Exercises for Intro Rational

(Review of Long Division):[1]

Problem 1 *Use polynomial long division to perform the indicated division. Write the polynomial in the form $p(x) = d(x)q(x) + r(x)$.*

$\left(4x^2 + 3x - 1\right) \div (x - 3)$

$4x^2 + 3x - 1 = (x - 3)(4x + 15) + 44$

Problem 2 *Use polynomial long division to perform the indicated division. Write the polynomial in the form $p(x) = d(x)q(x) + r(x)$.*

$\left(2x^3 - x + 1\right) \div \left(x^2 + x + 1\right)$

$2x^3 - x + 1 = \left(x^2 + x + 1\right)(2x - 2) + (-x + 3)$

Problem 3 *Use polynomial long division to perform the indicated division. Write the polynomial in the form $p(x) = d(x)q(x) + r(x)$.*

$\left(5x^4 - 3x^3 + 2x^2 - 1\right) \div \left(x^2 + 4\right)$

$5x^4 - 3x^3 + 2x^2 - 1 = \left(x^2 + 4\right)\left(5x^2 - 3x - 18\right) + (12x + 71)$

Problem 4 *Use polynomial long division to perform the indicated division. Write the polynomial in the form $p(x) = d(x)q(x) + r(x)$.*

$\left(-x^5 + 7x^3 - x\right) \div \left(x^3 - x^2 + 1\right)$

$-x^5 + 7x^3 - x = \left(x^3 - x^2 + 1\right)\left(-x^2 - x + 6\right) + \left(7x^2 - 6\right)$

Problem 5 *Use polynomial long division to perform the indicated division. Write the polynomial in the form $p(x) = d(x)q(x) + r(x)$.*

$\left(9x^3 + 5\right) \div (2x - 3)$

$9x^3 + 5 = (2x - 3)\left(\frac{9}{2}x^2 + \frac{27}{4}x + \frac{81}{8}\right) + \frac{283}{8}$

Problem 6 *Use polynomial long division to perform the indicated division. Write the polynomial in the form $p(x) = d(x)q(x) + r(x)$.*

$\left(4x^2 - x - 23\right) \div \left(x^2 - 1\right)$

$4x^2 - x - 23 = \left(x^2 - 1\right)(4) + (-x - 19)$

Problem 7 *Given the pair of functions f and F, sketch the graph of $y = F(x)$ by starting with the graph of $y = f(x)$ and using Theorem* **??**. *Track at least two points and the asymptotes. State the domain and range using interval notation.*

$f(x) = \frac{1}{x}$, $F(x) = \frac{1}{x - 2} + 1$

$F(x) = \frac{1}{x - 2} + 1$
Domain: $(-\infty, 2) \cup (2, \infty)$
Range: $(-\infty, 1) \cup (1, \infty)$
Vertical asymptote: $x = 2$
Horizontal asymptote: $y = 1$

Use the Desmos graph below with the settings $f(x) = \frac{1}{x}$, $h = 2$, $k = 1$, $a = 1$

[1] For more review, see Section **??**.

Desmos link: https://www.desmos.com/calculator/efqikakypj

Problem 8 *Given the pair of functions f and F, sketch the graph of $y = F(x)$ by starting with the graph of $y = f(x)$ and using Theorem* **??**. *Track at least two points and the asymptotes. State the domain and range using interval notation.*

$f(x) = \frac{1}{x}$, $F(x) = \frac{2x}{x+1}$

$F(x) = \frac{2x}{x+1} = \frac{-2}{x+1} + 2$

Domain: $(-\infty, -1) \cup (-1, \infty)$

Range: $(-\infty, 2) \cup (2, \infty)$

Vertical asymptote: $x = -1$

Horizontal asymptote: $y = 2$

Use the Desmos graph below with the settings $f(x) = \frac{1}{x},\ h = -1,\ k = 2,\ a = -2$

Desmos link: https://www.desmos.com/calculator/efqikakypj

Problem 9 *Given the pair of functions f and F, sketch the graph of $y = F(x)$ by starting with the graph of $y = f(x)$ and using Theorem* **??**. *Track at least two points and the asymptotes. State the domain and range using interval notation.*

$f(x) = x^{-1}$, $F(x) = 4x(2x+1)^{-1}$

$F(x) = 4x(2x+1)^{-1} = \frac{4x}{2x+1} = \frac{-1}{x+\frac{1}{2}} + 2$

Domain: $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, \infty\right)$

Range: $(-\infty, 2) \cup (2, \infty)$

Vertical asymptote: $x = -\frac{1}{2}$

Horizontal asymptote: $y = 2$

Use the Desmos graph below with the settings $f(x) = \frac{1}{x},\ h = -\frac{1}{2},\ k = 2,\ a = -1$

Desmos link: https://www.desmos.com/calculator/efqikakypj

Problem 10 *Given the pair of functions f and F, sketch the graph of $y = F(x)$ by starting with the graph of $y = f(x)$ and using Theorem* **??**. *Track at least two points and the asymptotes. State the domain and range using interval notation.*

$f(x) = x^{-2}$, $F(x) = -(x-1)^{-2} + 3$

$F(x) = -(x-1)^{-2} + 3 = \frac{-1}{(x-1)^2} + 3$

Domain: $(-\infty, 1) \cup (1, \infty)$

Range: $(-\infty, 3) \cup (3, \infty)$

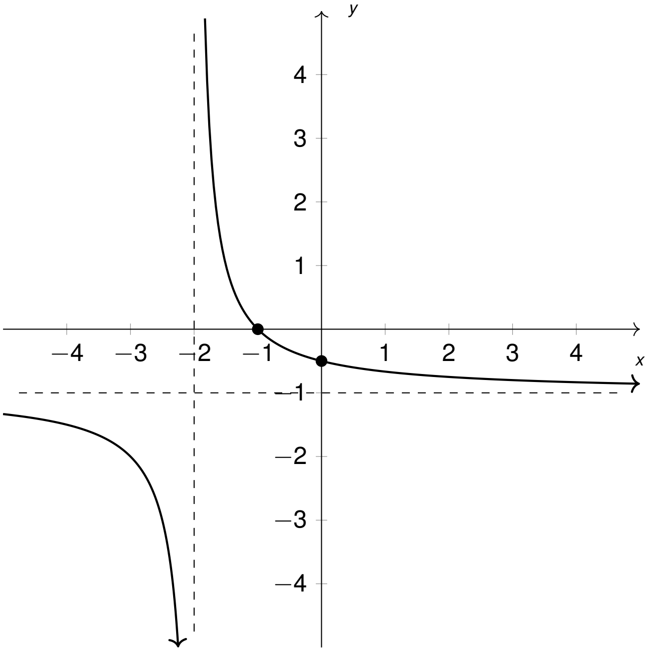
Vertical asymptote: $x = 1$

Horizontal asymptote: $y = 3$

Use the Desmos graph below with the settings $f(x) = \frac{1}{x^2},\ h = 1,\ k = 3,\ a = -1$

Desmos link: https://www.desmos.com/calculator/3ysihdzq86

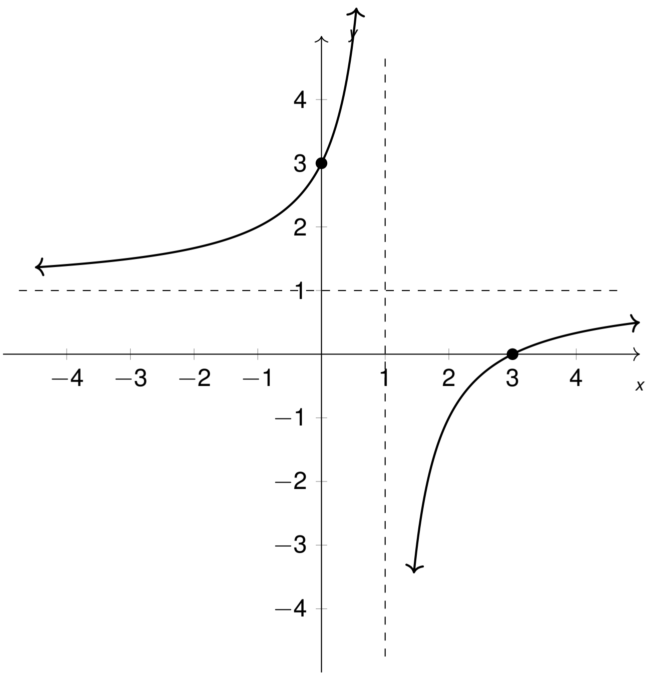
Problem 11 *Find a formula in the form* $F(x) = \dfrac{a}{x-h} + k$ *for the function.*



x-intercept $(-1, 0)$, *y-intercept* $\left(0, -\frac{1}{2}\right)$

$F(x) = \dfrac{1}{x+2} - 1$

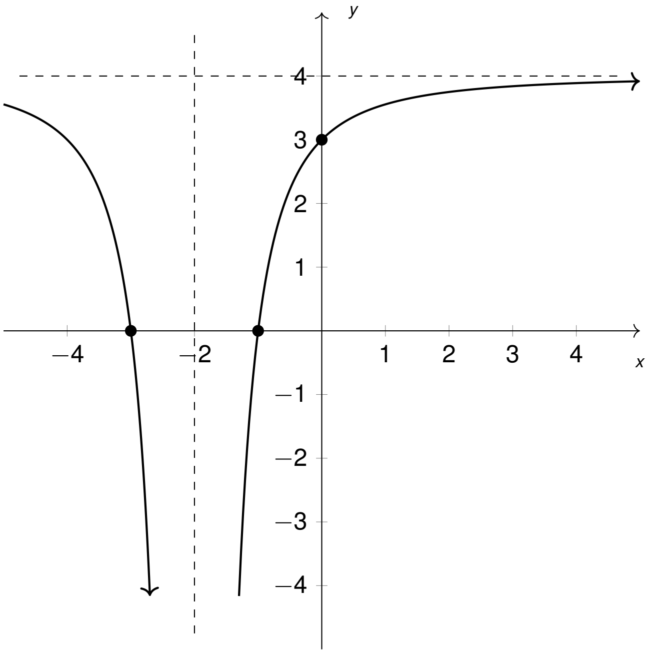
Problem 12 *Find a formula in the form* $F(x) = \dfrac{a}{x-h} + k$ *for the function.*



x-intercept $(3, 0)$, *y-intercept* $(0, 3)$

$F(x) = \dfrac{-2}{x-1} + 1$

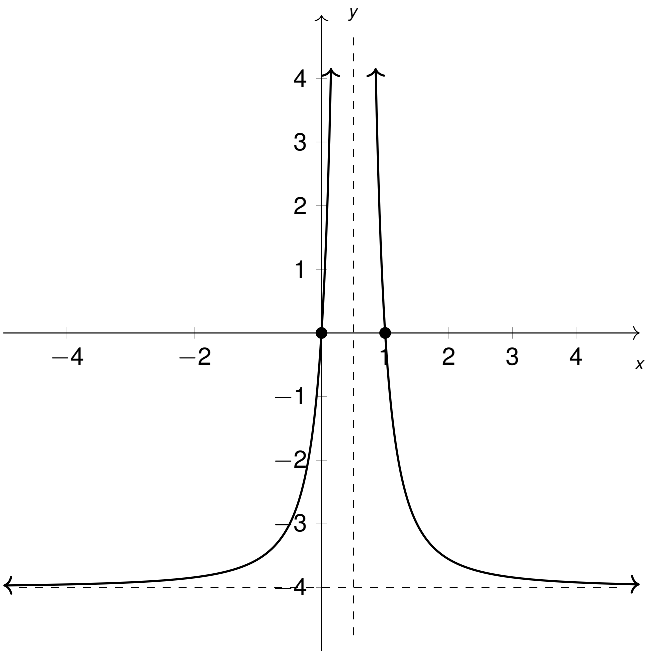
Problem 13 *Find a formula in the form* $F(x) = \dfrac{a}{(x-h)^2} + k$ *for the function.*



x-intercepts $(-3,0)$, $(-1,0)$, *y-intercept* $(0,3)$

$F(x) = \dfrac{-4}{(x+2)^2} + 4$

Problem 14 *Find a formula in the form* $F(x) = \dfrac{a}{(x-h)^2} + k$ *for the function.*



x-intercepts $(0,0)$, $(1,0)$, *Vertical Asymptote* $x = \dfrac{1}{2}$

$F(x) = \dfrac{1}{\left(x-\frac{1}{2}\right)^2} - 4$

Problem 15 *Consider the function* $f(x) = \dfrac{x}{3x-6}$

- *State the domain.* $(-\infty, 2) \cup (2, \infty)$
- *Identify any vertical asymptotes of the graph.* $x = 2$
- *Identify any holes in the graph.* *No holes in the graph*
- *Find the horizontal asymptote, if it exists.* $y = \dfrac{1}{3}$
- *Find the slant asymptote, if it exists.* *No slant asymptote*
- *Graph the function using a graphing utility and describe the behavior near the asymptotes.*

  Desmos link: `https://www.desmos.com/calculator/fmrfliu7sk`

  Vertical asymptote:

  $\lim\limits_{x\to 2^-} f(x) = -\infty$, $\lim\limits_{x\to 2^+} f(x) = \infty$

  Horizontal asymptote:

  $\lim\limits_{x\to -\infty} f(x) = \dfrac{1}{3}$ *More specifically: as* $x \to -\infty, f(x) \to \dfrac{1}{3}^-$

  $\lim\limits_{x\to \infty} f(x) = \dfrac{1}{3}$ *More specifically: as* $x \to \infty, f(x) \to \dfrac{1}{3}^+$

Problem 16 *Consider the function* $f(x) = \dfrac{3+7x}{5-2x}$

- *State the domain.* $(-\infty, \frac{5}{2}) \cup (\frac{5}{2}, \infty)$
- *Identify any vertical asymptotes of the graph.* $x = \dfrac{5}{2}$
- *Identify any holes in the graph.* *No holes in the graph*
- *Find the horizontal asymptote, if it exists.* $y = -\dfrac{7}{2}$
- *Find the slant asymptote, if it exists.* *No slant asymptote*
- *Graph the function using a graphing utility and describe the behavior near the asymptotes.*

  Vertical asymptote:

  $\lim\limits_{x\to \frac{5}{2}^-} f(x) = \infty$, $\lim\limits_{x\to \frac{5}{2}^+} f(x) = -\infty$

  Horizontal asymptote:

  $\lim\limits_{x\to -\infty} f(x) = -\dfrac{7}{2}$

  More specifically: as $x \to -\infty, f(x) \to -\dfrac{7}{2}^+$

  $\lim\limits_{x\to \infty} f(x) = -\dfrac{7}{2}$

  More specifically: as $x \to \infty, f(x) \to -\dfrac{7}{2}^-$

Problem 17 *Consider the function* $f(x) = \dfrac{x}{x^2+x-12}$

- *State the domain.* $(-\infty, -4) \cup (-4, 3) \cup (3, \infty)$
- *Identify any vertical asymptotes of the graph.* $x = -4, x = 3$
- *Identify any holes in the graph.* *No holes in the graph*
- *Find the horizontal asymptote, if it exists.* $y = 0$
- *Find the slant asymptote, if it exists.* *No slant asymptote*
- *Graph the function using a graphing utility and describe the behavior near the asymptotes.*

  Desmos link: `https://www.desmos.com/calculator/7jjwv7emh9`

  Vertical asymptotes:

  $\lim_{x\to-4^-} f(x) = -\infty$, $\lim_{x\to-4^+} f(x) = \infty$

  $\lim_{x\to3^-} f(x) = -\infty$, $\lim_{x\to3^+} f(x) = \infty$

  Horizontal asymptote:

  $\lim_{x\to-\infty} f(x) = 0$ *More specifically, as* $x \to -\infty, f(x) \to 0^-$

  $\lim_{x\to\infty} f(x) = 0$ *More specifically, as* $x \to \infty, f(x) \to 0^+$

Problem 18 *Consider the function* $g(t) = \dfrac{t}{t^2 + 1}$

- *State the domain.* $(-\infty, \infty)$
- *Identify any vertical asymptotes of the graph.* *No vertical asymptotes*
- *Identify any holes in the graph.* *No holes in the graph*
- *Find the horizontal asymptote, if it exists.* $y = 0$
- *Find the slant asymptote, if it exists.* *No slant asymptote*
- *Graph the function using a graphing utility and describe the behavior near the asymptotes.*

  Horizontal asymptote:

  $\lim_{t\to-\infty} g(t) = 0$ *More specifically, as* $t \to -\infty, g(t) \to 0^-$

  $\lim_{t\to\infty} g(t) = 0$ *More specifically, as* $t \to \infty, g(t) \to 0^+$

Problem 19 *Consider the function* $g(t) = \dfrac{t + 7}{(t + 3)^2}$

- *State the domain.* $(-\infty, -3) \cup (-3, \infty)$
- *Identify any vertical asymptotes of the graph.* $x = -3$
- *Identify any holes in the graph.* *No holes in the graph*
- *Find the horizontal asymptote, if it exists.* $y = 0$
- *Find the slant asymptote, if it exists.* *No slant asymptote*
- *Graph the function using a graphing utility and describe the behavior near the asymptotes.*

  Desmos link: `https://www.desmos.com/calculator/ixiribr6xw`

Vertical asymptote:

$\lim_{t \to -3} g(t) = \infty$

Horizontal asymptote:

$\lim_{t \to -\infty} g(t) = 0$ [2] *More specifically, as* $t \to -\infty, g(t) \to 0^-$

$\lim_{t \to \infty} g(t) = 0$

More specifically, as $t \to \infty, g(t) \to 0^+$

Problem 20 *Consider the function* $g(t) = \dfrac{t^3 + 1}{t^2 - 1}$

- *State the domain.* $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
- *Identify any vertical asymptotes of the graph.* $t = 1$
- *Identify any holes in the graph.* *Hole at* $(-1, -\dfrac{3}{2})$
- *Find the horizontal asymptote, if it exists.* *No horizontal asymptote*
- *Find the slant asymptote, if it exists.* $y = t$
- *Graph the function using a graphing utility and describe the behavior near the asymptotes.*

  Vertical asymptote:

  $\lim_{t \to 1^-} g(t) = -\infty$, $\lim_{t \to 1^+} g(t) = \infty$

  Slant asymptote:

  $\lim_{t \to -\infty} g(t) = -\infty$ *As* $t \to -\infty$*, the graph is below* $y = t$

  $\lim_{t \to \infty} g(t) = \infty$ *As* $t \to \infty$*, the graph is above* $y = t$

Problem 21 *Consider the function* $r(z) = \dfrac{4z}{z^2 + 4}$

- *State the domain.* $(-\infty, \infty)$
- *Identify any vertical asymptotes of the graph.* *No vertical asymptotes*
- *Identify any holes in the graph.* *No holes in the graph*
- *Find the horizontal asymptote, if it exists.* $y = 0$
- *Find the slant asymptote, if it exists.* *No slant asymptote*
- *Graph the function using a graphing utility and describe the behavior near the asymptotes.*

  Desmos link: `https://www.desmos.com/calculator/rbhqpfjbk2`

  Horizontal asymptote:

  $\lim_{z \to -\infty} r(z) = 0$ *More specifically, as* $z \to -\infty, r(z) \to 0^-$

  $\lim_{z \to \infty} r(z) = 0$ *More specifically, as* $z \to \infty, r(z) \to 0^+$

Problem 22 *Consider the function* $r(z) = \dfrac{4z}{z^2 - 4}$

[2] This is hard to see on the calculator, but trust me, the graph is below the t-axis to the left of $t = -7$.

- *State the domain.* $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$
- *Identify any vertical asymptotes of the graph.* $z=-2$, $z=2$
- *Identify any holes in the graph.* *No holes in the graph*
- *Find the horizontal asymptote, if it exists.* $y=0$
- *Find the slant asymptote, if it exists.* *No slant asymptote*
- *Graph the function using a graphing utility and describe the behavior near the asymptotes.*

  Vertical asymptotes:

  $\lim_{z\to-2^-} r(z)=-\infty,\ \lim_{z\to-2^+} r(z)=\infty$

  $\lim_{z\to2^-} r(z)=-\infty,\ \lim_{z\to2^+} r(z)=\infty$

  Horizontal asymptote:

  $\lim_{z\to-\infty} r(z)=0$ *More specifically, as* $z\to-\infty, r(z)\to 0^-$

  $\lim_{z\to\infty} r(z)=0$ *More specifically, as* $z\to\infty, r(z)\to 0^+$

Problem 23 *Consider the function* $r(z)=\dfrac{z^2-z-12}{z^2+z-6}$

- *State the domain.* $(-\infty,-3)\cup(-3,2)\cup(2,\infty)$
- *Identify any vertical asymptotes of the graph.* $z=2$
- *Identify any holes in the graph.*

  There is a hole in the graph at $(-3,\frac{7}{5})$
- *Find the horizontal asymptote, if it exists.* $y=1$
- *Find the slant asymptote, if it exists.* *No slant asymptote*
- *Graph the function using a graphing utility and describe the behavior near the asymptotes.*

  Desmos link: `https://www.desmos.com/calculator/z5wue2e7dc`

  Vertical asymptote:

  $\lim_{z\to2^-} r(z)=\infty,\ \lim_{z\to2^+} r(z)=-\infty$

  Horizontal asymptote:

  $\lim_{z\to-\infty} r(z)=1$ *More specifically, as* $z\to-\infty, r(z)\to 1^+$

  $\lim_{z\to\infty} r(z)=1$ *More specifically, as* $z\to\infty, r(z)\to 1^-$

Problem 24 *Consider the function* $f(x)=\dfrac{3x^2-5x-2}{x^2-9}$

- *State the domain.* $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$
- *Identify any vertical asymptotes of the graph.* $x=-3$, $x=3$
- *Identify any holes in the graph.* *No holes in the graph*
- *Find the horizontal asymptote, if it exists.* $y=3$

- *Find the slant asymptote, if it exists.* *No slant asymptote.*
- *Graph the function using a graphing utility and describe the behavior near the asymptotes.*

  Vertical asymptotes:

  $\lim_{x\to -3^-} f(x) = \infty$, $\lim_{x\to -3^+} f(x) = -\infty$

  $\lim_{x\to 3^-} f(x) = -\infty$, $\lim_{x\to 3^+} f(x) = \infty$

  Horizontal asymptote:

  $\lim_{x\to -\infty} f(x) = 3$ *More specifically, as* $x \to -\infty$, $f(x) \to 3^+$

  $\lim_{x\to \infty} f(x) = 3$ *More specifically, as* $x \to \infty$, $f(x) \to 3^-$

Problem 25 *Consider the function* $f(x) = \dfrac{x^3 + 2x^2 + x}{x^2 - x - 2}$

- *State the domain.* $(-\infty, -1) \cup (-1, 2) \cup (2, \infty)$
- *Identify any vertical asymptotes of the graph.* $x = 2$
- *Identify any holes in the graph.*

  There is a hole in the graph at $(-1, 0)$
- *Find the horizontal asymptote, if it exists.* *No horizontal asymptote*
- *Find the slant asymptote, if it exists.* $y = x + 3$
- *Graph the function using a graphing utility and describe the behavior near the asymptotes.*

  Desmos link: `https://www.desmos.com/calculator/wjqtjjrjvh`

  Vertical asymptote:

  $\lim_{x\to 2^-} f(x) = -\infty$, $\lim_{x\to 2^+} f(x) = \infty$

  Slant asymptote:

  $\lim_{x\to -\infty} f(x) = -\infty$ *As* $x \to -\infty$, *the graph is below* $y = x + 3$

  $\lim_{x\to \infty} f(x) = \infty$ *As* $x \to \infty$, *the graph is above* $y = x + 3$

Problem 26 *Consider the function* $f(x) = \dfrac{x^3 - 3x + 1}{x^2 + 1}$

- *State the domain.* $(-\infty, \infty)$
- *Identify any vertical asymptotes of the graph.* *No vertical asymptotes*
- *Identify any holes in the graph.* *No holes in the graph*
- *Find the horizontal asymptote, if it exists.* *No horizontal asymptote.*
- *Find the slant asymptote, if it exists.* $y = x$
- *Graph the function using a graphing utility and describe the behavior near the asymptotes.*

  Slant asymptote:

  $\lim_{x\to -\infty} f(x) = -\infty$ *As* $x \to -\infty$, *the graph is above* $y = x$

  $\lim_{x\to \infty} f(x) = \infty$ *As* $x \to \infty$, *the graph is below* $y = x$

Problem 27 *Consider the function* $g(t) = \dfrac{2t^2 + 5t - 3}{3t + 2}$

- *State the domain.* $\left(-\infty, -\dfrac{2}{3}\right) \cup \left(-\dfrac{2}{3}, \infty\right)$
- *Identify any vertical asymptotes of the graph.* $t = -\dfrac{2}{3}$
- *Identify any holes in the graph.* *No holes in the graph*
- *Find the horizontal asymptote, if it exists.* *No horizontal asymptote*
- *Find the slant asymptote, if it exists.* $y = \dfrac{2}{3}t + \dfrac{11}{9}$
- *Graph the function using a graphing utility and describe the behavior near the asymptotes.*

  Desmos link: `https://www.desmos.com/calculator/jzyukn8w8d`

  Vertical asymptote:

  $\lim_{t \to -\frac{2}{3}^-} g(t) = \infty$, $\lim_{t \to -\frac{2}{3}^+} g(t) = -\infty$

  Slant asymptote:

  $\lim_{t \to -\infty} g(t) = -\infty$ *As* $t \to -\infty$, *the graph is above* $y = \dfrac{2}{3}t + \dfrac{11}{9}$

  $\lim_{t \to \infty} g(t) = \infty$ *As* $t \to \infty$, *the graph is below* $y = \dfrac{2}{3}t + \dfrac{11}{9}$

Problem 28 *Consider the function* $g(t) = \dfrac{-t^3 + 4t}{t^2 - 9}$

- *State the domain.* $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
- *Identify any vertical asymptotes of the graph.* $t = -3$, $t = 3$
- *Identify any holes in the graph.* *No holes in the graph*
- *Find the horizontal asymptote, if it exists.* *No horizontal asymptote.*
- *Find the slant asymptote, if it exists.* $y = -t$
- *Graph the function using a graphing utility and describe the behavior near the asymptotes.*

  Vertical asymptotes:

  $\lim_{t \to -3^-} g(t) = \infty$, $\lim_{t \to -3^+} g(t) = -\infty$

  $\lim_{t \to 3^-} g(t) = \infty$, $\lim_{t \to 3^+} g(t) = -\infty$

  Slant asymptote:

  $\lim_{t \to -\infty} g(t) = \infty$ *As* $t \to -\infty$, *the graph is above* $y = -t$

  $\lim_{t \to \infty} g(t) = -\infty$ *As* $t \to \infty$, *the graph is below* $y = -t$

Problem 29 *Consider the function* $g(t) = \dfrac{-5t^4 - 3t^3 + t^2 - 10}{t^3 - 3t^2 + 3t - 1}$

- *State the domain.* $(-\infty, 1) \cup (1, \infty)$

- *Identify any vertical asymptotes of the graph.* $t = 1$
- *Identify any holes in the graph.* No holes in the graph
- *Find the horizontal asymptote, if it exists.* No horizontal asymptote
- *Find the slant asymptote, if it exists.* $y = -5t - 18$
- *Graph the function using a graphing utility and describe the behavior near the asymptotes.*

Desmos link: https://www.desmos.com/calculator/7uro9mqusv

Vertical asymptote:

$\lim_{t \to 1^-} g(t) = \infty, \lim_{t \to 1^+} g(t) = -\infty$

Slant asymptote:

$\lim_{t \to -\infty} g(t) = \infty$ As $t \to -\infty$, the graph is above $y = -5t - 18$

$\lim_{t \to \infty} g(t) = -\infty$ As $t \to \infty$, the graph is below $y = -5t - 18$

Problem 30 *Consider the function* $r(z) = \dfrac{z^3}{1 - z}$

- *State the domain.* $(-\infty, 1) \cup (1, \infty)$
- *Identify any vertical asymptotes of the graph.* $z = 1$
- *Identify any holes in the graph.* No holes in the graph.
- *Find the horizontal asymptote, if it exists.* No horizontal asymptote.
- *Find the slant asymptote, if it exists.* No slant asymptote.
- *Graph the function using a graphing utility and describe the behavior near the asymptotes.*

  Vertical asymptote:

  $\lim_{z \to 1^-} r(z) = \infty$

  $\lim_{z \to 1^+} r(z) = -\infty$

Problem 31 *Consider the function* $r(z) = \dfrac{18 - 2z^2}{z^2 - 9}$

- *State the domain.* $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
- *Identify any vertical asymptotes of the graph.* No vertical asymptotes
- *Identify any holes in the graph.* Holes in the graph at $(-3, -2)$ and $(3, -2)$
- *Find the horizontal asymptote, if it exists.* $y = -2$
- *Find the slant asymptote, if it exists.* No slant asymptote
- *Graph the function using a graphing utility and describe the behavior near the asymptotes.*

Desmos link: https://www.desmos.com/calculator/ybxxoog6c4

Horizontal asymptote:

$\lim_{z \to -\infty} r(z) = -2$ $\lim_{z \to \infty} r(z) = -2$

Problem 32 *Consider the function* $r(z) = \dfrac{z^3 - 4z^2 - 4z - 5}{z^2 + z + 1}$

- *State the domain.* $(-\infty, \infty)$
- *Identify any vertical asymptotes of the graph.* *No vertical asymptote.*
- *Identify any holes in the graph.* *No holes in the graph.*
- *Find the horizontal asymptote, if it exists.* *No horizontal asymptote.*
- *Find the slant asymptote, if it exists.* $y = z - 5$
- *Graph the function using a graphing utility and describe the behavior near the asymptotes.*

  Slant asymptote:

  $\lim\limits_{z \to -\infty} r(z) = -\infty$

  $\lim\limits_{z \to \infty} r(z) = \infty$

  $r(z) = z - 5$ *everywhere.*

Problem 33 *The cost* $C(p)$ *in dollars to remove* p% *of the invasive Ippizuti fish species from Sasquatch Pond is:*

$$C(p) = \frac{1770p}{100 - p}, \quad 0 \leq p < 100$$

1. *Find and interpret* $C(25)$ *and* $C(95)$.

   $C(25) = 590$ *means it costs \$590 to remove 25% of the fish and and* $C(95) = 33630$ *means it would cost \$33630 to remove 95% of the fish from the pond.*

2. *What does the vertical asymptote at* $x = 100$ *mean within the context of the problem?*

   The vertical asymptote at $x = 100$ *means that as we try to remove 100% of the fish from the pond, the cost increases without bound; i.e., it's impossible to remove all of the fish.*

3. *What percentage of the Ippizuti fish can you remove for \$40000?*

   For \$40000 you could remove about 95.76% of the fish.

Problem 34 *In the scenario of Example* **??**, $s(t) = -5t^2 + 100t$, $0 \leq t \leq 20$ *gives the height of a model rocket above the Moon's surface, in feet,* t *seconds after liftoff. For each of the times* t_0 *listed below, find and simplify a the formula for the average velocity* $\overline{v}(t)$ *between* t *and* t_0 *(see Definition* **??***) and use* $\overline{v}(t)$ *to find and interpret the instantaneous velocity of the rocket at* $t = t_0$ *(See Example* **??***).*

1. $t_0 = 5$

   $\overline{v}(t) = \dfrac{s(t) - s(5)}{t - 5} = \dfrac{-5t^2 + 100t - 375}{t - 5} = -5t + 75$, $t \neq 5$. *The instantaneous velocity of the rocket when* $t_0 = 5$ *is* $-5(5) + 75 = 50$ *meaning it is traveling* 50 *feet per second upwards.*

2. $t_0 = 9$

   $\overline{v}(t) = \dfrac{s(t) - s(9)}{t - 9} = \dfrac{-5t^2 + 100t - 495}{t - 9} = -5t + 55$, $t \neq 9$. *The instantaneous velocity of the rocket when* $t_0 = 9$ *is* $-5(9) + 55 = 10$*, so the rocket has slowed to* 10 *feet per second (but still heading up.)*

3. $t_0 = 10$

   $\overline{v}(t) = \dfrac{s(t) - s(10)}{t - 10} = \dfrac{-5t^2 + 100t - 495}{t - 10} = -5t + 50$, $t \neq 10$. *The instantaneous velocity of the rocket when* $t_0 = 10$ *is* $-5(10) + 50 = 0$*, so the rocket has momentarily stopped! In Example* **??***, we learned the rocket reaches its maximum height when* $t = 10$ *seconds, which means the rocket must change direction from heading up to coming back down, so it makes sense that for this instant, its velocity is* 0.

4. $t_0 = 11$
   $\overline{v}(t) = \dfrac{s(t) - s(11)}{t - 11} = \dfrac{-5t^2 + 100t - 495}{t - 11} = -5t + 45,\ t \neq 11.$ *The instantaneous velocity of the rocket when* $t_0 = 11$ *is* $-5(11) + 45 = -10$ *meaning the rocket has, indeed, changed direction and is heading downwards at a rate of* 10 *feet per second. (Note the symmetry here between this answer and our answer when* $t = 9$*.)*

Problem 35 *The population of Sasquatch in Portage County t years after the year 1803 is modeled by the function*

$$P(t) = \frac{150t}{t + 15}.$$

Find and interpret the horizontal asymptote of the graph of $y = P(t)$ *and explain what it means.*

The horizontal asymptote of the graph of $P(t) = \dfrac{150t}{t + 15}$ *is* $y = 150$ *and it means that the model predicts the population of Sasquatch in Portage County will never exceed 150.*

Problem 36 *The cost in dollars,* $C(x)$ *to make x dOpi media players is* $C(x) = 100x + 2000$, $x \geq 0$*. You may wish to review the concepts of fixed and variable costs introduced in Example* **??** *in Section* **??***.*

1. *Find a formula for the average cost* $\overline{C}(x)$*.*
   $\overline{C}(x) = \dfrac{100x + 2000}{x} = 100 + \dfrac{2000}{x},\ x > 0.$
2. *Find and interpret* $\overline{C}(1)$ *and* $\overline{C}(100)$*.*
   $\overline{C}(1) = 2100$ *and* $\overline{C}(100) = 120$*. When just* 1 *dOpi is produced, the cost per dOpi is* \$2100*, but when* 100 *dOpis are produced, the cost per dOpi is* \$120*.*
3. *How many dOpis need to be produced so that the average cost per dOpi is* \$200*?*
   $\overline{C}(x) = 200$ *when* $x = 20$*. So to get the cost per dOpi to* \$200*,* 20 *dOpis need to be produced.*
4. *Find and interpret* $\lim\limits_{x \to 0^+} \overline{C}(x)$*.*
   We find $\lim\limits_{x \to 0^+} \overline{C}(x) = \infty$*. This means that as fewer and fewer dOpis are produced, the cost per dOpi becomes unbounded. In this situation, there is a fixed cost of* \$2000 *(*$C(0) = 2000$*), we are trying to spread that* \$2000 *over fewer and fewer dOpis.*
5. *Interpret the behavior of* $\overline{C}(x)$ *as* $x \to \infty$*.*
   As $x \to \infty$, $\overline{C}(x) \to 100^+$*. This means that as more and more dOpis are produced, the cost per dOpi approaches* \$100*, but is always a little more than* \$100*. Since* \$100 *is the variable cost per dOpi (*$C(x) = \underline{100}x + 2000$*), it means that no matter how many dOpis are produced, the average cost per dOpi will always be a bit higher than the variable cost to produce a dOpi. As before, we can attribute this to the* \$2000 *fixed cost, which factors into the average cost per dOpi no matter how many dOpis are produced.*

Problem 37 *This exercise explores the relationships between fixed cost, variable cost, and average cost. The reader is encouraged to revisit Example* **??** *in Section* **??** *as needed. Suppose the cost in dollars* $C(x)$ *to make x items is given by* $C(x) = mx + b$ *where m and b are positive real numbers.*

1. *Show the fixed cost (the money spent even if no items are made) is b.*
   The cost to make 0 *items is* $C(0) = m(0) + b = b$*. Hence, so the fixed costs are b.*
2. *Show the variable cost (the increase in cost per item made) is m.*
   $C(x) = mx + b$ *is a linear function with slope* $m > 0$*. Hence, the cost increases at a rate of m dollars per item made. Hence, the variable cost is m.*

3. *Find a formula for the average cost when making x items, $\overline{C}(x)$.*

   $\overline{C}(x) = \dfrac{C(x)}{x} = \dfrac{mx+b}{x} = m + \dfrac{b}{x}$ *for* $x > 0$.

4. *Show $\overline{C}(x) > m$ for all $x > 0$ and, moreover, $\overline{C}(x) \to m^+$ as $x \to \infty$.*

   Since $b > 0$, $\overline{C}(x) = m + \dfrac{b}{x} > m$ *for* $x > 0$. *As* $x \to \infty$, $\dfrac{b}{x} \to 0$ *so* $\overline{C}(x) = m + \dfrac{b}{x} \to m$.

5. *Interpret $\overline{C}(x) \to m^+$ both geometrically and in terms of fixed, variable, and average costs.*

   Geometrically, the graph of $y = \overline{C}(x)$ has a horizontal asymptote $y = m$, the variable cost. In terms of costs, as more items are produced, the affect of the fixed cost on the average cost, $\dfrac{b}{x}$ falls away so that the average cost per item approaches the variable cost to make each item.

Problem 38 *Suppose the price-demand function for a particular product is given by $p(x) = mx + b$ where x is the number of items made and sold for $p(x)$ dollars. Here, $m < 0$ and $b > 0$. If the cost (in dollars) to make x of these products is also a linear function $C(x)$, show that the graph of the average profit function $\overline{P}(x)$ has a slant asymptote with slope m and interpret.*

If $p(x) = mx + b$ and $C(x)$ is linear, say $C(x) = rx + s$, then we can compute the the profit function (in general) as: $P(x) = xp(x) - C(x) = x(mx + b) - (rx + s)$ which simplifies to $P(x) = mx^2 + (b - r)x - s$. Hence, the average profit $\overline{P}(x) = \dfrac{P(x)}{x} = \dfrac{mx^2 + (b-r)x - s}{x} = mx + (b - r) - \dfrac{s}{x}$. We see that as $x \to \infty$, $\dfrac{s}{x} \to 0$ so $\overline{P}(x) \approx mx + (b - r)$. Hence, $y = mx + (b - r)$ is the slant asymptote to $y = \overline{P}(x)$. This means that as more items are sold, the average profit is decreasing at approximately the same rate as the price function is decreasing, m dollars per item. That is, to sell one additional item, we drop the price $p(x)$ by m dollars which results in a drop in the average profit by approximately m dollars.

Problem 39 *In Exercise **??** in Section **??**, we fit a few polynomial models to the following electric circuit data. The circuit was built with a variable resistor. For each of the following resistance values (measured in kilo-ohms, $k\Omega$), the corresponding power to the load (measured in milliwatts, mW) is given below.*[3]

Resistance: ($k\Omega$)	1.012	2.199	3.275	4.676	6.805	9.975
Power: (mW)	1.063	1.496	1.610	1.613	1.505	1.314

Using some fundamental laws of circuit analysis mixed with a healthy dose of algebra, we can derive the actual formula relating power $P(x)$ to resistance x:

$$P(x) = \frac{25x}{(x+3.9)^2}, \quad x \geq 0.$$

1. *Graph the data along with the function $y = P(x)$ using a graphing utility.*

2. *Use a graphing utility to approximate the maximum power that can be delivered to the load. What is the corresponding resistance value?*

   The maximum power is approximately 1.603 *mW which corresponds to* 3.9 *kΩ.*

3. *Find and interpret the end behavior of $P(x)$ as $x \to \infty$.*

   As $x \to \infty$, $P(x) \to 0^+$ which means as the resistance increases without bound, the power diminishes to zero.

Problem 40 *Let $f(x) = \dfrac{ax^2 - c}{x + 3}$. Find values for a and c so the graph of f has a hole at $(-3, 12)$.*

$a = -2$ *and* $c = -18$ *so* $f(x) = \dfrac{-2x^2 + 18}{x + 3}$.

[3]The authors wish to thank Don Anthan and Ken White of Lakeland Community College for devising this problem and generating the accompanying data set.

Problem 41 *Let* $f(x) = \dfrac{ax^n - 4}{2x^2 + 1}$.

1. *Find values for a and n so the graph of* $y = f(x)$ *has the horizontal asymptote* $y = 3$.

   $a = 6$ *and* $n = 2$ *so* $f(x) = \dfrac{6x^2 - 4}{2x^2 + 1}$

2. *Find values for a and n so the graph of* $y = f(x)$ *has the slant asymptote* $y = 5x$.

   $a = 10$ *and* $n = 3$ *so* $f(x) = \dfrac{10x^3 - 4}{2x^2 + 1}$.

Problem 42 *Suppose p is a polynomial function and a is a real number. Define* $r(x) = \dfrac{p(x) - p(a)}{x - a}$. *Use the Factor Theorem, Theorem* **??**, *to prove the graph of* $y = r(x)$ *has a hole at* $x = a$.

If we define $f(x) = p(x) - p(a)$ *then f is a polynomial function with* $f(a) = p(a) - p(a) = 0$. *The Factor Theorem guarantees* $(x - a)$ *is a factor of* $f(x)$, *that is,* $f(x) = p(x) - p(a) = (x - a)q(x)$ *for some polynomial* $q(x)$. *Hence,* $r(x) = \dfrac{p(x) - p(a)}{x - a} = \dfrac{(x - a)q(x)}{x - a} = q(x)$ *so the graph of* $y = r(x)$ *is the same as the graph of the polynomial* $y = q(x)$ *except for a hole when* $x = a$.

Problem 43 *For each function* $f(x)$ *listed below, compute the average rate of change over the indicated interval.*[4] *What trends do you observe? How do your answers manifest themselves graphically? How do you results compare with those of Exercise* **??** *in Section* **??**?

$f(x)$	$[0.9, 1.1]$	$[0.99, 1.01]$	$[0.999, 1.001]$	$[0.9999, 1.0001]$
x^{-1}				
x^{-2}				
x^{-3}				
x^{-4}				

The slope of the curves near $x = 1$ *matches the exponent on* x. *This exactly what we saw in Exercise* **??** *in Section* **??**.

$f(x)$	$[0.9, 1.1]$	$[0.99, 1.01]$	$[0.999, 1.001]$	$[0.9999, 1.0001]$
x^{-1}	-1.0101	-1.0001	≈ -1	≈ -1
x^{-2}	-2.0406	-2.0004	≈ -2	≈ -2
x^{-3}	-3.1021	-3.0010	≈ -3	≈ -3
x^{-4}	-4.2057	-4.0020	≈ -4	≈ -4

Problem 44 *In his now famous 1919 dissertation The Learning Curve Equation, Louis Leon Thurstone presents a rational function which models the number of words a person can type in four minutes as a function of the number of pages of practice one has completed.*[5] *Using his original notation and original language, we have* $Y = \dfrac{L(X + P)}{(X + P) + R}$ *where L is the predicted practice limit in terms of speed units, X is pages written, Y is writing speed in terms of words in four minutes, P is equivalent previous practice in terms of pages and R is the rate of learning. In Figure 5 of the paper, he graphs a scatter plot and the curve* $Y = \dfrac{216(X + 19)}{X + 148}$. *Discuss this equation with your classmates. How would you update the notation? Explain what the horizontal asymptote of the graph means. You should take some time to look at the original paper. Skip over the computations you don't understand yet and try to get a sense of the time and place in which the study was conducted.*

[4] See Definition **??** in Section **??** for a review of this concept, as needed.

[5] This paper, which is now in the public domain and can be found here, is from a bygone era when students at business schools took typing classes on manual typewriters.