Introduction to Rational Functions

If we add, subtract, or multiply polynomial functions, the result is another polynomial function. When we divide polynomial functions, however, we may not get a polynomial function. The result of dividing two polynomials is a **rational function**, so named because rational functions are **ratios** of polynomials.

Definition 1. A **rational function** is a function which is the ratio of polynomial functions. Said differently, r is a rational function if it is of the form

$$r(x) = \frac{p(x)}{q(x)},$$

where p and q are polynomial functions.[1]

1 Laurent Monomial Functions

As with polynomial functions, we begin our study of rational functions with what are, in some sense, the building blocks of rational functions, **Laurent monomial functions**.

Definition 2. A **Laurent monomial function** is either a monomial function (see Definition **??**) or a function of the form $f(x) = \frac{a}{x^n} = ax^{-n}$ for $n \in \mathbb{N}$.

Laurent monomial functions are named in honor of Pierre Alphonse Laurent and generalize the notion of 'monomial function' from Chapter **??** to terms with negative exponents. Our study of these functions begins with an analysis of $r(x) = \frac{1}{x} = x^{-1}$, the reciprocal function. The first item worth noting is that $r(0)$ is not defined owing to the presence of x in the denominator. That is, the domain of r is $\{x \in \mathbb{R} \mid x \neq 0\}$ or, using interval notation, $(-\infty, 0) \cup (0, \infty)$. Of course excluding 0 from the domain of r serves only to pique our curiosity about the behavior of $r(x)$ when $x \approx 0$. Thinking from a number sense perspective, the closer the denominator of $\frac{1}{x}$ is to 0, the larger the value of the fraction (in absolute value.)[2] So it stands to reason that as x gets closer and closer to 0, the values for $r(x) = \frac{1}{x}$ should grow larger and larger (in absolute value.) This is borne out in the table below on the left where it is apparent that for $x \approx 0$, $r(x)$ is becoming unbounded.

As we investigate the end behavior of r, we find that as $x \to -\infty$ and as $x \to \infty$, $r(x) \approx 0$. Again, number sense agrees here with the data, since as the denominator of $\frac{1}{x}$ becomes unbounded, the value of the fraction should diminish.[3] That being said, we could ask if the graph ever reaches the x-axis. If we attempt to solve $y = r(x) = \frac{1}{x} = 0$. we arrive at the contradiction $1 = 0$ hence, 0 is not in the range of r. Every other real number besides 0 is in the range of r, however. To see this, let $c \neq 0$ be a real number. Then $\frac{1}{c}$ is defined and, moreover, $r\left(\frac{1}{c}\right) = \frac{1}{(1/c)} = c$. This shows c is in the range of r. Hence, the range of r is $\{y \in \mathbb{R} \mid y \neq 0\}$ or, using interval notation, $(-\infty, 0) \cup (0, \infty)$.

[1] According to this definition, all polynomial functions are also rational functions. (Take $q(x) = 1$).

[2] Technically speaking, -1×10^{117} is a 'small' number (since it is very far to the left on the number line.) However, it's absolute value, 1×10^{117} is very large. If you read footnotes, you've seen this clarification before . . . (see Section **??**.)

[3] We'll talk more about this end behavior shortly - stay tuned!

x	$r(x) = \frac{1}{x}$
-0.01	-100
-0.001	-1000
-0.0001	-10000
-0.00001	-100000
0	undefined
0.00001	100000
0.0001	10000
0.001	1000
0.01	100

x	$r(x) = \frac{1}{x}$
-100000	-0.00001
-10000	-0.0001
-1000	-0.001
-100	-0.01
0	undefined
100	0.01
1000	0.001
10000	0.0001
100000	0.00001

GeoGebra link: `https://www.geogebra.org/m/tuaxyby6`

GeoGebra link: `https://www.geogebra.org/m/vfvtz6gs`

Like we did in Section **??**, we'll borrow some notation from Calculus in order for us to codify the behavior as $x \to 0$. First off, note that the behavior of r differs depending on which direction we approach 0. We describe the values $x < 0$ but $x \to 0$ (such as $x = -0.01, -0.001$, etc.) as 'x approaching 0 **from the left**,' written as $x \to 0^-$. If we think of these numbers as all being x-values where x = '0$-$ a little bit', the the '$-$' in the notation '$x \to 0^-$' makes better sense. For these values, the function values $r(x) \to -\infty$. Using the limit notation introduced in Section **??**, we'd write: $\lim\limits_{x\to 0^-} r(x) = -\infty$.

Similarly, we say 'as x approaches 0 **from the right**,' that is as $x \to 0^+$, $r(x) \to \infty$, or, more succinctly, $\lim\limits_{x\to 0^+} r(x) = \infty$. As before, we understand 'from the right' means we are using x values slightly to the **right** of 0 on the number line: numbers such as $x = 0.001$. These numbers could described as '0 + a little bit,' which justifies the '+' in the notation '$x \to 0^+$.'

We can also use this notation to describe the end behavior, but here the numerical roles are reversed. We see as $x \to -\infty$, $r(x) \to 0^-$ and as $x \to \infty$, $r(x) \to 0^+$. When it comes to codifying these results using Calculus, we write $\lim\limits_{x\to -\infty} r(x) = 0$ and $\lim\limits_{x\to\infty} r(x) = 0$. Note that, unfortunately, we lose the directionality here on the limiting value - that is, we do not write $\lim\limits_{x\to -\infty} r(x) = 0^-$ or $\lim\limits_{x\to\infty} r(x) = 0^+$. Without getting too much into formal definitions, the reason is that if limiting values are finite, we express them as real numbers.[4] Period.

The way we describe what is happening graphically is to say the line $x = 0$ is a **vertical asymptote** to the graph of $y = r(x)$ and the line $y = 0$ is a **horizontal asymptote** to the graph of $y = r(x)$. Roughly speaking, asymptotes are lines which approximate functions as either the inputs or outputs become unbounded.

Definition 3. The line $x = c$ is called a **vertical asymptote** of the graph of a function $y = f(x)$ if either of the limits $\lim\limits_{x\to c^-} f(x)$ or $\lim\limits_{x\to c^+} f(x)$ (or both) result in ∞ or $-\infty$.

Definition 4. The line $y = c$ is called a **horizontal asymptote** of the graph of a function $y = f(x)$ either $\lim\limits_{x\to -\infty} f(x) = c$ or $\lim\limits_{x\to\infty} f(x) = c$ (or both).

The behaviors illustrated in the graph $r(x) = \frac{1}{x}$ are typical of functions of the form $f(x) = \frac{1}{x^n} = x^{-n}$ for natural numbers, n. As with the monomial functions discussed in Section **??**, the patterns that develop primarily depend on whether n is odd or even. Having thoroughly discussed the graph of $y = \frac{1}{x} = x^{-1}$, we graph it along with $y = \frac{1}{x^3} = x^{-3}$ and $y = \frac{1}{x^5} = x^{-5}$ below. Note the points $(-1, -1)$ and $(1, 1)$ are common to all

[4] We will, of course, offer more detail if the situation presents itself.

three graphs as are the asymptotes $x = 0$ and $y = 0$. As the n increases, the graphs become steeper for $|x| < 1$ and flatten out more quickly for $|x| > 1$. Both the domain and range in each case appears to be $(-\infty, 0) \cup (0, \infty)$. Indeed, owing to the x in the denominator of $f(x) = \frac{1}{x^n}$, $f(0)$, and only $f(0)$, is undefined. Hence the domain is $(-\infty, 0) \cup (0, \infty)$. When thinking about the range, note the equation $f(x) = \frac{1}{x^n} = c$ has the solution $x = \sqrt[n]{\frac{1}{c}}$ as long as $c \neq 0$. Thus means $f\left(\sqrt[n]{\frac{1}{c}}\right) = c$ for every nonzero real number c. If $c = 0$, we are in the same situation as before: $\frac{1}{x^n} = 0$ has no real solution. This establishes the range is $(-\infty, 0) \cup (0, \infty)$. Finally, each of the graphs appear to be symmetric about the origin. Indeed, since n is odd, $f(-x) = (-x)^{-n} = (-1)^{-n}x^{-n} = -x^{-n} = -f(x)$, proving every member of this function family is odd.

x	$\frac{1}{x} = x^{-1}$	$\frac{1}{x^3} = x^{-3}$	$\frac{1}{x^5} = x^{-5}$
-10	-0.1	-0.001	-0.00001
-1	-1	-1	-1
-0.1	-10	-1000	-100000
0	undefined	undefined	undefined
0.1	10	1000	100000
1	1	1	1
10	0.1	0.001	0.00001

Desmos link: `https://www.desmos.com/calculator/qdhc5piwqf`

We repeat the same experiment with functions of the form $f(x) = \frac{1}{x^n} = x^{-n}$ where n is even. $y = \frac{1}{x^2} = x^{-2}$, $y = \frac{1}{x^4} = x^{-4}$ and $y = \frac{1}{x^6} = x^{-6}$. These graphs all share the points $(-1, 1)$ and $(1, 1)$, and asymptotes $x = 0$ and $y = 0$. Note here that both $\lim_{x \to 0^-} f(x) = \infty$ and $\lim_{x \to 0^+} f(x) = \infty$, so we may simply write $\lim_{x \to 0} f(x) = \infty$.

The same remarks about the steepness for $|x| < 1$ and the flattening for $|x| > 1$ also apply. For the same reasons as given above, the domain of each of these functions is $(-\infty, 0) \cup (0, \infty)$. When it comes to the range, the fact n is even tells us there are solutions to $\frac{1}{x^n} = c$ only if $c > 0$. It follows that the range is $(0, \infty)$ for each of these functions. Concerning symmetry, as n is even, $f(-x) = (-x)^{-n} = (-1)^{-n}x^{-n} = x^{-n} = f(x)$, proving each member of this function family is even. Hence, the graphs of these functions are symmetric about the y-axis.

x	$\frac{1}{x^2} = x^{-2}$	$\frac{1}{x^4} = x^{-4}$	$\frac{1}{x^6} = x^{-6}$
-10	0.01	0.0001	1×10^{-6}
-1	1	1	1
-0.1	100	10000	1×10^{6}
0	undefined	undefined	undefined
0.1	100	10000	1×10^{6}
1	1	1	1
10	0.01	0.0001	1×10^{-6}

Desmos link: `https://www.desmos.com/calculator/fbzv77wpjb`

Not surprisingly, we have an analog to Theorem **??** for this family of Laurent monomial functions.

Theorem 1. For real numbers a, h, and k with $a \neq 0$, the graph of $F(x) = \frac{a}{(x-h)^n} + k = a(x-h)^{-n} + k$ can be obtained from the graph of $f(x) = \frac{1}{x^n} = x^{-n}$ by performing the following operations, in sequence:

1. add h to each of the x-coordinates of the points on the graph of f. This results in a horizontal shift to the right if $h > 0$ or left if $h < 0$.

   NOTE: This transforms the graph of $y = x^{-n}$ to $y = (x - h)^{-n}$.

   The vertical asymptote moves from $x = 0$ to $x = h$.

2. multiply the y-coordinates of the points on the graph obtained in Step 1 by a. This results in a vertical scaling, but may also include a reflection about the x-axis if $a < 0$.

   NOTE: This transforms the graph of $y = (x - h)^{-n}$ to $y = a(x - h)^{-n}$.

3. add k to each of the y-coordinates of the points on the graph obtained in Step 2. This results in a vertical shift up if $k > 0$ or down if $k < 0$.

   NOTE: This transforms the graph of $y = a(x - h)^{-n}$ to $y = a(x - h)^{-n} + k$.

   The horizontal asymptote moves from $y = 0$ to $y = k$.

The proof of Theorem 1 is **identical** to the proof of Theorem **??** - just replace x^n with x^{-n}. We nevertheless encourage the reader to work through the details[5] and compare the results of this theorem with Theorems **??**, **??**, and **??**.

We put Theorem 1 to good use in the following example.

Example 1. Use Theorem 1 to graph the following. Label at least two points and the asymptotes. State the domain and range using interval notation.

1. $f(x) = (2x - 3)^{-2}$

2. $g(t) = \dfrac{2t - 1}{t + 1}$

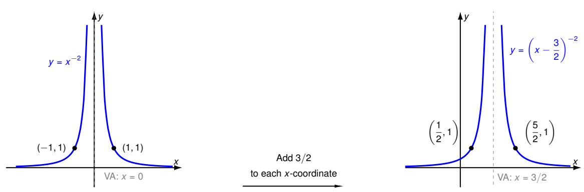
Solution.

1. In order to use Theorem 1, we first must put $f(x) = (2x - 3)^{-2}$ into the form prescribed by the theorem. To that end, we factor:

$$f(x) = \left(2\left[x - \frac{3}{2}\right]\right)^{-2} = 2^{-2}\left(x - \frac{3}{2}\right)^{-2} = \frac{1}{4}\left(x - \frac{3}{2}\right)^{-2}$$

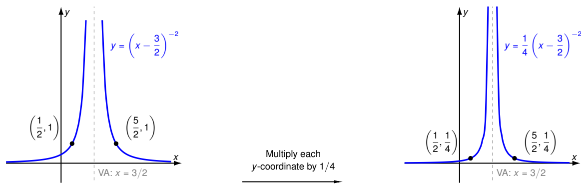
   We identify $n = 2$, $a = \frac{1}{4}$ and $h = \frac{3}{2}$ (and $k = 0$.) Per the theorem, we begin with the graph of $y = x^{-2}$ and track the two points $(-1, 1)$ and $(1, 1)$ along with the vertical and horizontal asymptotes $x = 0$ and $y = 0$, respectively through each step.

   Step 1: add $\frac{3}{2}$ to each of the x-coordinates of each of the points on the graph of $y = x^{-2}$. This moves the vertical asymptote from $x = 0$ to $x = \frac{3}{2}$ (which we represent by a dashed line.)



[5] We are, in fact, building to Theorem **??** in Section **??**, so the more you see the forest for the trees, the better off you'll be when the time comes to generalize these moves to all functions.

Step 2: multiply each of the y-coordinates of each of the points on the graph of $y = \left(x - \dfrac{3}{2}\right)^{-2}$ by $\dfrac{1}{4}$.



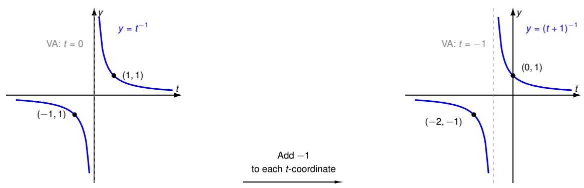
Since we did not shift the graph vertically, the horizontal asymptote remains $y = 0$. We can determine the domain and range of f by tracking the changes to the domain and range of our progenitor function, $y = x^{-2}$. We get the domain and range of f is $\left(-\infty, \dfrac{3}{2}\right) \cup \left(\dfrac{3}{2}, \infty\right)$ and the range of f is $(-\infty, 0) \cup (0, \infty)$.

2. Using either long or synthetic division, we get

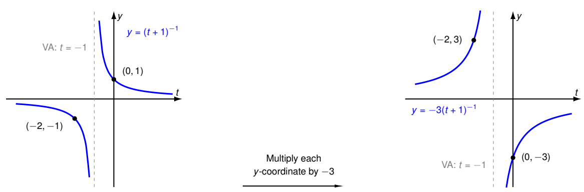
$$g(t) = \frac{2t - 1}{t + 1} = -\frac{3}{t + 1} + 2 = \frac{-3}{(t - (-1))^1} + 2$$

so we identify $n = 1$, $a = -3$, $h = -1$, and $k = 2$. We start with the graph of $y = \dfrac{1}{t}$ with points $(-1, -1)$, $(1, 1)$ and asymptotes $t = 0$ and $y = 0$ and track these through each of the steps.

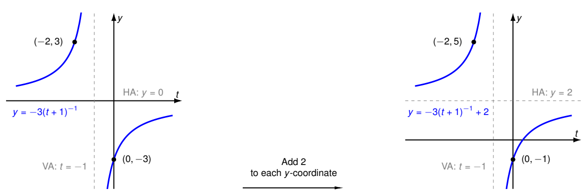
Step 1: Add -1 to each of the t-coordinates of each of the points on the graph of $y = \dfrac{1}{t}$. This moves the vertical asymptote from $t = 0$ to $t = -1$.



Step 2: multiply each of the y-coordinates of each of the points on the graph of $y = \dfrac{1}{t + 1}$ by -3.



Step 3: add 2 to each of the y-coordinates of each of the points on the graph of $y = \dfrac{-3}{t + 1}$. This moves the horizontal asymptote from $y = 0$ to $y = 2$.



As above, we determine the domain and range of g by tracking the changes in the domain and range of $y = \frac{1}{t}$. We find the domain of g is $(-\infty, -1) \cup (-1, \infty)$ and the range is $(-\infty, 2) \cup (2, \infty)$. ■

In Example 1, we once again see the benefit of changing the form of a function to make use of an important result. A natural question to ask is to what extent general rational functions can be rewritten to use Theorem 1. In the same way polynomial functions are sums of monomial functions, it turns out, allowing for non-real number coefficients, that every rational function can be written as a sum of (possibly shifted) Laurent monomial functions.[6]

2 Local Behavior near Excluded Values

We take time now to focus on behaviors of the graphs of rational functions near excluded values. We've already seen examples of one type of behavior: vertical asymptotes. Our next example gives us a physical interpretation of a vertical asymptote. This type of model arises from a family of equations cheerily named 'doomsday' equations.[7]

Example 2. A mathematical model for the population $P(t)$, in thousands, of a certain species of bacteria, t days after it is introduced to an environment is given by $P(t) = \frac{100}{(5-t)^2}$, $0 \leq t < 5$.

1. Find and interpret $P(0)$.
2. When will the population reach 100,000?
3. Graph $y = P(t)$.
4. Find and interpret $\lim\limits_{t \to 5^-} P(t)$.

Solution.

1. Substituting $t = 0$ gives $P(0) = \frac{100}{(5-0)^2} = 4$. Since t represents the number of days **after** the bacteria are introduced into the environment, $t = 0$ corresponds to the day the bacteria are introduced. Since $P(t)$ is measured in **thousands**, $P(t) = 4$ means 4000 bacteria are initially introduced into the environment.
2. To find when the population reaches 100,000, we first need to remember that $P(t)$ is measured in **thousands**. In other words, 100,000 bacteria corresponds to $P(t) = 100$. Hence, we need to solve $P(t) = \frac{100}{(5-t)^2} = 100$. Clearing denominators and dividing by 100 gives $(5-t)^2 = 1$, which, after extracting square roots, produces $t = 4$ or $t = 6$. Of these two solutions, only $t = 4$ in our domain, so this is the solution we keep. Hence, it takes 4 days for the population of bacteria to reach 100,000.
3. After a slight re-write, we have $P(t) = \frac{100}{(5-t)^2} = \frac{100}{[(-1)(t-5)]^2} = \frac{100}{(t-5)^2}$. Using Theorem 1, we start with the graph of $y = \frac{1}{t^2}$ below on the left. After shifting the graph to the right 5 units and stretching it vertically by a factor of 100 (note, the graphs are not to scale!), we restrict the domain to $0 \leq t < 5$ to arrive at the graph of $y = P(t)$ below on the right.
4. From the graph, we see as $t \to 5^-$, $P(t) \to \infty$, so $\lim\limits_{t \to 5^-} P(t) = \infty$. This means that the population of bacteria is increasing without bound as we near 5 days, which cannot actually happen. For this reason, $t = 5$ is called the 'doomsday' for this population. There is no way any environment can support infinitely many bacteria, so shortly before $t = 5$ the environment would collapse.

[6] i.e., Laurent 'Polynomials.' This result is a combination of Theorems **??** in Section **??** and Theorem **??** in Section **??**.

[7] These functions arise in Differential Equations. The unfortunate name will make sense shortly.

Desmos link: https://www.desmos.com/calculator/gjrtnqumns

■

Will all values excluded from the domain of a rational function produce vertical asymptotes in the graph? The short answer is 'no.' There are milder interruptions that can occur - holes in the graph - which we explore in our next example.

To this end, we formalize the notion of **average velocity** - a concept we first encountered in Example **??** in Section **??**. In that example, the function $s(t) = -5t^2 + 100t$, $0 \leq t \leq 20$ gives the height of a model rocket above the Moon's surface, in feet, t seconds after liftoff. The function s is an example of a **position function** since it provides information about **where** the rocket is at time t. In that example, we interpreted the average rate of change of s over an interval as the average velocity of the rocket over that interval. The average velocity provides two pieces of information: the average speed of the rocket along with the rocket's direction.

Suppose we have a position function s defined over an interval containing some fixed time t_0. We can define the average velocity as a function of any time t other than t_0:

Definition 5. Let $s(t)$ be the position of an object at time t and t_0 be a fixed time in the domain of s. The **average velocity** between time t and time t_0 for $t \neq t_0$ is given by

$$\overline{v}(t) = \frac{\Delta[s(t)]}{\Delta t} = \frac{s(t) - s(t_0)}{t - t_0}.$$

We must exclude $t = t_0$ from the domain of $\overline{v}$ in Definition **??** since, otherwise, we would have a 0 in the denominator. What is interesting in this case however, is that substituting $t = t_0$ also produces 0 in the **numerator**. (Do you see why?) While '$\frac{0}{0}$' is undefined, it is more precisely called an 'indeterminate form' and is studied extensively in Calculus. We explore this phenomenon in the next example.

Example 3. Let $s(t) = -5t^2 + 100t$, $0 \leq t \leq 20$ give the height of a model rocket above the Moon's surface, in feet, t seconds after liftoff.

1. Find and simplify an expression for the average velocity of the rocket between times t and 15, $\overline{v}(t)$.
2. Find and interpret $\overline{v}(14)$.
3. Find and interpret $\lim\limits_{t \to 15} \overline{v}(t)$.
4. Graph $y = \overline{v}(t)$. Interpret your answer to part **??** graphically.

Solution.

1. Using Definition **??** with $t_0 = 15$, we get:

$$\begin{aligned}
\overline{v}(t) &= \frac{s(t) - s(15)}{t - 15}, \qquad t \neq 15 \\
&= \frac{(-5t^2 + 100t) - 375}{t - 15} \\
&= \frac{-5(t^2 - 20t + 75)}{t - 15} \\
&= \frac{-5(t - 15)(t - 5)}{t - 15} \\
&= \frac{-5\cancel{(t - 15)}(t - 5)}{\cancel{(t - 15)}} \\
&= -5(t - 5) = -5t + 25, \quad t \neq 15
\end{aligned}$$

Since the domain of s is $0 \leq t \leq 20$, our final answer is $\overline{v}(t) = -5t + 25$, for $t \in [0, 15) \cup (15, 20]$.

2. We find $\overline{v}(14) = -5(14) + 25 = -45$. This means that between 14 and 15 seconds after launch, the rocket was traveling, on average, a speed 45 feet per second **downwards**, or falling back to the Moon's surface.

3. To find $\lim\limits_{t \to 15} \overline{v}(t)$, we need to analyze the outputs from $\overline{v}(t)$ as $t \to 15$. Using the table below on the left, it certainly **appears** to be the case that $\lim\limits_{t \to 15} \overline{v}(t) = -50$. Of course, we could be choosing the number -50 because of a bias towards nice, integer answers.[8]

   However, we can argue a stronger case algebraically using the simplified formula $\overline{v}(t) = -5t + 25$. As $t \to 15$, $-5t \to -5(15) = -75$ so $-5t + 25 \to -50$. It stands to reason, then, that $\lim\limits_{t \to 15} \overline{v}(t) = -50$.

   This means our average velocity approaches -50 feet per second as we sample times closer and closer to $t = 15$ seconds after liftoff. Since we're pushing the t-values to $t = 15$, instead of viewing the limit -50 as an **average** velocity calculated **between** two time values, we view -50 feet per second as the **instantaneous velocity** of the rocket **at** $t = 15$. That is, **at** $t = 15$ seconds after liftoff, the rocket **is traveling** downwards at a rate of 50 feet per second.[9]

4. From part **??**, we know $\overline{v}(t) = -5t + 25$, for $t \in [0, 15) \cup (15, 20]$ Hence the graph of $\overline{v}(t)$ is a portion of the line $y = -5t + 25$. Since $\overline{v}(t)$ is not defined when $t = 15$, our graph is the line segment starting at $(0, 25)$ and ending at $(20, 75)$ which skips over the point $(15, -50)$, creating a hole in the graph.[10]

t	$\overline{v}(t)$
14.9	-49.5
14.99	-49.95
14.999	-49.995
15	undefined
15.001	-50.005
15.01	-50.05
15.1	-50.5

Desmos link: `https://www.desmos.com/calculator/9v2iaznyny`

■

Some notes about Example **??** are in order. First, excluded values from the domain of a rational function don't necessarily cause vertical asymptotes in the graph. Even though $\overline{v}(15)$ doesn't exist, the fact that $\lim\limits_{t \to 15} \overline{v}(t) = -50$ means that we **expect** $\overline{v}(15)$ to be -50.This sentiment is exactly what a hole in the graph at $(15, -50)$ communicates.

Second, in finding $\lim\limits_{t \to 15} \overline{v}(t) = -50$, we've taken some (more) steps into Calculus. Specifically, we used properties of the limit process that we've not yet formalized, let alone justified. The main idea is that the '$\frac{0}{0}$' indeterminate form which occurs when we attempt to evaluate $\overline{v}(15)$ using the formula $\overline{v}(t) = \dfrac{-5t^2 + 100t - 375}{t - 15}$ is resolved when the factor $(t - 15)$ cancels from the denominator. We can algebraically reason what to expect out of the expression $\overline{v}(t) = -5t + 25$ as $t \to 15$ because there is no longer any division by 0.

We will revisit these sorts of machinations later in the text in a bit more generality.[11] For now, we'll work to build some intuition with some classic hand-waving which we hope will do more good than harm.

Our next theorem generalizes our reasoning from this last example.

[8] Or maybe the limit is actually -49.999999865 which, owing to calculation limitations of the graphing utility rounds to -50.
[9] It is worth noting that Exercise **??** from Section **??** has you arrive at this limit via a table.
[10] We've seen and discussed such holes in the graph as far back as Example **??** in Section **??** ...
[11] See Section **??**.

Theorem 2. Location of Vertical Asymptotes and Holes:[12] Suppose r is a rational function and c is not in the domain of r.

- If $\lim_{x \to c} r(x) = L$ where L is a real number, then the graph of r has a hole at (c, L).
- Otherwise, the graph of $y = r(x)$ has a vertical asymptote $x = c$.

Of course the first question to ask is how do we know if $\lim_{x \to c} r(x)$ results in a real number, L, and if so, how do we find L? It turns out that if the limit exists, then the same sort of algebraic cancellation which occurred Example **??** is guaranteed to happen.

Let's consider a generic rational function $r(x) = \dfrac{p(x)}{q(x)}$ where p and q are polynomial functions. The values 'c' excluded from the domain of r are the zeros of q: $q(c) = 0$. We now have two cases to consider.

If $p(c) \neq 0$, then as $x \to c$, $r(x) = \dfrac{p(x)}{q(x)} \to \dfrac{p(c), \text{ a nonzero number}}{0}$ which results in unbounded behavior (graphically, a vertical asymptote.)

If $p(c) = 0$, then as $x \to c$, $r(x) = \dfrac{p(x)}{q(x)} \to \dfrac{0}{0}$, an indeterminate form. The Factor Theorem,[13] guarantees both $p(x)$ and $q(x)$ contain factors of $(x - c)$. This means we can simplify the expression $r(x)$ by cancelling common factors of $(x - c)$. If all of the factors of $(x - c)$ in the denominator, $q(x)$, cancel with factors in the numerator, $p(x)$, then the division by 0 is eliminated and we can proceed as in Example **??** to determine the limit.[14] If some factors of $(x - c)$ remain in the denominator, then we're back to the first scenario and the graph will have a vertical asymptote.

We practice this methodology in the following example.

Example 4. For each function below:

- determine the values excluded from the domain.
- determine whether each excluded value corresponds to a vertical asymptote or hole in the graph.
- verify your answers using a graphing utility.
- describe the behavior of the graph near each excluded value using proper notation.
- investigate any apparent symmetry of the graph about the y-axis or origin.

1. $f(x) = \dfrac{2x}{x^2 - 3}$
2. $g(t) = \dfrac{t^2 - t - 6}{t^2 - 9}$
3. $h(t) = \dfrac{t^2 - t - 6}{t^2 + 9}$
4. $r(t) = \dfrac{t^2 - t - 6}{t^2 + 4t + 4}$

Solution.

[12] Or, 'How to tell your asymptote from a hole in the graph.'
[13] Theorem **??** in Section **??**
[14] Using the vocabulary from Section **??**, the limit will exist if the multiplicity of c as a zero for p is greater than or equal to the multiplicity of c as a zero for q.

1. We begin by finding the values excluded from the domain by setting the denominator equal to 0. Solving $x^2 - 3 = 0$, we get $x = \pm\sqrt{3}$ which factors the denominator as $(x - \sqrt{3})(x + \sqrt{3})$. Since $f(x)$ is in lowest terms (can you see why?), no cancellation occurs, hence, the lines $x = -\sqrt{3}$ and $x = \sqrt{3}$ are vertical asymptotes to the graph of $y = f(x)$.

   A graphing utility verifies this claim, and from the graph, we see that $\lim\limits_{x\to-\sqrt{3}^-} f(x) = -\infty$, $\lim\limits_{x\to-\sqrt{3}^+} f(x) = \infty$, $\lim\limits_{x\to\sqrt{3}^-} f(x) = -\infty$, and $\lim\limits_{x\to\sqrt{3}^+} f(x) = \infty$.

   As a side note, the graph of f appears to be symmetric about the origin. Sure enough, we find: $f(-x) = \dfrac{2(-x)}{(-x)^2 - 3} = -\dfrac{2x}{x^2 - 3} = -f(x)$, proving f is odd.

   Desmos link: `https://www.desmos.com/calculator/df6apyni8k`

2. As above, we find the values excluded from the domain of g finding the zeros of the denominator. Solving $t^2 - 9 = 0$ gives $t = \pm 3$. In this case, we can simplify the formula for $g(t)$: $\dfrac{t^2 - t - 6}{t^2 - 9} = \dfrac{(t-3)(t+2)}{(t-3)(t+3)} = \dfrac{\cancel{(t-3)}(t+2)}{\cancel{(t-3)}(t+3)} = \dfrac{t+2}{t+3}$. Hence, $g(t) = \dfrac{t+2}{t+3}$ provided $t \neq 3$.

   Since the factor $(t + 3)$, which corresponds to the zero $t = -3$, did not cancel from the denominator of $g(t)$, we expect a vertical asymptote to the graph at $t = -3$. On the other hand, as $t \to 3$, $t + 2 \to 5$ and $t + 3 \to 6$ so $\dfrac{t+2}{t+3} \to \dfrac{3+2}{3+3} = \dfrac{5}{6}$. This gives $\lim\limits_{t\to3} g(t) = \dfrac{5}{6}$. Hence we have a hole in the graph of $y = g(t)$ at $\left(3, \dfrac{5}{6}\right)$.

   Graphing g we can definitely see the vertical asymptote $t = -3$: as $\lim\limits_{t\to-3^-} g(t) = \infty$ and $\lim\limits_{t\to-3^+} g(t) = -\infty$. Near $t = 3$, the graph seems to have no interruptions, but we know g is undefined at $t = 3$. Depending on the graphing utility, we may or may not convince the display to show us the hole at $\left(3, \dfrac{5}{6}\right)$.

   Desmos link: `https://www.desmos.com/calculator/rpkble6dtp`

3. Setting the denominator of the expression for $h(t)$ to 0 gives $t^2 + 9 = 0$, which has no real solutions. Accordingly, the graph of $y = h(t)$ (at least as much as we can discern from the technology) is devoid of both vertical asymptotes and holes. Using terms defined in Section **??**, the function h is both continuous and smooth.[15]

   Desmos link: `https://www.desmos.com/calculator/azhqh1fq88`

4. Setting the denominator of $r(t)$ to zero gives the equation $t^2 + 4t + 4 = 0$. We get the (repeated!) solution $t = -2$. Simplifying, we get $\dfrac{t^2 - t - 6}{t^2 + 4t + 4} = \dfrac{(t-3)(t+2)}{(t+2)(t+2)} = \dfrac{(t-3)\cancel{(t+2)}}{(t+2)\cancel{(t+2)}} = \dfrac{t-3}{t+2}$. Since not all factors of $(t + 2)$ cancelled from the denominator, $t = -2$ continues to produce a 0 in the denominator. Hence $t = -2$ is a vertical asymptote to the graph. A graphing utility bears this out. Specifically, $\lim\limits_{t\to-2^-} r(t) = \infty$ and $\lim\limits_{t\to-2^+} r(t) = -\infty$.

   Desmos link: `https://www.desmos.com/calculator/zfbk9xwhqp`

■

[15] We'll remind you more about continuous functions in Section **??** . . .

3 End Behavior

Now that we've discussed behavior near values excluded from the domains of rational functions, let's focus our attention on end behavior. We have already seen one example of this in the form of horizontal asymptotes. Our next example of the section gives us a real-world application of a horizontal asymptote.[16]

Example 5. The number of students $N(t)$ at local college who have had the flu t months after the semester begins can be modeled by:

$$N(t) = \frac{1500t + 50}{3t + 1}, \quad t \geq 0.$$

1. Find and interpret $N(0)$.
2. How long will it take until 300 students will have had the flu?
3. Use Theorem 1 to graph $y = N(t)$.
4. Find and interpret $\lim\limits_{t \to \infty} N(t)$.

Solution.

1. Substituting $t = 0$ gives $N(0) = \dfrac{1500(0) + 50}{1 + 3(0)} = 50$. Since t represents the number of months since the beginning of the semester, $t = 0$ describes the state of the flu outbreak at the beginning of the semester. Hence, at the beginning of the semester, 50 students have had the flu.
2. We set $N(t) = \dfrac{1500t + 50}{3t + 1} = 300$ and solve. Clearing denominators gives $1500t + 50 = 300(3t + 1)$ from which we get $t = \dfrac{5}{12}$. This means it will take $\dfrac{5}{12}$ months, or about 13 days, for 300 students to have had the flu.
3. To graph $y = N(t)$, we first use long division to rewrite $N(t) = \dfrac{-450}{3t + 1} + 500$. From there, we get

$$N(t) = -\frac{450}{3t + 1} + 500 = \frac{-450}{3\left(t + \frac{1}{3}\right)} + 500 = \frac{-150}{t + \frac{1}{3}} + 500$$

   Using Theorem 1, we start with the graph of $y = \dfrac{1}{t}$ below on the left and perform the following steps: shift the graph to the left by $\dfrac{1}{3}$ units, stretch the graph vertically by a factor of 150, reflect the graph across the t-axis, and finally, shift the graph up 500 units. As the domain of N is $t \geq 0$, we obtain the graph below on the right.

   Desmos link: `https://www.desmos.com/calculator/oceht0nz7b`

4. Owing to the horizontal asymptote, $y = 500$, we have $\lim\limits_{t \to \infty} N(t) = 500$. (More specifically, as $t \to \infty$, $N(t) \to 500^{-}$.) This means as time goes by, only a total of 500 students will have ever had the flu. ∎

We determined the horizontal asymptote to the graph of $y = N(t)$ in Example **??** by rewriting $N(t)$ into a form compatible with Theorem 1, and while there is nothing wrong with this approach, it will simply not work for general rational functions which cannot be rewritten this way. To that end, we revisit this problem using Theorem **??** from Section **??**. The end behavior of the numerator of $N(t) = \dfrac{1500t + 50}{3t + 1}$ is determined by its

[16] Though the population below is more accurately modeled with the functions in Chapter **??**, we approximate it (using Calculus, of course!) using a rational function.

leading term, $1500t$, and the end behavior of the denominator is likewise determined by its leading term, $3t$. Hence, as $t \to \infty$:

$$N(t) = \frac{1500t + 50}{3t + 1} \approx \frac{1500t}{3t} = 500.$$

Hence $\lim\limits_{t\to\infty} N(t) = 500$ so $y = 500$ is the horizontal asymptote. This same reasoning can be used in general to argue the following theorem.

Theorem 3. Location of Horizontal Asymptotes: Suppose r is a rational function and $r(x) = \dfrac{p(x)}{q(x)}$, where p and q are polynomial functions with leading coefficients a and b, respectively.

- If the degree of $p(x)$ is the same as the degree of $q(x)$, then $\lim\limits_{x\to\infty} r(x) = \dfrac{a}{b}$ so $y = \dfrac{a}{b}$ is the[17] horizontal asymptote of the graph of $y = r(x)$.
- If the degree of $p(x)$ is less than the degree of $q(x)$, then $\lim\limits_{x\to\infty} r(x) = 0$ so $y = 0$ (the x-axis) is the horizontal asymptote of the graph of $y = r(x)$.
- If the degree of $p(x)$ is greater than the degree of $q(x)$, then the graph of $y = r(x)$ has no horizontal asymptotes.

So see why Theorem **??** works, suppose $r(x) = \dfrac{p(x)}{q(x)}$ where a is the leading coefficient of $p(x)$ and b is the leading coefficient of $q(x)$. As $x \to -\infty$ or $x \to \infty$, Theorem **??** gives $r(x) \approx \dfrac{ax^n}{bx^m}$, where n and m are the degrees of $p(x)$ and $q(x)$, respectively.

If the degree of $p(x)$ and the degree of $q(x)$ are the same, then $n = m$ so that $r(x) \approx \dfrac{ax^n}{bx^n} = \dfrac{a}{b}$. Hence $\lim\limits_{x\to-\infty} r(x) = \dfrac{a}{b}$ and $\lim\limits_{x\to\infty} r(x) = \dfrac{a}{b}$ which means $y = \dfrac{a}{b}$ is the horizontal asymptote in this case.

If the degree of $p(x)$ is less than the degree of $q(x)$, then $n < m$, so $m - n$ is a positive number, and hence, $r(x) \approx \dfrac{ax^n}{bx^m} = \dfrac{a}{bx^{m-n}} \to 0$. As $x \to -\infty$ or $x \to \infty$, $r(x)$ is more or less a fraction with a constant numerator, a, but a denominator which is unbounded. Hence, $\lim\limits_{x\to-\infty} r(x) = 0$ and $\lim\limits_{x\to\infty} r(x) = 0$ producing the horizontal asymptote $y = 0$.

If the degree of $p(x)$ is greater than the degree of $q(x)$, then $n > m$, and hence $n - m$ is a positive number and $r(x) \approx \dfrac{ax^n}{bx^m} = \dfrac{ax^{n-m}}{b}$, which is a monomial function from Section **??**. As such, r becomes unbounded as $x \to -\infty$ or $x \to \infty$.

Note that in the two cases which produce horizontal asymptotes, the behavior of r is identical as $x \to -\infty$ and $x \to \infty$. Hence, if the graph of a rational function has a horizontal asymptote, there is only one.[18]

We put Theorem **??** to good use in the following example.

Example 6. For each function below:

- use Theorem **??** to analytically determine the horizontal asymptotes to the graph, if any.
- check your answers using Theorem **??** and a graphing utility.
- describe the end behavior of the graph using proper notation.
- investigate any apparent symmetry of the graph about the y-axis or origin.

[17] The use of the definite article will be justified momentarily.

[18] We will (first) encounter functions with more than one horizontal asymptote in Chapter **??**.

1. $F(s) = \dfrac{5s}{s^2+1}$
2. $g(x) = \dfrac{x^2-4}{x+1}$
3. $h(t) = \dfrac{6t^3-3t+1}{5-2t^3}$
4. $r(x) = 2 - \dfrac{3x^2}{1-x^2}$

Solution.

1. Using Theorem **??**, we get as $s \to -\infty$ or $s \to \infty$, $F(s) = \dfrac{5s}{s^2+1} \approx \dfrac{5s}{s^2} = \dfrac{5}{s}$. Hence, $\lim\limits_{s\to-\infty} F(s) = 0$ and $\lim\limits_{s\to\infty} F(s) = 0$ so $y = 0$ is a horizontal asymptote to the graph.

   Alternatively, to use Theorem **??** note the degree of the numerator of $F(s)$, 1, is less than the degree of the denominator, 2, so $y = 0$ as the horizontal asymptote using this approach as well.

   Graphically, as $s \to -\infty$ or $s \to \infty$, the graph $y = F(s)$ approaches the s-axis ($y = 0$). More specifically, as $s \to -\infty$, $F(s) \to 0^-$ and as $s \to \infty$, $F(s) \to 0^+$.

   As a side note, the graph of F appears to be symmetric about the origin. Indeed, $F(-s) = \dfrac{5(-s)}{(-s)^2+1} = -\dfrac{5s}{s^2+1}$ proving F is odd.

   Desmos link: `https://www.desmos.com/calculator/xymlnyklim`

2. As $x \to -\infty$ or $x \to \infty$, $g(x) = \dfrac{x^2-4}{x+1} \approx \dfrac{x^2}{x} = x$, and while $y = x$ is a line, it is not a horizontal line. Hence, we conclude the graph of $y = g(x)$ has no horizontal asymptotes. Sure enough, Theorem **??** supports this since the degree of the numerator of $g(x)$ is 2 which is greater than the degree of the denominator, 1. From the graph, we see that the graph of $y = g(x)$ doesn't appear to level off to a constant value, confirming there is no horizontal asymptote.[19]

   Desmos link: `https://www.desmos.com/calculator/rnoyel4hsv`

3. As $t \to -\infty$ or $t \to \infty$, $h(t) = \dfrac{6t^3-3t+1}{5-2t^3} \approx \dfrac{6t^3}{-2t^3} = -3$. Hence, $\lim\limits_{t\to-\infty} h(t) = -3$ and $\lim\limits_{t\to\infty} h(t) = -3$, indicating a horizontal asymptote $y = -3$. Sure enough, since the degrees of the numerator and denominator of $h(t)$ are both three, Theorem **??** tells us $y = \dfrac{6}{-2} = -3$ is the horizontal asymptote. We see from the graph of $y = h(t)$ that as $t \to -\infty$, $h(t) \to -3^+$, and as $t \to \infty$, $h(t) \to -3^-$.

   Desmos link: `https://www.desmos.com/calculator/l0cy41qgxa`

4. If we apply Theorem **??** to the term $\dfrac{3x^2}{1-x^2}$ in the expression for $r(x)$, we find $\dfrac{3x^2}{1-x^2} \approx \dfrac{3x^2}{-x^2} = -3$ as $x \to -\infty$ or $x \to \infty$. It seems reasonable to conclude, then, that $\lim\limits_{x\to-\infty} r(x) = 2 - (-3) = 5$ and likewise $\lim\limits_{x\to\infty} r(x) = 2 - (-3) = 5$ so $y = 5$ is our horizontal asymptote.

   In order to double check this calculation using Theorem **??**, however, we need to rewrite the expression $r(x)$ with a single denominator: $r(x) = 2 - \dfrac{3x^2}{1-x^2} = \dfrac{2(1-x^2)-3x^2}{1-x^2} = \dfrac{2-5x^2}{1-x^2}$. Now we apply

[19] Sit tight! We'll revisit this function and its end behavior shortly.

Theorem **??** and note since the numerator and denominator have the same degree, we are guaranteed the horizontal asymptote is $y = \frac{-5}{-1} = 5$.

Both calculations are borne out graphically below where it appears as if as $x \to -\infty$ or $x \to \infty$, $r(x) \to 5^{+}$.

As a final note, the graph of r appears to be symmetric about the y axis. We find $r(-x) = 2 - \frac{3(-x)^2}{1-(-x)^2} = 2 - \frac{3x^2}{1-x^2} = r(x)$, proving r is even.

Desmos link: https://www.desmos.com/calculator/6xi4rywk3p

■

We close this section with a discussion of the **third** (and final!) kind of asymptote which can be associated with the graphs of rational functions. Let us return to the function $g(x) = \frac{x^2-4}{x+1}$ in Example **??**. Performing long division,[20] we get $g(x) = \frac{x^2-4}{x+1} = x - 1 - \frac{3}{x+1}$. Since the term $\frac{3}{x+1} \to 0$ as $x \to -\infty$ and as $x \to \infty$, it stands to reason that as x becomes unbounded, the function values $g(x) = x - 1 - \frac{3}{x+1} \approx x - 1$. Geometrically, this means that the graph of $y = g(x)$ should resemble the line $y = x - 1$ as $x \to -\infty$ and $x \to \infty$. We see this play out both numerically and graphically below. (As usual, the asymptote $y = x - 1$ is denoted by a dashed line.)

x	$g(x)$	$x-1$
-10	≈ -10.6667	-11
-100	≈ -100.9697	-101
-1000	≈ -1000.9970	-1001
-10000	≈ -10000.9997	-10001

x	$g(x)$	$x-1$
10	≈ 8.7273	9
100	≈ 98.9703	99
1000	≈ 998.9970	999
10000	≈ 9998.9997	9999

Desmos link: https://www.desmos.com/calculator/zyeycofx4r

The way we symbolize the relationship between the end behavior of $y = g(x)$ with that of the line $y = x - 1$ is to write 'as $x \to -\infty$ and $x \to \infty$, $g(x) \to x - 1$' in order to have some notational consistency with what we have done earlier in this section when it comes to end behavior.[21] In this case, we say the line $y = x - 1$ is a **slant asymptote**[22] to the graph of $y = g(x)$. Informally, the graph of a rational function has a slant asymptote if, as $x \to -\infty$ or as $x \to \infty$, the graph resembles a non-horizontal, or 'slanted' line. More formally, we define a slant asymptote as follows.

Definition 6. The line $y = mx + b$ where $m \neq 0$ is called a **slant asymptote** of the graph of a function $y = f(x)$ if as $x \to -\infty$ or as $x \to \infty$, $f(x) \to mx + b$.

A few remarks are in order. First, note that the stipulation $m \neq 0$ in Definition **??** is what makes the 'slant' asymptote 'slanted' as opposed to the case when $m = 0$ in which case we'd have a horizontal asymptote.

Secondly, while we have motivated what me mean intuitively by the notation '$f(x) \to mx + b$,' like so many ideas in this section, the formal definition requires Calculus. Another way to express this sentiment, however, is to rephrase '$f(x) \to mx + b$' as '$[f(x) - (mx + b)] \to 0$.' In other words, the graph of $y = f(x)$ has the **slant** asymptote $y = mx + b$ if and only if the graph of $y = f(x) - (mx + b)$ has a **horizontal** asymptote $y = 0$. This last sentiment can be encoded using limit notation as follows.

[20] See the remarks following Theorem **??**.
[21] Other notations include $g(x) \asymp x - 1$ or $g(x) \sim x - 1$.
[22] Also called an 'oblique' asymptote in some, ostensibly higher class (and more expensive), texts.

Definition 7. The line $y = mx + b$ where $m \neq 0$ is called a **slant asymptote** of the graph of a function $y = f(x)$ if either $\lim_{x\to-\infty}[f(x) - (mx + b)] = 0$ or $\lim_{x\to\infty}[f(x) - (mx + b)] = 0$ (or both!).

Our next task is to determine the conditions under which the graph of a rational function has a slant asymptote, and if it does, how to find it. In the case of $g(x) = \dfrac{x^2 - 4}{x + 1}$, the degree of the numerator $x^2 - 4$ is 2, which is **exactly one more** than the degree if its denominator $x + 1$ which is 1. This results in a **linear** quotient polynomial, and it is this quotient polynomial which is the slant asymptote. Generalizing this situation gives us the following theorem.[23]

Theorem 4. Determination of Slant Asymptotes: Suppose r is a rational function and $r(x) = \dfrac{p(x)}{q(x)}$, where the degree of p is **exactly** one more than the degree of q. Then the graph of $y = r(x)$ has the slant asymptote $y = L(x)$ where $L(x)$ is the quotient obtained by dividing $p(x)$ by $q(x)$.

In the same way that Theorem **??** gives us an easy way to see if the graph of a rational function $r(x) = \dfrac{p(x)}{q(x)}$ has a horizontal asymptote by comparing the degrees of the numerator and denominator, Theorem **??** gives us an easy way to check for slant asymptotes. Unlike Theorem **??**, which gives us a quick way to **find** the horizontal asymptotes (if any exist), Theorem **??** gives us no such 'short-cut'. If a slant asymptote exists, we have no recourse but to use long division to find it.[24]

Example 7. For each of the following functions:

- find the slant asymptote, if it exists.
- verify your answer using a graphing utility.
- investigate any apparent symmetry of the graph about the y-axis or origin.

1. $f(x) = \dfrac{x^2 - 4x + 2}{1 - x}$
2. $g(t) = \dfrac{t^2 - 4}{t - 2}$
3. $h(x) = \dfrac{x^3 + 1}{x^2 - 4}$
4. $r(t) = 2t - 1 + \dfrac{4t^3}{1 - t^2}$

Solution.

1. The degree of the numerator is 2 and the degree of the denominator is 1, so Theorem **??** guarantees us a slant asymptote. To find it, we divide $1 - x = -x + 1$ into $x^2 - 4x + 2$ and get a quotient of $-x + 3$, so our slant asymptote is $y = -x + 3$. We confirm this graphically below.

Desmos link: `https://www.desmos.com/calculator/5guwnabrtb`

[23] Once again, this theorem is brought to you courtesy of Theorem **??** and Calculus.
[24] That's OK, though. In the next section, we'll use long division to analyze end behavior and it's worth the effort!

2. As with the previous example, the degree of the numerator $g(t) = \dfrac{t^2-4}{t-2}$ is 2 and the degree of the denominator is 1, so Theorem **??** applies. In this case,

$$g(t) = \frac{t^2-4}{t-2} = \frac{(t+2)(t-2)}{(t-2)} = \frac{(t+2)\cancel{(t-2)}}{\cancel{(t-2)}^{\,1}} = t+2, \quad t \neq 2$$

so we have that the slant asymptote $y = t+2$ is identical to the graph of $y = g(t)$ except at $t = 2$ (where the latter has a 'hole' at $(2,4)$.) While the word 'asymptote' has the connotation of 'approaching but not equaling,' Definitions 4 and **??** allow for these extreme cases.

Desmos link: `https://www.desmos.com/calculator/fan3rfclao`

3. For $h(x) = \dfrac{x^3+1}{x^2-4}$, the degree of the numerator is 3 and the degree of the denominator is 2 so again, we are guaranteed the existence of a slant asymptote. The long division $\left(x^3+1\right) \div \left(x^2-4\right)$ gives a quotient of just x, so our slant asymptote is the line $y = x$. The graphing utility confirms this.

Note the graph of h appears to be symmetric about the origin. We check $h(-x) = \dfrac{(-x)^3+1}{(-x)^2-4} = \dfrac{-x^3+1}{x^2-4} = -\dfrac{x^3-1}{x^2-4}$. However, $-h(x) = -\dfrac{x^3+1}{x^2-4}$, so it appears as if $h(-x) \neq -h(x)$ for all x. Checking $x = 1$, we find $h(1) = -\dfrac{2}{3}$ but $h(-1) = 0$ which shows the graph of h, is in fact, **not** symmetric about the origin.

Desmos link: `https://www.desmos.com/calculator/8d7qp8aqbz`

4. For our last example, $r(t) = 2t - 1 + \dfrac{4t^3}{1-t^2}$, the expression $r(t)$ is not in the form to apply Theorem **??** directly. We can, nevertheless, appeal to the spirit of the theorem and use long division to rewrite the term $\dfrac{4t^3}{1-t^2} = -4t + \dfrac{4t}{1-t^2}$. We then get:

$$\begin{aligned} r(t) &= 2t - 1 + \frac{4t^3}{1-t^2} \\ &= 2t - 1 - 4t + \frac{4t}{1-t^2} \\ &= -2t - 1 + \frac{4t}{1-t^2} \end{aligned}$$

As $t \to -\infty$ or $x \to \infty$, Theorem **??** gives $\dfrac{4t}{1-t^2} \approx \dfrac{4t}{-t^2} = -\dfrac{4}{t} \to 0$. Hence, as $t \to -\infty$ or $t \to \infty$, $r(t) \to -2t-1$, so $y = -2t-1$ is the slant asymptote to the graph as confirmed by the graphing utility below. From a distance, the graph of r appears to be symmetric about the origin. However, if we look carefully, we see the y-intercept is $(0,-1)$, as borne out by the computation $r(0) = -1$. Hence r cannot be odd.[25]

Desmos link: `https://www.desmos.com/calculator/ba1wjcg3yx`

■

Our last example gives a real-world application of a slant asymptote. The problem features the concept of **average profit**. The average profit, denoted $\overline{P}(x)$, is the total profit, $P(x)$, divided by the number of items sold, x. In English, the average profit tells us the profit made per item sold. It, along with average cost, is defined below.

[25] Do you see why?

Definition 8. Let $C(x)$ and $P(x)$ represent the cost and profit to make and sell x items, respectively.

- The **average cost**, $\overline{C}(x) = \dfrac{C(x)}{x}$, $x > 0$.

  NOTE: The average cost is the cost per item produced.

- The **average profit**, $\overline{P}(x) = \dfrac{P(x)}{x}$, $x > 0$.

  NOTE: The average profit is the profit per item sold.

You'll explore average cost (and its relation to variable cost) in Exercise **??**. For now, we refer the reader to to Example **??** in Section **??**.

Example 8. Recall the profit (in dollars) when x PortaBoy game systems are produced and sold is given by $P(x) = -1.5x^2 + 170x - 150$, $0 \le x \le 166$.

1. Find and simplify an expression for the average profit, $\overline{P}(x)$. What is the domain of $\overline{P}$?
2. Find and interpret $\overline{P}(50)$.
3. Determine the slant asymptote to the graph of $y = \overline{P}(x)$. Check your answer using a graphing utility.
4. Interpret the slope of the slant asymptote.

Solution.

1. We find $\overline{P}(x) = \dfrac{P(x)}{x} = \dfrac{-1.5x^2 + 170x - 150}{x} = -1.5x + 170 - \dfrac{150}{x}$. Since the domain of P is $[0, 166]$ but $x \neq 0$, the domain of $\overline{P}$ is $(0, 166]$.
2. We find $\overline{P}(50) = -1.5(50) + 170 - \dfrac{150}{50} = 92$. This means that when 50 PortaBoy systems are sold, the average profit is \$92 per system.
3. Technically, the graph of $y = \overline{P}(x)$ has no slant asymptote since the domain of the function is restricted to $(0, 166]$. That being said, if we were to let $x \to \infty$, the term $\dfrac{150}{x} \to 0$, so we'd have $\overline{P}(x) \to -1.5x + 170$. This means the slant asymptote would be $y = -1.5x + 170$. We graph $y = \overline{P}(x)$ and $y = -1.5x + 170$.

   Desmos link: `https://www.desmos.com/calculator/hnnzvyr0yf`

4. The slope of the slant asymptote $y = -1.5x + 170$ is -1.5. Allowing for $x \to \infty$, $\overline{P}(x) \approx -1.5x + 170$ which means as we sell more systems, the average profit is **decreasing** at about a rate of \$1.50 per system. If the number 1.5 sounds familiar to this problem situation, it should. In Example **??** in Section **??**, we determined the slope of the demand function to be -1.5. In that situation, the -1.5 meant that in order to sell an additional system, the price had to drop by \$1.50. The fact the average profit is decreasing at more or less this same rate means the loss in profit per system can be attributed to the reduction in price needed to sell each additional system.[26] ■

[26] We generalize this result in Exercise **??**.

4 Exercises