Rational Inequalities

In this section, we solve equations and inequalities involving rational functions and explore associated application problems. Our first example showcases the critical difference in procedure between solving equations and inequalities.

Example 1.

1. Solve $\dfrac{x^3-2x+1}{x-1} = \dfrac{1}{2}x - 1$.
2. Solve $\dfrac{x^3-2x+1}{x-1} \geq \dfrac{1}{2}x - 1$.
3. Verify your solutions using a graphing utility.

Solution.

1. To solve the equation, we clear denominators

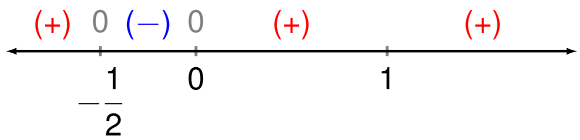
$$\begin{array}{rcll}
\dfrac{x^3-2x+1}{x-1} & = & \dfrac{1}{2}x-1 & \\
\left(\dfrac{x^3-2x+1}{x-1}\right)\cdot 2(x-1) & = & \left(\dfrac{1}{2}x-1\right)\cdot 2(x-1) & \\
2x^3-4x+2 & = & x^2-3x+2 & \text{expand} \\
2x^3-x^2-x & = & 0 & \\
x(2x+1)(x-1) & = & 0 & \text{factor} \\
x & = & -\dfrac{1}{2},\ 0,\ 1 &
\end{array}$$

Since we cleared denominators, we need to check for extraneous solutions. Sure enough, we see that $x = 1$ does not satisfy the original equation, so our only solutions are $x = -\frac{1}{2}$ and $x = 0$.

2. To solve the inequality, it may be tempting to begin as we did with the equation — namely by multiplying both sides by the quantity $(x-1)$. The problem is that, depending on x, $(x-1)$ may be positive (which doesn't affect the inequality) or $(x-1)$ could be negative (which would reverse the inequality). Instead of working by cases, we collect all of the terms on one side of the inequality with 0 on the other and make a sign diagram using the technique given on page **??** in Section **??**.

$$\begin{array}{rcll}
\dfrac{x^3-2x+1}{x-1} & \geq & \dfrac{1}{2}x-1 & \\
\dfrac{x^3-2x+1}{x-1}-\dfrac{1}{2}x+1 & \geq & 0 & \\
\dfrac{2\left(x^3-2x+1\right)}{2(x-1)}-\dfrac{x(x-1)}{2(x-1)}+\dfrac{2(x-1)}{2(x-1)} & \geq & 0 & \text{get a common denominator} \\
\dfrac{2\left(x^3-2x+1\right)-x(x-1)+2(x-1)}{2(x-1)} & \geq & 0 & \\
\dfrac{2x^3-x^2-x}{2x-2} & \geq & 0 & \text{expand}
\end{array}$$

Viewing the left hand side as a rational function $r(x)$ we make a sign diagram. The only value excluded from the domain of r is $x = 1$ which is the solution to $2x - 2 = 0$. The zeros of r are the solutions to $2x^3 - x^2 - x = 0$, which we have already found to be $x = 0$, $x = -\frac{1}{2}$ and $x = 1$, the latter was discounted as a zero because it is not in the domain. Choosing test values in each test interval, we construct the sign diagram below.



We are interested in where $r(x) \geq 0$. We see $r(x) > 0$, or $(+)$, on the intervals $\left(-\infty, -\frac{1}{2}\right)$, $(0,1)$ and $(1, \infty)$. We know $r(x) = 0$ when $x = -\frac{1}{2}$ and $x = 0$. Hence, $r(x) \geq 0$ on $\left(-\infty, -\frac{1}{2}\right] \cup [0,1) \cup (1, \infty)$.

3. To check our answers graphically, let $f(x) = \frac{x^3 - 2x + 1}{x - 1}$ and $g(x) = \frac{1}{2}x - 1$. The solutions to $f(x) = g(x)$ are the x-coordinates of the points where the graphs of $y = f(x)$ and $y = g(x)$ intersect. We graph both f and g using Desmos below. We find only two intersection points, $(-0.5, -1.25)$ and $(0, -1)$ which correspond to our solutions $x = -\frac{1}{2}$ and $x = 0$. The solution to $f(x) \geq g(x)$ represents not only where the graphs meet, but the intervals over which the graph of $y = f(x)$ is above $(>)$ the graph of $g(x)$. From the graph, this happens on $\left(-\infty, -\frac{1}{2}\right] \cup [0,1) \cup (1, \infty)$, where we are careful to skip over $x = 1$ since $f(1)$ is not defined.[1]

Desmos link: `https://www.desmos.com/calculator/qrwiqyv2bk`

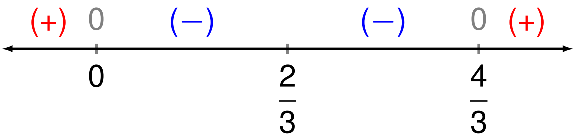
The important take-away from Example **??** is not to clear fractions when working with an inequality unless you know for certain the sign of the denominators. We offer another example.

Example 2. Solve: $2t(3t-2)^{-1} \leq 3t^2(3t-2)^{-2}$. Check your answer using a graphing utility.

Solution. We begin by rewriting the terms with negative exponents as fractions and gathering all nonzero terms to one side of the inequality:

$$\begin{aligned}
2t(3t-2)^{-1} &\leq 3t^2(3t-2)^{-2} \\
\frac{2t}{3t-2} &\leq \frac{3t^2}{(3t-2)^2} \\
\frac{2t}{3t-2} - \frac{3t^2}{(3t-2)^2} &\leq 0 \\
\frac{2t(3t-2)}{(3t-2)^2} - \frac{3t^2}{(3t-2)^2} &\leq 0 && \text{get a common denominator} \\
\frac{2t(3t-2) - 3t^2}{(3t-2)^2} &\leq 0 \\
\frac{3t^2 - 4t}{(3t-2)^2} &\leq 0 && \text{expand}
\end{aligned}$$

We define $r(t) = \frac{3t^2 - 4t}{(3t-2)^2}$ and set about constructing a sign diagram for r. Solving $(3t-2)^2 = 0$ gives $t = \frac{2}{3}$, our sole excluded value. To find the zeros of r, we set $r(t) = \frac{3t^2 - 4t}{(3t-2)^2} = 0$ and solve $3t^2 - 4t = 0$. Factoring gives $t(3t-4) = 0$ so our solutions are $t = 0$ and $t = \frac{4}{3}$. After choosing test values, we get the sign diagram below.



Since we are looking for where $r(t) \leq 0$, we are looking for where $r(t)$ is $(-)$ or $r(t) = 0$. Hence, our final answer is $\left[0, \frac{2}{3}\right) \cup \left(\frac{2}{3}, \frac{4}{3}\right]$.

[1] We invite the reader to show the graph of $y = f(x)$ has a hole at $(1, 1)$.

To check, we use Desmos to graph $f(t) = 2t(3t-1)^{-1}$, $g(t) = 3t^2(3t-2)^{-2}$. Sure enough, the graph of f intersects the graph of g when $t = 0$ and $t = \frac{4}{3}$. Moreover, the graph of f is below the graph of g everywhere they are defined between these values, in accordance with our algebraic solution.

Desmos link: `https://www.desmos.com/calculator/mngh4jcrvl`

One thing to note about Example **??** is that the quantity $(3t-2)^2 \geq 0$ for all values of t. Hence, as long as we remember $t = \frac{2}{3}$ is excluded from consideration, we could actually multiply both sides of the inequality in Example **??** by $(3t-2)^2$ to obtain $2t(3t-2) \leq 3t^2$. We could then solve this (slightly easier) inequality using the methods of Section **??** as long as we remember to exclude $t = \frac{2}{3}$ from our solution. Once again, the more you *understand*, the less you have to *memorize*. If you know the '*why*' behind an algorithm instead of just the '*how*,' you will know when you can short-cut it.

Our next example is an application of average cost. Recall from Definition **??** if $C(x)$ represents the cost to make x items then the average cost per item is given by $\overline{C}(x) = \frac{C(x)}{x}$, for $x > 0$.

Example 3. Recall from Example **??** that the cost, $C(x)$, in dollars, to produce x PortaBoy game systems for a local retailer is $C(x) = 80x + 150$, $x \geq 0$.

1. Find an expression for the average cost function, $\overline{C}(x)$.
2. Solve $\overline{C}(x) < 100$ and interpret.
3. Find and interpret $\lim\limits_{x\to\infty} \overline{C}(x)$.

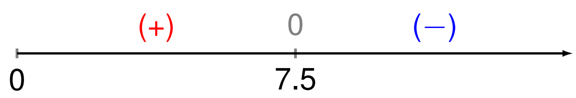
Solution.

1. From $\overline{C}(x) = \frac{C(x)}{x}$, we obtain $\overline{C}(x) = \frac{80x+150}{x}$. The domain of C is $x \geq 0$, but since $x = 0$ causes problems for $\overline{C}(x)$, we get our domain to be $x > 0$, or $(0, \infty)$.

2. Solving $\overline{C}(x) < 100$ means we solve $\frac{80x+150}{x} < 100$. We proceed as in the previous example.

$$\begin{array}{rcll} \dfrac{80x+150}{x} & < & 100 & \\ \dfrac{80x+150}{x} - 100 & < & 0 & \\ \dfrac{80x+150-100x}{x} & < & 0 & \text{common denominator} \\ \dfrac{150-20x}{x} & < & 0 & \end{array}$$

   If we take the left hand side to be a rational function $r(x)$, we need to keep in mind that the applied domain of the problem is $x > 0$. This means we consider only the positive half of the number line for our sign diagram. On $(0, \infty)$, r is defined everywhere so we need only look for zeros of r. Setting $r(x) = 0$ gives $150 - 20x = 0$, so that $x = \frac{15}{2} = 7.5$. The test intervals on our domain are $(0, 7.5)$ and $(7.5, \infty)$. We find $r(x) < 0$ on $(7.5, \infty)$.



In the context of the problem, x represents the number of PortaBoy games systems produced and $\overline{C}(x)$ is the average cost to produce each system. Solving $\overline{C}(x) < 100$ means we are trying to find how many systems we need to produce so that the average cost is less than \$100 per system. Our solution, $(7.5, \infty)$ tells us that we need to produce more than 7.5 systems to achieve this. Since it doesn't make sense to produce half a system, our final answer is $[8, \infty)$.

3. To find $\lim_{x\to\infty} \overline{C}(x)$, we note that we can rewrite $\overline{C}(x) = \frac{80x + 150}{x} = 80 + \frac{150}{x}$. As $x \to \infty$, note that $\frac{150}{x} \to 0$ so $\lim_{x\to\infty} \overline{C}(x) = 80 + 0 = 80$. Thus the average cost per system is getting closer to \$80 per system. Since $\frac{150}{x} > 0$ for all $x > 0$, we have that $\overline{C}(x) > 80$ for all $x > 0$. This means that the average cost per system is always greater than \$80 per system, but the average cost is approaching this amount as more and more systems are produced. Looking back at Example **??**, we realize \$80 is the variable cost per system $-$ the cost per system above and beyond the fixed initial cost of \$150. Another way to interpret our answer is that 'infinitely' many systems would need to be produced to effectively 'zero out' the fixed cost. ■

Note that number **??** in Example **??** is another opportunity to short-cut the standard algorithm and obtain the solution more quickly if we take stock of the situation. Since the applied domain is $x > 0$, we can multiply through the inequality $\frac{80x + 150}{x} < 100$ by x without worrying about changing the sense of the inequality. This reduces the problem to $80x + 150 < 100x$, a basic linear inequality whose solution is readily seen to be $x > 7.5$. It is absolutely critical here that $x > 0$. Indeed, any time you decide to multiply an inequality by a variable expression, it is necessary to justify why the inequality is preserved. Our next example is another classic 'box with no top' problem. The reader is encouraged to compare and contrast this problem with Example **??** in Section **??**.

Example 4. A box with a square base and no top is to be constructed so that it has a volume of 1000 cubic centimeters. Let x denote the width of the box, in centimeters as seen in the interactive below. Feel free to move the slider for the width, x, and observe the base of the box remains a square while the height changes to maintain a volume of 1000 cubic centimeters.

GeoGebra link: `https://www.geogebra.org/m/zv4shrqn`

1. Explain why the height of the box (in centimeters) is a function of the width x. Call this function h and find an expression for $h(x)$, complete with an appropriate applied domain.
2. Solve $h(x) \geq x$ and interpret.
3. Find and interpret $\lim_{x\to 0^+} h(x)$ and $\lim_{x\to\infty} h(x)$.
4. Express the surface area of the box as a function of x, $S(x)$ and state the applied domain.
5. Use a graphing utility to approximate (to two decimal places) the dimensions of the box which minimize the surface area.

Solution.

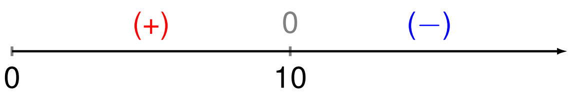
1. We are told that the volume of the box is 1000 cubic centimeters and that x represents the width, in centimeters. Since x represents a physical dimension of a box, we have that $x > 0$. From geometry, we know volume = width $\times$ height $\times$ depth. Since the base of the box is a square, the width and the depth are both x centimeters. Hence, $1000 = x^2$(height). Solving for the height, we get height $= \frac{1000}{x^2}$. In other words, for each width $x > 0$, we are able to compute **the**[2] corresponding height using the formula $\frac{1000}{x^2}$. Hence, the height is a function of x. Using function notation, we write $h(x) = \frac{1000}{x^2}$. As mentioned before, our only restriction is $x > 0$ so the domain of h is $(0, \infty)$.

[2]that is, the one and only one

2. To solve $h(x) \geq x$, we proceed as before and collect all nonzero terms on one side of the inequality in order to use a sign diagram.

$$\begin{array}{rcll} h(x) & \geq & x & \\ \dfrac{1000}{x^2} & \geq & x & \\ \dfrac{1000}{x^2} - x & \geq & 0 & \\ \dfrac{1000 - x^3}{x^2} & \geq & 0 & \text{common denominator} \end{array}$$

We consider the left hand side of the inequality as our rational function $r(x)$. We see immediately the only value excluded from the domain of r is 0, but since our applied domain is $x > 0$, we restrict our attention to the interval $(0, \infty)$. The sole zero of r comes when $1000 - x^3 = 0$, or when $x = 10$. Choosing test values in the intervals $(0, 10)$ and $(10, \infty)$ gives the following:



We see $r(x) > 0$ on $(0, 10)$, and since $r(x) = 0$ at $x = 10$, our solution is $(0, 10]$. In the context of the problem, $h(x)$ represents the height of the box while x represents the width (and depth) of the box. Solving $h(x) \geq x$ is tantamount to finding the values of x which result in a box where the height is at least as big as the width (and, in this case, depth.) Our answer tells us the width of the box can be at most 10 centimeters for this to happen.[3]

On the interactive below, as we adjust the slider for the width, x, we can see the box is taller than it is wide precisely when $0 < x \leq 10$.

GeoGebra link: `https://www.geogebra.org/m/qdacpjpp`

3. Since $h(x) = \dfrac{1000}{x^2}$, $\lim\limits_{x \to 0^+} h(x) = \infty$. This means that the smaller the width x (and, in this case, depth), the larger the height h has to be in order to maintain a volume of 1000 cubic centimeters. On the other hand, $\lim\limits_{x \to \infty} h(x) = 0$, which means that in order to maintain a volume of 1000 cubic centimeters, the width and depth must get bigger as the height becomes smaller.

4. Since the box has no top, the surface area can be found by adding the area of each of the sides to the area of the base. The base is a square of dimensions x by x, and each side has dimensions x by $h(x)$. We get the surface area, $S(x) = x^2 + 4xh(x)$. Since $h(x) = \dfrac{1000}{x^2}$, we have $S(x) = x^2 + 4x\left(\dfrac{1000}{x^2}\right) = x^2 + \dfrac{4000}{x}$. The domain of S is the same as h, namely $(0, \infty)$, for the same reasons as above.

5. Graphing $y = S(x)$ using Desmos, we find a local minimum when $x \approx 12.60$. As far as we can tell,[4] this is the only local extremum, so it is the (absolute) minimum as well. This means that the width and depth of the box should each measure approximately 12.60 centimeters. To determine the height, we find $h(12.60) \approx 6.30$, so the height of the box should be approximately 6.30 centimeters.[5]

On the GeoGebra interactive below, we have the graph of $y = S(x)$ along with two diagrams of the box. The top diagram has the box 'flattened' out as a square base with four attached rectangular tabs. This is one way to visualize the surface area of the box being the sum of the area of the (square) base plus

[3] As with the previous example, knowing $x > 0$ means $x^2 > 0$ so we can clear denominators right away and solve $x^3 \leq 1000$, or $x \leq 10$. Coupled with our applied domain, $x > 0$, we would arrive at the same solution, $(0, 10]$.

[4] without Calculus, that is...

[5] The y-coordinate here, 476.22 means the minimum surface area possible is 476.22 square centimeters. Minimizing the surface area minimizes the material required to make the box, therein helping to reduce the cost of the box.

the area of the four (rectangular) sides. The other diagram is the box as we first visualized it, with the four tabs being folded up to create the sides of the box. We invite the reader to adjust the slider for the width of the base of the box, x, to explore the relationships between all three diagrams dynamically.

GeoGebra link: https://www.geogebra.org/m/hwtdhxss

■

Our last example uses regression to verify a very famous scientific law.

Example 5. Boyle's Law states that when temperature is held constant, the pressure of a gas is inversely proportional to the volume of the gas.[6] According to this website the actual data relating the volume V of a gas and its pressure P used by Boyle and his assistant in 1662 to formulate this law is given below. (NOTE: both pressure and volume here are given in 'arbitrary units.')

V	48	46	44	42	40	38	36	34	32	30	28	26	24
P	29.13	30.56	31.94	33.5	35.31	37	39.31	41.63	44.19	47.06	50.31	54.31	58.81

V	23	22	21	20	19	18	17	16	15	14	13	12
P	61.31	64.06	67.06	70.69	74.13	77.88	82.75	87.88	93.06	100.44	107.81	117.56

1. Assuming P and V are inversely proportional, estimate the constant of proportionality, k.

2. Use a graphing utility to fit a curve of the form $P = \dfrac{k}{V}$ to these data.

Solution.

1. Recall if P and V are inversely proportional, there is a real number k so $PV = k$ for all values of P and V. Multiplying the corresponding P and V values from the data together result in numbers which are consistently approximately 1400. This gives us confidence in the claim P and V are inversely proportional and suggests $k \approx 1400$.

2. We plot the pairs (V, P) and run a regression, the results of which are below. To our amazement, the graphing utility, in this case Desmos, reports $k \approx 1406.9$ with $R^2 \approx 1$. This means the data are a very good fit to the model $P = \dfrac{k}{V}$, or $PV = k$, hence verifying Boyle's Law for this set of data.

Desmos link: https://www.desmos.com/calculator/ezeh4v35lo

■

[6] For a review of what this means, see Section **??**.