- planes.

GeoGebra link: `https://www.geogebra.org/m/fmvv5rhy`

We see that as θ ranges from 0 to $\frac{\pi}{2}$, r decreases from 6 to 2. Plotting this on the xy-plane, we start 6 units out from the origin on the positive x-axis and slowly pull in towards the positive y-axis.

On the interval $\left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$, r decreases from 2 to 0, which means the graph is heading into (and will eventually cross through) the origin.

Not only do we reach the origin when $\theta = \frac{2\pi}{3}$, a theorem from Calculus[5] states that the curve hugs the line $\theta = \frac{2\pi}{3}$ as it approaches the origin.

On the interval $\left[\frac{2\pi}{3}, \pi\right]$, r ranges from 0 to -2. Since $r \leq 0$, the curve passes through the origin in the xy-plane,[6] following the line $\theta = \frac{2\pi}{3}$. Since $|r|$ is increasing from 0 to 2, the curve pulls away from the origin and continues upwards through Quadrant IV to finish at a point on the positive x-axis.

Next, as θ progresses from π to $\frac{4\pi}{3}$, r ranges from -2 to 0. Since $r \leq 0$, we continue our graph in the first quadrant, heading into the origin along the line $\theta = \frac{4\pi}{3}$.

On the interval $\left[\frac{4\pi}{3}, \frac{3\pi}{2}\right]$, r returns to positive values and increases from 0 to 2. We hug the line $\theta = \frac{4\pi}{3}$ as we move through the origin and head towards the negative y-axis.

As we round out the interval, we find that as θ runs through $\frac{3\pi}{2}$ to 2π, r increases from 2 out to 6, and we end up back where we started, 6 units from the origin on the positive x-axis.

Again, we invite the reader to show that plotting the curve for values of θ outside $[0, 2\pi]$ results in retracing a portion of the curve already traced. Our final graph is below.

3. As usual, we start by considering how we would graph $r = 5\sin(2\theta)$ in the θr-plane. Since the frequency of this sinusoid is 2, a fundamental cycle of this sinusoid would be traced out as θ ranges from 0 to π. We partition our interval into subintervals to help us with the graphing, namely $\left[0, \frac{\pi}{4}\right]$, $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$, $\left[\frac{\pi}{2}, \frac{3\pi}{4}\right]$ and $\left[\frac{3\pi}{4}, \pi\right]$.

   As we investigate the graph of $r = 5\sin(2\theta)$ in both the xy-plane and θr plane using the GeoGebra interactive below, we notice that in order to trace out the full curve in the xy-plane, we need to extend beyond the values $[0, \pi]$.

   GeoGebra link: `https://www.geogebra.org/m/nevwynxt`

[5] The 'tangents at the pole' theorem from second semester Calculus.

[6] Recall that one way to visualize plotting polar coordinates (r, θ) with $r < 0$ is to start the rotation from the left side of the pole, in this case, the negative x-axis. Rotating between $\frac{2\pi}{3}$ and π radians from the negative x-axis in this case determines the region between the line $\theta = \frac{2\pi}{3}$ and the x-axis in Quadrant IV.

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To start, as θ ranges from 0 to $\frac{\pi}{4}$, r increases from 0 to 5. Hence the graph of $r = 5\sin(2\theta)$ in the xy-plane starts at the origin and gradually sweeps out so it is 5 units away from the origin on the line $\theta = \frac{\pi}{4}$.

Next, we see that r decreases from 5 to 0 as θ runs through $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$. Moreover, r is becomes negative as θ crosses $\frac{\pi}{2}$. Hence, we draw the curve hugging the line $\theta = \frac{\pi}{2}$ (the y-axis) as the curve heads to the origin.

As θ runs from $\frac{\pi}{2}$ to $\frac{3\pi}{4}$, r becomes negative and ranges from 0 to -5. Since $r \leq 0$, the curve pulls away from the negative y-axis into Quadrant IV.

For $\frac{3\pi}{4} \leq \theta \leq \pi$, r increases from -5 to 0, so the curve pulls back to the origin.

Even though we have finished with one complete cycle of $r = 5\sin(2\theta)$ at this point, if we continue plotting beyond $\theta = \pi$, we find that the curve continues into the third quadrant and eventually finishes up in Quadrant II.

Indeed, we need to trace through two periods of the sinusoid $r = 5\sin(2\theta)$ in the θr plane in order to complete the graph of $r = 5\sin(2\theta)$ in the xy-plane.

4. Graphing $r^2 = 16\cos(2\theta)$ is complicated by the r^2, so we solve for r by extracting square roots and get $r = \pm\sqrt{16\cos(2\theta)} = \pm 4\sqrt{\cos(2\theta)}$.

   In the GeoGebra interactive below, note that as we move the slider for θ, on both the plot in the xy-plane and the plot in the θr plane, two portions of the curve are being traced out at once, then nothing at all, then two more portions of the graph, then nothing at all. We'll explain these phenomena at length momentarily, but we'll offer a brief explanation first. The '$\pm$' accounts for tracing out two portions of the curve at the same time while the gaps in sketching for certain θ values is due to the presence of the square root.

   GeoGebra link: `https://www.geogebra.org/m/q7edszea`

   How would we approach this analytically? First off, we would sketch a fundamental period of $r = \cos(2\theta)$. When $\cos(2\theta) < 0$, $\sqrt{\cos(2\theta)}$ is undefined, so we wouldn't have any values on those intervals. In this particular case, $\cos(2\theta) < 0$ on the interval $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$, so there would be no graph for either $r = 4\sqrt{\cos(2\theta)}$ or $r = -4\sqrt{\cos(2\theta)}$ there.

   On the intervals which remain, $\cos(2\theta)$ ranges from 0 to 1, inclusive. Hence, $\sqrt{\cos(2\theta)}$ ranges from 0 to 1 as well.[7] From this, we know $r = \pm 4\sqrt{\cos(2\theta)}$ ranges continuously from 0 to ± 4, respectively.

   These observations allow us to sketch $r = 4\sqrt{\cos(2\theta)}$ and $r = -4\sqrt{\cos(2\theta)}$ on the θr plane as in the interactive above on the right and use them to sketch the corresponding pieces of the curve $r^2 = 16\cos(2\theta)$ in the xy-plane.

   As we have seen in earlier examples, the lines $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$, which are the zeros of the functions $r = \pm 4\sqrt{\cos(2\theta)}$, serve as guides for us to draw the curve as is passes through the origin.

   As we plot points corresponding to values of θ outside of the interval $[0, \pi]$, we find ourselves retracing parts of the curve,[8]

A few remarks are in order. First, there is no relation, in general, between the period of the function $f(\theta)$ and the length of the interval required to sketch the complete graph of $r = f(\theta)$ in the xy-plane.

[7] Owing to the relationship between $y = x$ and $y = \sqrt{x}$ over $[0, 1]$, we know $\sqrt{\cos(2\theta)} \geq \cos(2\theta)$ wherever the former is defined.

[8] In this case, we could have generated the entire graph by using just the plot $r = 4\sqrt{\cos(2\theta)}$, but graphed over the interval $[0, 2\pi]$ in the θr-plane. We leave the details to the reader.

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As we saw in the run-up to Example **??**, despite the period of the sinusoid $f(\theta) = 6\cos(\theta)$ being 2π, we sketched the complete graph of $r = 6\cos(\theta)$ in the xy-plane just using the values of θ as θ ranged from 0 to π.

On the other hand, in Example **??**, number **??**, the period of $f(\theta) = 5\sin(2\theta)$ is π, but in order to obtain the complete graph of $r = 5\sin(2\theta)$, we needed to run θ from 0 to 2π.

Second, the symmetry seen in the examples is also a common occurrence when graphing polar equations. In addition symmetry about each axis and the origin, it is possible to talk about *rotational* symmetry with these curves. We leave the exploration of symmetry to Exercises **??** - **??**.

Last we note that while many of the 'common' polar graphs can be grouped into families,[9] the authors truly feel that taking the time to work through each graph in the manner presented here is the best way to not only understand the polar coordinate system, but also prepare you for what is needed in Calculus.

Next we turn our attention to finding the intersection points of polar curves. What complicates matters in polar coordinates is that any given point has infinitely many representations. As a result, if a point P is on the graph of two different polar equations, it is entirely possible that the representation $P(r, \theta)$ which satisfies one of the equations does not satisfy the other equation.

In our next example, we see the need to rely on Geometry as much as Algebra to solve each problem.

Example 3. Find the points of intersection of the graphs of the following polar equations.

1. $r = 2\sin(\theta)$ and $r = 2 - 2\sin(\theta)$
2. $r = 2$ and $r = 3\cos(\theta)$
3. $r = 3$ and $r = 6\cos(2\theta)$
4. $r = 3\sin\left(\dfrac{\theta}{2}\right)$ and $r = 3\cos\left(\dfrac{\theta}{2}\right)$

Explanation. 1. Following the procedure in Example **??**, we graph $r = 2\sin(\theta)$ and find it to be a circle centered at the point with rectangular coordinates $(0, 1)$ with a radius of 1. The graph of $r = 2 - 2\sin(\theta)$ is a special kind of limaçon called a 'cardioid.'[10]

GeoGebra link: `https://www.geogebra.org/m/qkhkthvy`

It appears as if there are three intersection points: one in the first quadrant, one in the second quadrant, and the origin. Our next task is to find polar representations of these points.

In order for a point P to be on the graph of $r = 2\sin(\theta)$, it must have a representation $P(r, \theta)$ which satisfies $r = 2\sin(\theta)$. If P is also on the graph of $r = 2 - 2\sin(\theta)$, then P has a (possibly different) representation $P(r', \theta')$ which satisfies $r' = 2\sin(\theta')$. We first try to see if we can find any points which have a single representation $P(r, \theta)$ that satisfies both $r = 2\sin(\theta)$ and $r = 2 - 2\sin(\theta)$.

Assuming such a pair (r, θ) exists, then equating[11] the expressions for r gives $2\sin(\theta) = 2 - 2\sin(\theta)$ or $\sin(\theta) = \dfrac{1}{2}$. From this, we get $\theta = \dfrac{\pi}{6} + 2\pi k$ or $\theta = \dfrac{5\pi}{6} + 2\pi k$ for integers k.

Plugging $\theta = \dfrac{\pi}{6}$ into $r = 2\sin(\theta)$, we get $r = 2\sin\left(\dfrac{\pi}{6}\right) = 2\left(\dfrac{1}{2}\right) = 1$, which is also the value we obtain when we substitute it into $r = 2 - 2\sin(\theta)$. Hence, $\left(1, \dfrac{\pi}{6}\right)$ is one representation for the point of intersection in the first quadrant.

[9] Numbers **??** and **??** in Example **??** are examples of 'limaçons,' number **??** is an example of a 'polar rose,' and number **??** is the famous 'Lemniscate of Bernoulli.'

[10] Presumably, the name is derived from its resemblance to a stylized human heart.

[11] We are really using the technique of substitution to solve the system of equations $\left\{\begin{array}{rcl} r & = & 2\sin(\theta) \\ r & = & 2 - 2\sin(\theta) \end{array}\right.$

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For the point of intersection in the second quadrant, we try $\theta = \dfrac{5\pi}{6}$. Both equations give us the point $\left(1, \dfrac{5\pi}{6}\right)$, so this is our answer here.

We now turn our attention to the origin. We know from Section **??** that the pole may be represented as $(0, \theta)$ for any angle θ. On the graph of $r = 2\sin(\theta)$, we start at the origin when $\theta = 0$ and return to it at $\theta = \pi$, and as the reader can verify, we are at the origin exactly when $\theta = \pi k$ for integers k.

On the curve $r = 2 - 2\sin(\theta)$, however, we reach the origin when $\theta = \dfrac{\pi}{2}$, and more generally, when $\theta = \dfrac{\pi}{2} + 2\pi k$ for integers k. There is no integer value of k for which $\pi k = \dfrac{\pi}{2} + 2\pi k$ which means while the origin is on both graphs, the point is never reached simultaneously. In any case, we have determined the three points of intersection to be $\left(1, \dfrac{\pi}{6}\right)$, $\left(1, \dfrac{5\pi}{6}\right)$ and the origin.

2. As before, we make a quick sketch of $r = 2$ and $r = 3\cos(\theta)$ to get feel for the number and location of the intersection points. The graph of $r = 2$ is a circle, centered at the origin, with a radius of 2.

   The graph of $r = 3\cos(\theta)$ is also a circle - but this one is centered at the point with rectangular coordinates $\left(\dfrac{3}{2}, 0\right)$ and has a radius of $\dfrac{3}{2}$.

   GeoGebra link: https://www.geogebra.org/m/fhvk3s89

   We have two intersection points to find, one in Quadrant I and one in Quadrant IV. Proceeding as above, we first determine if any of the intersection points P have a representation (r, θ) which satisfies both $r = 2$ and $r = 3\cos(\theta)$.

   Equating $r = 2$ and $r = 3\cos(\theta)$, we get $2 = 3\cos(\theta)$, or $\cos(\theta) = \dfrac{2}{3}$. To solve this equation, we need the arccosine function: $\theta = \arccos\left(\dfrac{2}{3}\right) + 2\pi k$ or $\theta = 2\pi - \arccos\left(\dfrac{2}{3}\right) + 2\pi k$ for integers k.

   From these solutions, we get $\left(2, \arccos\left(\dfrac{2}{3}\right)\right)$ as one representation for our answer in Quadrant I, and $\left(2, 2\pi - \arccos\left(\dfrac{2}{3}\right)\right)$ as one representation for our answer in Quadrant IV.

   The reader is encouraged to check these results algebraically and geometrically.

3. Proceeding as above, we first graph $r = 3$ and $r = 6\cos(2\theta)$ to get an idea of how many intersection points to expect and where they lie.

   The graph of $r = 3$ is a circle centered at the origin with a radius of 3 and the graph of $r = 6\cos(2\theta)$ is another four-leafed rose.[12]

   GeoGebra link: https://www.geogebra.org/m/rzkvcy2y

   It appears as if there are eight points of intersection - two in each quadrant. We first look to see if there any points $P(r, \theta)$ with a representation that satisfies both $r = 3$ and $r = 6\cos(2\theta)$.

   Solving $6\cos(2\theta) = 3$, we get $\cos(2\theta) = \dfrac{1}{2}$, so $\theta = \dfrac{\pi}{6} + \pi k$ or $\theta = \dfrac{5\pi}{6} + \pi k$ for integers k. From these, we obtain four distinct points represented by $\left(3, \dfrac{\pi}{6}\right)$, $\left(3, \dfrac{5\pi}{6}\right)$, $\left(3, \dfrac{7\pi}{6}\right)$ and $\left(3, \dfrac{11\pi}{6}\right)$.

   To determine the coordinates of the remaining four points, we have to consider how the representations of the points of intersection can differ. We know from Section **??** that if (r, θ) and (r', θ') represent the same point and $r \neq 0$, then either $r = r'$ or $r = -r'$.

[12] See Example **??** number **??**.

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If $r = r'$, then $\theta' = \theta + 2\pi k$, so one possibility is that an intersection point P has a representation (r, θ) which satisfies $r = 3$ and another representation $(r, \theta + 2\pi k)$ for some integer, k which satisfies $r = 6\cos(2\theta)$. At this point,[13] we replace every occurrence of θ in the equation $r = 6\cos(2\theta)$ with $(\theta + 2\pi k)$ to see if, by equating the resulting expressions for r, we get any more solutions for θ.

Doing so, we get $\cos(2(\theta + 2\pi k)) = \cos(2\theta + 4\pi k) = \cos(2\theta)$ for every integer k. Hence, the equation $r = 6\cos(2(\theta + 2\pi k))$ reduces to the same equation we had before, $r = 6\cos(2\theta)$, which means we get no additional solutions. Moving on to the case where $r = -r'$, we have that $\theta' = \theta + (2k + 1)\pi$ for integers k. We look to see if we can find points P which have a representation (r, θ) that satisfies $r = 3$ and another, $(-r, \theta + (2k + 1)\pi)$, that satisfies $r = 6\cos(2\theta)$.

Substituting[14] $(-r)$ for r and $(\theta + (2k + 1)\pi)$ for θ in $r = 6\cos(2\theta)$ gives $-r = 6\cos(2(\theta + (2k + 1)\pi))$. Since $\cos(2(\theta + (2k + 1)\pi)) = \cos(2\theta + (2k + 1)(2\pi)) = \cos(2\theta)$ for all integers k, the equation $-r = 6\cos(2(\theta + (2k + 1)\pi))$ reduces to $-r = 6\cos(2\theta)$, or $r = -6\cos(2\theta)$.

Coupling $r = -6\cos(2\theta)$ with $r = 3$ gives $-6\cos(2\theta) = 3$ or $\cos(2\theta) = -\frac{1}{2}$. Solving, we get $\theta = \frac{\pi}{3} + \pi k$ or $\theta = \frac{2\pi}{3} + \pi k$. From these solutions, we obtain[15] the remaining four intersection points with representations $\left(-3, \frac{\pi}{3}\right)$, $\left(-3, \frac{2\pi}{3}\right)$, $\left(-3, \frac{4\pi}{3}\right)$ and $\left(-3, \frac{5\pi}{3}\right)$, which check graphically.

4. As usual, we begin by graphing $r = 3\sin\left(\frac{\theta}{2}\right)$ and $r = 3\cos\left(\frac{\theta}{2}\right)$. Using the techniques presented in Example **??**, we plot both functions as θ ranges from 0 to 4π to obtain the complete graph. To our surprise and/or delight, it appears as if these two equations describe the *same* curve, as demonstrated in the GeoGebra interactive below.

   GeoGebra link: `https://www.geogebra.org/m/rbug78nb`

   To verify this incredible claim,[16] we need to show that, in fact, the graphs of these two equations intersect at all points on the plane.

   Suppose P has a representation (r, θ) which satisfies both $r = 3\sin\left(\frac{\theta}{2}\right)$ and $r = 3\cos\left(\frac{\theta}{2}\right)$. Equating these two expressions for r gives the equation $3\sin\left(\frac{\theta}{2}\right) = 3\cos\left(\frac{\theta}{2}\right)$. While normally we discourage dividing by a variable expression (in case it could be 0), we use the same logic here as we did in the solution to Example **??** number **??** in Section **??**.

   If $3\cos\left(\frac{\theta}{2}\right) = 0$, then $\cos\left(\frac{\theta}{2}\right) = 0$ and for the equation $3\sin\left(\frac{\theta}{2}\right) = 3\cos\left(\frac{\theta}{2}\right)$ to hold, $\sin\left(\frac{\theta}{2}\right) = 0$ as well. Since no angles have both cosine and sine equal to zero, we are safe to divide both sides of the equation $3\sin\left(\frac{\theta}{2}\right) = 3\cos\left(\frac{\theta}{2}\right)$ by $3\cos\left(\frac{\theta}{2}\right)$ to get $\tan\left(\frac{\theta}{2}\right) = 1$. Solving this equation gives $\theta = \frac{\pi}{2} + 2\pi k$ for integers k which corresponds to just *one* intersection point: $\left(\frac{3\sqrt{2}}{2}, \frac{\pi}{2}\right)$. We now investigate other representations for the intersection points.

[13] The authors have chosen to replace θ with $\theta + 2\pi k$ in the equation $r = 6\cos(2\theta)$ for illustration purposes only. We could have just as easily chosen to do this substitution in the equation $r = 3$. Since there is no θ in $r = 3$, however, this case would reduce to the previous case instantly. The reader is encouraged to follow this latter procedure in the interests of efficiency.

[14] Again, we could have easily chosen to substitute these into $r = 3$ which would give $-r = 3$, or $r = -3$.

[15] We obtain these representations by substituting the values for θ into $r = 6\cos(2\theta)$, once again, for illustration purposes. Again, we could 'plug' these values for θ into $r = 3$ (where there is no θ) and get the list of points: $\left(3, \frac{\pi}{3}\right)$, $\left(3, \frac{2\pi}{3}\right)$, $\left(3, \frac{4\pi}{3}\right)$ and $\left(3, \frac{5\pi}{3}\right)$. While it is not true that $\left(3, \frac{\pi}{3}\right)$ represents the same point as $\left(-3, \frac{\pi}{3}\right)$, we still get the same *set* of solutions.

[16] Graphing $r = 3\sin\left(\frac{\theta}{2}\right)$ and $r = 3\cos\left(\frac{\theta}{2}\right)$ in the θr-plane show that viewed as functions of r, these are two different animals.

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Suppose P is an intersection point with a representation (r,θ) which satisfies $r = 3\sin\left(\frac{\theta}{2}\right)$ and a different representation $(r, \theta + 2\pi k)$ for some integer k which satisfies $r = 3\cos\left(\frac{\theta}{2}\right)$.

Substituting $(r, \theta + 2\pi k)$ into $r = 3\cos\left(\frac{\theta}{2}\right)$, we get $r = 3\cos\left(\frac{1}{2}[\theta + 2\pi k]\right) = 3\cos\left(\frac{\theta}{2} + \pi k\right)$. Using the sum formula for cosine, we expand $3\cos\left(\frac{\theta}{2} + \pi k\right) = 3\cos\left(\frac{\theta}{2}\right)\cos(\pi k) - 3\sin\left(\frac{\theta}{2}\right)\sin(\pi k)$. Since $\sin(\pi k) = 0$ for all integers k, $r = 3\cos\left(\frac{\theta}{2} + \pi k\right)$ reduces to $r = 3\cos\left(\frac{\theta}{2}\right)\cos(\pi k)$.

If k is an even integer, $\cos(\pi k) = 1$, so we get the same equation $r = 3\cos\left(\frac{\theta}{2}\right)$ as before, and hence any new solutions come from the case when k is odd.

If k is odd, $r = 3\cos\left(\frac{\theta}{2}\right)\cos(\pi k)$ reduces to $r = -3\cos\left(\frac{\theta}{2}\right)$. Coupling $r = -3\cos\left(\frac{\theta}{2}\right)$ with the equation $r = 3\sin\left(\frac{\theta}{2}\right)$ gives $3\sin\left(\frac{\theta}{2}\right) = -3\cos\left(\frac{\theta}{2}\right)$, or $\tan\left(\frac{\theta}{2}\right) = -1$. Solving, we get $\theta = -\frac{\pi}{2} + 2\pi k$ for integers k, which again produces just one intersection point: $\left(\frac{3\sqrt{2}}{2}, -\frac{\pi}{2}\right)$.

Next, we assume P has a representation (r,θ) which satisfies $r = 3\sin\left(\frac{\theta}{2}\right)$ and a representation $(-r, \theta + (2k+1)\pi)$ which satisfies $r = 3\cos\left(\frac{\theta}{2}\right)$ for some integer k.

Substituting $(-r)$ for r and $(\theta + (2k+1)\pi)$ in for θ into $r = 3\cos\left(\frac{\theta}{2}\right)$ gives $-r = 3\cos\left(\frac{1}{2}[\theta + (2k+1)\pi]\right)$ or $r = -3\cos\left(\frac{1}{2}[\theta + (2k+1)\pi]\right)$. Once again, we use the sum formula for cosine to get

$$\begin{aligned} r &= -3\cos\left(\frac{1}{2}[\theta + (2k+1)\pi]\right) \\ &= -3\cos\left(\frac{\theta}{2} + \frac{(2k+1)\pi}{2}\right) \\ &= -3\left[\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{(2k+1)\pi}{2}\right) - \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{(2k+1)\pi}{2}\right)\right] \\ &= 3\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{(2k+1)\pi}{2}\right) \end{aligned}$$

where the last equality is true since $\cos\left(\frac{(2k+1)\pi}{2}\right) = 0$.

Note when $k = 0$, $\sin\left(\frac{(2k+1)\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$, and the equation $r = -3\cos\left(\frac{1}{2}[\theta + (2k+1)\pi]\right)$ reduces to $r = 3\sin\left(\frac{\theta}{2}\right)$, which is the other equation under consideration!

What this means is that if a polar representation (r,θ) for the point P satisfies $r = 3\sin\left(\frac{\theta}{2}\right)$, then the representation $(-r, \theta + \pi)$ for P automatically satisfies $r = 3\cos\left(\frac{\theta}{2}\right)$. Hence the equations $r = 3\sin\left(\frac{\theta}{2}\right)$ and $r = 3\cos\left(\frac{\theta}{2}\right)$ determine the same set of points in the plane.

Our work in Example **??** justifies the following.

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Guidelines for Finding Points of Intersection of Graphs of Polar Equations:

To find the points of intersection of the graphs of two polar equations E_1 and E_2:

- Sketch the graphs of E_1 and E_2. Check to see if the curves intersect at the origin (pole).
- Solve for pairs (r, θ) which satisfy both E_1 and E_2.
- Substitute $(\theta + 2\pi k)$ for θ in either one of E_1 or E_2 (but not both) and solve for pairs (r, θ) which satisfy both equations. Keep in mind that k is an integer.
- Substitute $(-r)$ for r and $(\theta + (2k + 1)\pi)$ for θ in either one of E_1 or E_2 (but not both) and solve for pairs (r, θ) which satisfy both equations. Keep in mind that k is an integer.

Our last example ties together graphing and points of intersection to describe regions in the plane.

Example 4. Sketch the region in the xy-plane described by the following sets.

1. $\left\{(r, \theta) \,|\, 0 \leq r \leq 5\sin(2\theta),\ 0 \leq \theta \leq \dfrac{\pi}{2}\right\}$
2. $\left\{(r, \theta) \,|\, 3 \leq r \leq 6\cos(2\theta),\ 0 \leq \theta \leq \dfrac{\pi}{6}\right\}$
3. $\left\{(r, \theta) \,|\, 2 + 4\cos(\theta) \leq r \leq 0,\ \dfrac{2\pi}{3} \leq \theta \leq \dfrac{4\pi}{3}\right\}$
4. $\left\{(r, \theta) \,|\, 0 \leq r \leq 2\sin(\theta),\ 0 \leq \theta \leq \dfrac{\pi}{6}\right\} \cup \left\{(r, \theta) \,|\, 0 \leq r \leq 2 - 2\sin(\theta),\ \dfrac{\pi}{6} \leq \theta \leq \dfrac{\pi}{2}\right\}$

Explanation. Our first step in these problems is to sketch the graphs of the polar equations involved to get a sense of the geometric situation. Since all of the equations in this example are found in either Example **??** or Example **??**, most of the work is done for us.

1. We know from Example **??** number **??** that the graph of $r = 5\sin(2\theta)$ is a rose. Moreover, we know as $0 \leq \theta \leq \dfrac{\pi}{2}$, we trace out the 'leaf' of the rose which lies in the first quadrant.

   The inequality $0 \leq r \leq 5\sin(2\theta)$ means we want all of the points between the origin ($r = 0$) and the curve $r = 5\sin(2\theta)$ as θ runs through $\left[0, \dfrac{\pi}{2}\right]$.

   Not only does the GeoGebra interactive below show us the shaded region, we can also adjust the slider to see how the ray from the origin out to the curve sweeps out the region as θ varies from 0 to $\dfrac{\pi}{2}$.

   GeoGebra link: https://www.geogebra.org/m/fsfj2xha

2. We know from Example **??** number **??** that $r = 3$ and $r = 6\cos(2\theta)$ intersect at $\theta = \dfrac{\pi}{6}$, so the region that is being described here is the set of points whose directed distance r from the origin is at least 3 but no more than $6\cos(2\theta)$ as θ runs from 0 to $\dfrac{\pi}{6}$.

   In other words, we are looking at the points outside or on the circle (since $r \geq 3$) but inside or on the rose (since $r \leq 6\cos(2\theta)$) as seen below.

   GeoGebra link: https://www.geogebra.org/m/uyr74p9m

3. From Example **??** number **??**, we know that the graph of $r = 2 + 4\cos(\theta)$ is a limaçon whose 'inner loop' is traced out as θ runs through the given values $\dfrac{2\pi}{3}$ to $\dfrac{4\pi}{3}$.

Since the values r takes on in this interval are non-positive, the inequality $2 + 4\cos(\theta) \leq r \leq 0$ makes sense, and we are looking for all of the points between the pole $r = 0$ and the limaçon as θ ranges over the interval $\left[\frac{2\pi}{3}, \frac{4\pi}{3}\right]$. In other words, we shade in the inner loop of the limaçon.

Once again, we can see not only the region in question in the GeoGebra interactive below, but how the region is swept out as θ ranges from $\frac{2\pi}{3}$ to $\frac{4\pi}{3}$

GeoGebra link: `https://www.geogebra.org/m/ecvqpk84`

4. We have two regions described here connected with the union symbol '$\cup$.' We shade each in turn and find our final answer by combining the two.

   In Example **??**, number **??**, we found that the curves $r = 2\sin(\theta)$ and $r = 2 - 2\sin(\theta)$ intersect when $\theta = \frac{\pi}{6}$. Hence, for the first region, $\left\{(r, \theta) \,|\, 0 \leq r \leq 2\sin(\theta), 0 \leq \theta \leq \frac{\pi}{6}\right\}$, we are shading the region between the origin ($r = 0$) out to the circle ($r = 2\sin(\theta)$) as θ ranges from 0 to $\frac{\pi}{6}$, which is the angle of intersection of the two curves.

   For the second region, $\left\{(r, \theta) \,|\, 0 \leq r \leq 2 - 2\sin(\theta), \frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}\right\}$, θ picks up where it left off at $\frac{\pi}{6}$ and continues to $\frac{\pi}{2}$. In this case, however, we are shading from the origin ($r = 0$) out to the cardioid $r = 2 - 2\sin(\theta)$ which pulls into the origin at $\theta = \frac{\pi}{2}$.

   We combine these two regions to obtain our final answer depicted below.

GeoGebra link: `https://www.geogebra.org/m/gms7kgys`

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