Exercises for Complex Zeros

Problem 1 *Find all of the zeros of the polynomial then completely factor it over the real numbers and completely factor it over the complex numbers.*

$f(x) = x^2 - 4x + 13$

$f(x) = x^2 - 4x + 13 = (x - (2 + 3i))(x - (2 - 3i))$
Zeros: $x = 2 \pm 3i$

Problem 2 *Find all of the zeros of the polynomial then completely factor it over the real numbers and completely factor it over the complex numbers.*

$f(x) = x^2 - 2x + 5$

$f(x) = x^2 - 2x + 5 = (x - (1 + 2i))(x - (1 - 2i))$
Zeros: $x = 1 \pm 2i$

Problem 3 *Find all of the zeros of the polynomial then completely factor it over the real numbers and completely factor it over the complex numbers.*

$p(z) = 3z^2 + 2z + 10$

$p(z) = 3z^2 + 2z + 10 = 3\left(z - \left(-\frac{1}{3} + \frac{\sqrt{29}}{3}i\right)\right)\left(z - \left(-\frac{1}{3} - \frac{\sqrt{29}}{3}i\right)\right)$

Zeros: $z = -\frac{1}{3} \pm \frac{\sqrt{29}}{3}i$

Problem 4 *Find all of the zeros of the polynomial then completely factor it over the real numbers and completely factor it over the complex numbers.*

$p(z) = z^3 - 2z^2 + 9z - 18$

$p(z) = z^3 - 2z^2 + 9z - 18 = (z - 2)\left(z^2 + 9\right) = (z - 2)(z - 3i)(z + 3i)$
Zeros: $z = 2, \pm 3i$

Problem 5 *Find all of the zeros of the polynomial then completely factor it over the real numbers and completely factor it over the complex numbers.*

$g(t) = t^3 + 6t^2 + 6t + 5$

$g(t) = t^3 + 6t^2 + 6t + 5 = (t + 5)(t^2 + t + 1) = (t + 5)\left(t - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right)\left(t - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right)$

Zeros: $t = -5,\ t = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

Problem 6 *Find all of the zeros of the polynomial then completely factor it over the real numbers and completely factor it over the complex numbers.*

$g(t) = 3t^3 - 13t^2 + 43t - 13$

$g(t) = 3t^3 - 13t^2 + 43t - 13 = (3t - 1)(t^2 - 4t + 13) = (3t - 1)(t - (2 + 3i))(t - (2 - 3i))$
Zeros: $t = \frac{1}{3},\ t = 2 \pm 3i$

Problem 7 *Find all of the zeros of the polynomial then completely factor it over the real numbers and completely factor it over the complex numbers.*

$f(x) = x^3 + 3x^2 + 4x + 12$

$f(x) = x^3 + 3x^2 + 4x + 12 = (x + 3)\left(x^2 + 4\right) = (x + 3)(x + 2i)(x - 2i)$
Zeros: $x = -3,\ \pm 2i$

Problem 8 *Find all of the zeros of the polynomial then completely factor it over the real numbers and completely factor it over the complex numbers.*

$f(x) = 4x^3 - 6x^2 - 8x + 15$

$$f(x) = 4x^3 - 6x^2 - 8x + 15 = \left(x + \frac{3}{2}\right)\left(4x^2 - 12x + 10\right)$$
$$= 4\left(x + \frac{3}{2}\right)\left(x - \left(\frac{3}{2} + \frac{1}{2}i\right)\right)\left(x - \left(\frac{3}{2} - \frac{1}{2}i\right)\right)$$

Zeros: $x = -\frac{3}{2},\ x = \frac{3}{2} \pm \frac{1}{2}i$

Problem 9 *Find all of the zeros of the polynomial then completely factor it over the real numbers and completely factor it over the complex numbers.*

$p(z) = z^3 + 7z^2 + 9z - 2$

$$p(z) = z^3 + 7z^2 + 9z - 2 = (z + 2)\left(z - \left(-\frac{5}{2} + \frac{\sqrt{29}}{2}\right)\right)\left(z - \left(-\frac{5}{2} - \frac{\sqrt{29}}{2}\right)\right)$$

Zeros: $z = -2,\ z = -\frac{5}{2} \pm \frac{\sqrt{29}}{2}$

Problem 10 *Find all of the zeros of the polynomial then completely factor it over the real numbers and completely factor it over the complex numbers.*

$p(z) = 9z^3 + 2z + 1$

$$p(z) = 9z^3 + 2z + 1 = \left(z + \frac{1}{3}\right)\left(9z^2 - 3z + 3\right)$$
$$= 9\left(z + \frac{1}{3}\right)\left(z - \left(\frac{1}{6} + \frac{\sqrt{11}}{6}i\right)\right)\left(z - \left(\frac{1}{6} - \frac{\sqrt{11}}{6}i\right)\right)$$

Zeros: $z = -\frac{1}{3},\ z = \frac{1}{6} \pm \frac{\sqrt{11}}{6}i$

Problem 11 *Find all of the zeros of the polynomial then completely factor it over the real numbers and completely factor it over the complex numbers.*

$g(t) = 4t^4 - 4t^3 + 13t^2 - 12t + 3$

$$g(t) = 4t^4 - 4t^3 + 13t^2 - 12t + 3 = \left(t - \frac{1}{2}\right)^2\left(4t^2 + 12\right) = 4\left(t - \frac{1}{2}\right)^2(t + i\sqrt{3})(t - i\sqrt{3})$$

Zeros: $t = \frac{1}{2},\ t = \pm\sqrt{3}i$

Problem 12 *Find all of the zeros of the polynomial then completely factor it over the real numbers and completely factor it over the complex numbers.*

$g(t) = 2t^4 - 7t^3 + 14t^2 - 15t + 6$

$$g(t) = 2t^4 - 7t^3 + 14t^2 - 15t + 6 = (t - 1)^2\left(2t^2 - 3t + 6\right)$$
$$= 2(t - 1)^2\left(t - \left(\frac{3}{4} + \frac{\sqrt{39}}{4}i\right)\right)\left(t - \left(\frac{3}{4} - \frac{\sqrt{39}}{4}i\right)\right)$$

Zeros: $t = 1,\ t = \frac{3}{4} \pm \frac{\sqrt{39}}{4}i$

Problem 13 *Find all of the zeros of the polynomial then completely factor it over the real numbers and completely factor it over the complex numbers.*

$f(x) = x^4 + x^3 + 7x^2 + 9x - 18$

$$f(x) = x^4 + x^3 + 7x^2 + 9x - 18 = (x + 2)(x - 1)\left(x^2 + 9\right) = (x + 2)(x - 1)(x + 3i)(x - 3i)$$

Zeros: $x = -2,\ 1,\ \pm 3i$

Problem 14 *Find all of the zeros of the polynomial then completely factor it over the real numbers and completely factor it over the complex numbers.*

$f(x) = 6x^4 + 17x^3 - 55x^2 + 16x + 12$

$f(x) = 6x^4 + 17x^3 - 55x^2 + 16x + 12 = 6\left(x + \frac{1}{3}\right)\left(x - \frac{3}{2}\right)\left(x - \left(-2 + 2\sqrt{2}\right)\right)\left(x - \left(-2 - 2\sqrt{2}\right)\right)$

Zeros: $x = -\frac{1}{3},\ x = \frac{3}{2},\ x = -2 \pm 2\sqrt{2}$

Problem 15 *Find all of the zeros of the polynomial then completely factor it over the real numbers and completely factor it over the complex numbers.*

$p(z) = -3z^4 - 8z^3 - 12z^2 - 12z - 5$

$p(z) = -3z^4 - 8z^3 - 12z^2 - 12z - 5 = (z+1)^2\left(-3z^2 - 2z - 5\right)$
$= -3(z+1)^2\left(z - \left(-\frac{1}{3} + \frac{\sqrt{14}}{3}i\right)\right)\left(z - \left(-\frac{1}{3} - \frac{\sqrt{14}}{3}i\right)\right)$

Zeros: $z = -1,\ z = -\frac{1}{3} \pm \frac{\sqrt{14}}{3}i$

Problem 16 *Find all of the zeros of the polynomial then completely factor it over the real numbers and completely factor it over the complex numbers.*

$p(z) = 8z^4 + 50z^3 + 43z^2 + 2z - 4$

$p(z) = 8z^4 + 50z^3 + 43z^2 + 2z - 4 = 8\left(z + \frac{1}{2}\right)\left(z - \frac{1}{4}\right)(z - (-3 + \sqrt{5}))(z - (-3 - \sqrt{5}))$

Zeros: $z = -\frac{1}{2},\ \frac{1}{4},\ z = -3 \pm \sqrt{5}$

Problem 17 *Find all of the zeros of the polynomial then completely factor it over the real numbers and completely factor it over the complex numbers.*

$g(t) = t^4 + 9t^2 + 20$

$g(t) = t^4 + 9t^2 + 20 = \left(t^2 + 4\right)\left(t^2 + 5\right) = (t - 2i)(t + 2i)\left(t - i\sqrt{5}\right)\left(t + i\sqrt{5}\right)$

Zeros: $t = \pm 2i, \pm i\sqrt{5}$

Problem 18 *Find all of the zeros of the polynomial then completely factor it over the real numbers and completely factor it over the complex numbers.*

$g(t) = t^4 + 5t^2 - 24$

$g(t) = t^4 + 5t^2 - 24 = \left(t^2 - 3\right)\left(t^2 + 8\right) = (t - \sqrt{3})(t + \sqrt{3})\left(t - 2i\sqrt{2}\right)\left(t + 2i\sqrt{2}\right)$

Zeros: $t = \pm\sqrt{3}, \pm 2i\sqrt{2}$

Problem 19 *Find all of the zeros of the polynomial then completely factor it over the real numbers and completely factor it over the complex numbers.*

$f(x) = x^5 - x^4 + 7x^3 - 7x^2 + 12x - 12$

$f(x) = x^5 - x^4 + 7x^3 - 7x^2 + 12x - 12 = (x - 1)\left(x^2 + 3\right)\left(x^2 + 4\right)$
$= (x - 1)(x - i\sqrt{3})(x + i\sqrt{3})(x - 2i)(x + 2i)$

Zeros: $x = 1,\ \pm\sqrt{3}i,\ \pm 2i$

Problem 20 *Find all of the zeros of the polynomial then completely factor it over the real numbers and completely factor it over the complex numbers.*

$f(x) = x^6 - 64$

$$f(x) = x^6 - 64 = (x-2)(x+2)\left(x^2+2x+4\right)\left(x^2-2x+4\right)$$
$$= (x-2)(x+2)\left(x-\left(-1+i\sqrt{3}\right)\right)\left(x-\left(-1-i\sqrt{3}\right)\right)\left(x-\left(1+i\sqrt{3}\right)\right)\left(x-\left(1-i\sqrt{3}\right)\right)$$

Zeros: $x = \pm 2$, $x = -1 \pm i\sqrt{3}$, $x = 1 \pm i\sqrt{3}$

Problem 21 *Find all of the zeros of the polynomial then completely factor it over the real numbers and completely factor it over the complex numbers.*

$f(x) = x^4 - 2x^3 + 27x^2 - 2x + 26$

Hint: *$x = i$ is one of the zeros.*

$f(x) = x^4 - 2x^3 + 27x^2 - 2x + 26 = (x^2 - 2x + 26)(x^2 + 1) = (x - (1+5i))(x - (1-5i))(x+i)(x-i)$
Zeros: $x = 1 \pm 5i,\ \ x = \pm i$

Problem 22 *Find all of the zeros of the polynomial then completely factor it over the real numbers and completely factor it over the complex numbers.*

$p(z) = 2z^4 + 5z^3 + 13z^2 + 7z + 5$

Hint: *$z = -1 + 2i$ is a zero.*

$$p(z) = 2z^4 + 5z^3 + 13z^2 + 7z + 5 = \left(z^2+2z+5\right)\left(2z^2+z+1\right)$$
$$= 2(z-(-1+2i))(z-(-1-2i))\left(z-\left(-\frac{1}{4}+i\frac{\sqrt{7}}{4}\right)\right)\left(z-\left(-\frac{1}{4}-i\frac{\sqrt{7}}{4}\right)\right)$$

Zeros: $z = -1 \pm 2i, -\dfrac{1}{4} \pm i\dfrac{\sqrt{7}}{4}$

Problem 23 *Use Theorem* **??** *to create a polynomial function with real number coefficients which has all of the desired characteristics. You may leave the polynomial in factored form.*

- *The zeros of f are $c = \pm 2$ and $c = \pm 1$.*
- *The leading term of $f(x)$ is $117x^4$.*

$f(x) = 117(x+2)(x-2)(x+1)(x-1)$

Problem 24 *Use Theorem* **??** *to create a polynomial function with real number coefficients which has all of the desired characteristics. You may leave the polynomial in factored form.*

- *The zeros of p are $c = 1$ and $c = 3$.*
- *$c = 3$ is a zero of multiplicity 2.*
- *The leading term of $p(z)$ is $-5z^3$.*

$p(z) = -5(z-1)(z-3)^2$

Problem 25 *Use Theorem* **??** *to create a polynomial function with real number coefficients which has all of the desired characteristics. You may leave the polynomial in factored form.*

- *The solutions to $g(t) = 0$ are $t = \pm 3$ and $t = 6$.*
- *The leading term of $g(t)$ is $7t^4$.*

- *The point* $(-3, 0)$ *is a local minimum on the graph of* $y = g(t)$*.*

$g(t) = 7(t+3)^2(t-3)(t-6)$

Problem 26 *Use Theorem* **??** *to create a polynomial function with real number coefficients which has all of the desired characteristics. You may leave the polynomial in factored form.*

- *The solutions to* $f(x) = 0$ *are* $x = \pm 3$*,* $x = -2$*, and* $x = 4$*.*
- *The leading term of* $f(x)$ *is* $-x^5$*.*
- *The point* $(-2, 0)$ *is a local maximum on the graph of* $y = f(x)$*.*

$f(x) = -(x+2)^2(x-3)(x+3)(x-4)$

Problem 27 *Use Theorem* **??** *to create a polynomial function with real number coefficients which has all of the desired characteristics. You may leave the polynomial in factored form.*

- p *is degree 4.*
- $\lim_{z \to \infty} p(z) = -\infty$*.*
- p *has exactly three* z*-intercepts:* $(-6, 0)$*,* $(1, 0)$ *and* $(117, 0)$*.*
- *The graph of* $y = p(z)$ *crosses through the* z*-axis at* $(1, 0)$*.*

$p(z) = (z+6)^2(z-1)(z-117)$ *where* a *can be any real number as long as* $a < 0$

Problem 28 *Use Theorem* **??** *to create a polynomial function with real number coefficients which has all of the desired characteristics. You may leave the polynomial in factored form.*

- *The zeros of* g *are* $c = \pm 1$ *and* $c = \pm i$*.*
- *The leading term of* $g(t)$ *is* $42t^4$*.*

$g(t) = 42(t-1)(t+1)(t-i)(t+i)$

Problem 29 *Use Theorem* **??** *to create a polynomial function with real number coefficients which has all of the desired characteristics. You may leave the polynomial in factored form.*

- $c = 2i$ *is a zero.*
- *the point* $(-1, 0)$ *is a local minimum on the graph of* $y = f(x)$*.*
- *the leading term of* $f(x)$ *is* $117x^4$*.*

$f(x) = 117(x+1)^2(x-2i)(x+2i)$

Problem 30 *Use Theorem* **??** *to create a polynomial function with real number coefficients which has all of the desired characteristics. You may leave the polynomial in factored form.*

- *The solutions to* $p(z) = 0$ *are* $z = \pm 2$ *and* $z = \pm 7i$*.*
- *The leading term of* $p(z)$ *is* $-3z^5$*.*

- *The point* $(2,0)$ *is a local maximum on the graph of* $y = p(z)$.

$p(z) = -3(z-2)^2(z+2)(z-7i)(z+7i)$

Problem 31 *Use Theorem* **??** *to create a polynomial function with real number coefficients which has all of the desired characteristics. You may leave the polynomial in factored form.*

- g *is degree* 5.
- $t = 6$, $t = i$ *and* $t = 1 - 3i$ *are zeros of* g.
- $\lim_{t \to -\infty} g(t) = \infty$

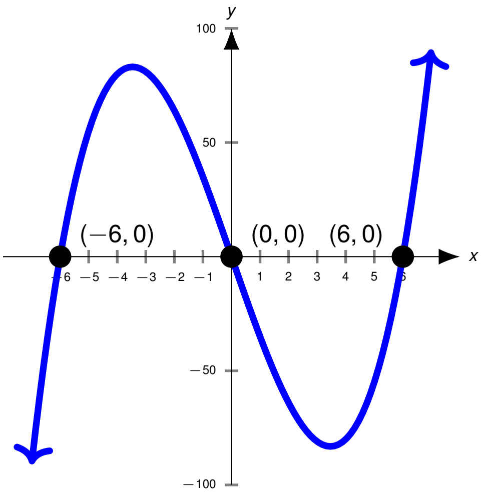
$g(t) = a(t-6)(t-i)(t+i)(t-(1-3i))(t-(1+3i))$ *where* a *is any real number,* $a < 0$

Problem 32 *Use Theorem* **??** *to create a polynomial function with real number coefficients which has all of the desired characteristics. You may leave the polynomial in factored form.*

- *The leading term of* $f(x)$ *is* $-2x^3$.
- $c = 2i$ *is a zero.*
- $f(0) = -16$.

$f(x) = -2(x-2i)(x+2i)(x+2)$

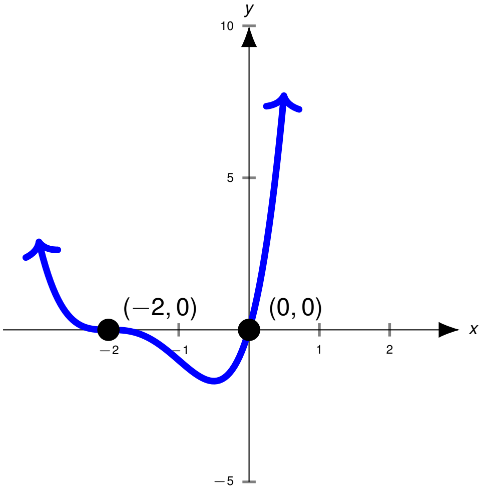
Problem 33 *Find a possible formula for the polynomial function given its graph. You may leave the polynomial in factored form.*



The graph of $y = f(x)$.

$f(x) = x(x+6)(x-6)$

Problem 34 *Find a possible formula for the polynomial function given its graph. You may leave the polynomial in factored form.*

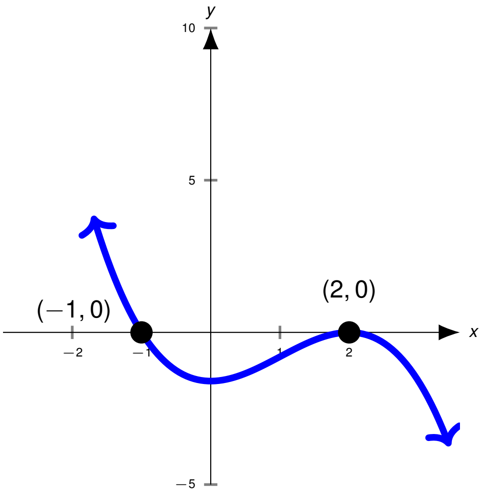


The graph of $y = g(t)$.

$g(t) = t(t + 2)^3$

Problem 35 *Find a possible formula for the polynomial function given its graph. You may leave the polynomial in factored form.*

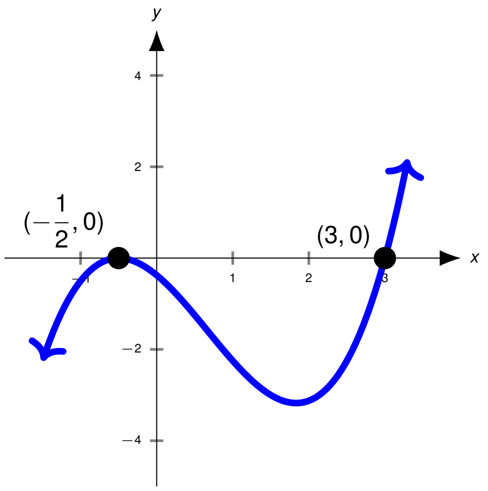
$y = p(z)$



The graph of $y = p(z)$.

$p(z) = -2(z + 1)(z - 2)^2$

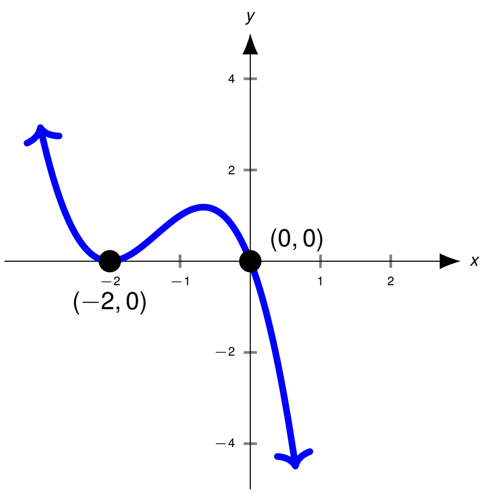
Problem 36 *Find a possible formula for the polynomial function given its graph. You may leave the polynomial in factored form.*



The graph of $y = f(x)$.

$f(x) = 4\left(x + \frac{1}{2}\right)^2 (x - 3)$

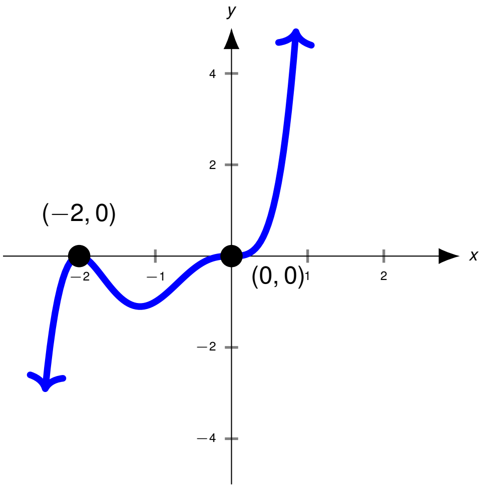
Problem 37 *Find a possible formula for the polynomial function given its graph. You may leave the polynomial in factored form.*



The graph of $y = F(s)$.

$F(s) = -s(s+2)^2$

Problem 38 *Find a possible formula for the polynomial function given its graph. You may leave the polynomial in factored form.*



The graph of $y = G(t)$.

$G(t) = t^3(t + 2)^2$

Problem 39 *With help from your classmates, choose several nonzero complex numbers z, find their complex conjugates $\overline{z}$. Plot each pair z and $\overline{z}$ in the Complex Plane. What appears to be the relationship between these numbers geometrically? State and prove a general result.*

If $z = a + bi$, then z corresponds to the point (a, b) in the xy-plane. Hence, $\overline{z} = \overline{a + bi} = a - bi$ corresponds to the point $(a, -b)$. Hence, the points corresponding to z and $\overline{z}$ are reflections about the x-axis.

Problem 40 *With help from your classmates, choose several nonzero complex numbers z and find $-z$. Plot each pair z and $-z$ in the Complex Plane. What appears to be the relationship between these numbers geometrically? State and prove a general result.*

If $z = a + bi$, then z corresponds to the point (a, b) in the xy-plane. Hence, $-z = -(a + bi) = -a - bi$ corresponds to the point $(-a, -b)$. Hence, the points corresponding to z and $-z$ are reflections through the origin.

Problem 41 *With help from your classmates, choose several different complex numbers z and find the product of i and z, iz. Plot each pair of z and iz in the Complex Plane. In each case, show the line containing the origin and the point corresponding to z is perpendicular[1] to the line containing the origin and the point corresponding to iz. Show this result holds in general for every nonzero complex number.*

*If $z = a + bi$, then z corresponds to the point (a, b) in the xy-plane. Writing out the product iz, we get: $iz = i(a + bi) = ia + bi^2 = ia - b = -b + ia$. Hence, iz corresponds to the point $(-b, a)$. If $z \neq 0$, then neither a nor b is 0 (do you see why?) Hence, the slope of the line containing $(0, 0)$ and (a, b) is $\frac{b}{a}$ and the slope of the line containing $(0, 0)$ and $(-b, a)$ is $-\frac{a}{b}$. Per Theorem **??**, since the slopes of these lines are negative reciprocals, the lines themselves are perpendicular.[2]*

Problem 42 *Given a complex number $z = a + bi$, we define the **modulus** of z, $|z|$, by $|z| = \sqrt{a^2 + b^2}$. With help from your classmates, calculate $|z|$ for several different complex numbers, z. What does z measure geometrically? Show that if x is a real number, then the modulus of x is the same as the absolute value of x, and comment how all this relates to Definition **??** in Section **??**.*

*$|z| = \sqrt{a^2 + b^2}$ measures the distance from the origin to the point (a, b). Hence, $|z|$ measures the distance from z to 0 in the Complex Plane. This is exactly how $|x|$ is defined in Definition **??** in Section **??**. In that section, however, the only part of the Complex Plane under discussion is the real number line.*

[1]See Theorem **??** in Section **??** of you need a refresher on how to do this.

[2]We'll be able to show in Section **??** that, more precisely, multiplication by i rotates the complex number counter-clockwise by 90°.

Problem 43 *Let z and w be arbitrary complex numbers. Show that $\overline{z}\,\overline{w} = \overline{zw}$ and $\overline{\overline{z}} = z$.*