Exercises for Root Radical Functions

Question 1 *In Exercises **??** - **??**, given the pair of functions f and F, sketch the graph of $y = F(x)$ by starting with the graph of $y = f(x)$ and using Theorem **??**. Track at least two points and state the domain and range using interval notation.*

Problem 1.1 $f(x) = \sqrt{x}$, $F(x) = \sqrt{x+3} - 2$

Use the Desmos graph below with the settings $f(x) = \sqrt{x},\ h = -3,\ k = -2,\ a = 1,\ b = 1$

Desmos link: https://www.desmos.com/calculator/zleydlirwe

Domain: $[-3, \infty)$*, Range:* $[-2, \infty)$

Problem 1.2 $f(x) = \sqrt{x}$, $F(x) = \sqrt{4-x} - 1$

$F(x) = \sqrt{4-x} - 1 = \sqrt{-x+4} - 1$

Use the Desmos graph below with the settings $f(x) = \sqrt{x},\ h = -4,\ k = -1,\ a = 1,\ b = -1$

Desmos link: https://www.desmos.com/calculator/zleydlirwe

Domain: $(-\infty, 4]$*, Range:* $[-1, \infty)$

Problem 1.3 $f(x) = \sqrt[3]{x}$, $F(x) = \sqrt[3]{x-1} - 2$

$F(x) = \sqrt[3]{x-1} - 2$

Use the Desmos graph below with the settings $f(x) = \sqrt[3]{x},\ h = 1,\ k = -2,\ a = 1,\ b = 1$

Desmos link: https://www.desmos.com/calculator/ljc0enezou

Domain: $(-\infty, \infty)$*, Range:* $(-\infty, \infty)$

Problem 1.4 $f(x) = \sqrt[3]{x}$, $F(x) = -\sqrt[3]{8x+8} + 4$

$F(x) = -\sqrt[3]{8x+8} + 4$

Use the Desmos graph below with the settings $f(x) = \sqrt[3]{x},\ h = -8,\ k = 4,\ a = -1,\ b = 8$

Desmos link: https://www.desmos.com/calculator/ljc0enezou

Domain: $(-\infty, \infty)$*, Range:* $(-\infty, \infty)$

Problem 1.5 $f(x) = \sqrt[4]{x}$, $F(x) = \sqrt[4]{x-1} - 2$ $F(x) = \sqrt[4]{x-1} - 2$

Use the Desmos graph below with the settings $f(x) = \sqrt[4]{x},\ h = 1,\ k = -2,\ a = 1,\ b = 1$

Desmos link: https://www.desmos.com/calculator/qt32scgh6u

Domain: $[1, \infty)$*, Range:* $[-2, \infty)$

Problem 1.6 $f(x) = \sqrt[4]{x}$, $F(x) = -3\sqrt[4]{x-7} + 1$

$F(x) = -3\sqrt[4]{x-7} + 1$

Use the Desmos graph below with the settings $f(x) = \sqrt[4]{x},\ h = 7,\ k = 1,\ a = -3,\ b = 1$

Desmos link: https://www.desmos.com/calculator/qt32scgh6u

Domain: $[7, \infty)$, *Range:* $(-\infty, 1]$

Problem 1.7 $f(x) = \sqrt[5]{x}$, $F(x) = \sqrt[5]{x+2} + 3$ $F(x) = \sqrt[5]{x+2} + 3$

Use the Desmos graph below with the settings $f(x) = \sqrt[5]{x}$, $h = -2$, $k = 3$, $a = 1$, $b = 1$

Desmos link: https://www.desmos.com/calculator/bstxexnlow

Domain: $(-\infty, \infty)$, *Range:* $(-\infty, \infty)$

Problem 1.8 $f(x) = \sqrt[8]{x}$, $F(x) = \sqrt[8]{-x} - 2$ $F(x) = \sqrt[8]{-x} - 2$

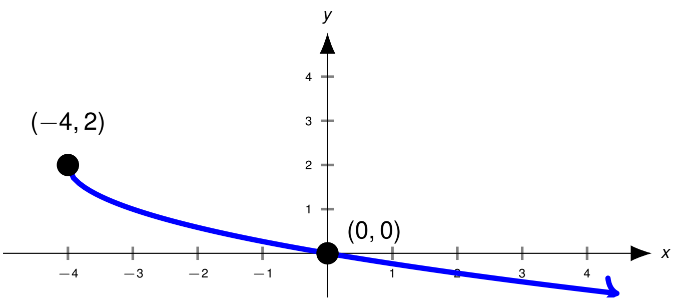
Use the Desmos graph below with the settings $f(x) = \sqrt[8]{x}$, $h = 0$, $k = -2$, $a = 1$, $b = -1$

Desmos link: https://www.desmos.com/calculator/xcow3jqvoh

Domain: $(-\infty, 0]$, *Range:* $[-2, \infty)$

Question 2 *In Exercises* **??** - **??**, *find a formula for each function below in the form* $F(x) = a\sqrt{bx - h} + k$.
NOTE: *There may be more than one solution!*

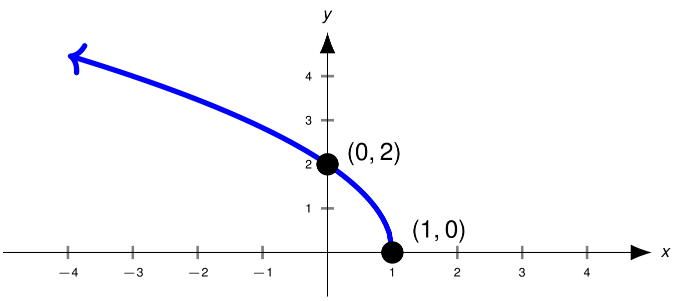
Problem 2.1 $y = F(x)$



The graph of $y = F(x)$.

One solution is: $F(x) = -\sqrt{x+4} + 2$

Problem 2.2 $y = F(x)$

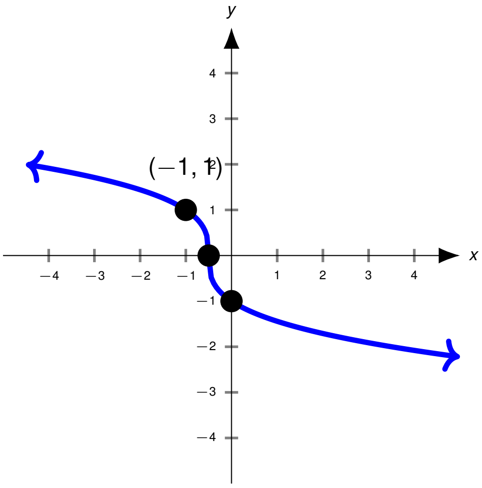


The graph of $y = F(x)$.

One solution is: $F(x) = 2\sqrt{-x+1}$

Question 3 *In Exercises **??** - **??**, find a formula for each function below in the form* $F(x) = a\sqrt[3]{bx - h} + k$.
NOTE: *There may be more than one solution!*

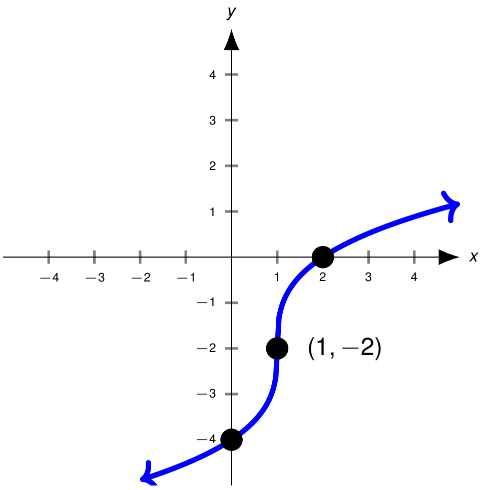
Problem 3.1 $y = F(x)$



x-intercept $(-\frac{1}{2}, 0)$, *y-intercept* $(0, -1)$

One solution is: $F(x) = -\sqrt[3]{2x + 1}$

Problem 3.2 $y = F(x)$



x-intercept $(2, 0)$, *y-intercept* $(0, -4)$

One solution is: $F(x) = 2\sqrt[3]{x - 1} - 2$

Problem 4 *Use the fact that the nth root functions are increasing to solve the following polynomial inequalities:*

- $x^3 \leq 64$

  $(-\infty, 4]$

- $2 - t^5 < 34$

  $(-2, \infty)$

- $\dfrac{(2z+1)^3}{4} \geq 2$

  $\left[\dfrac{1}{2}, \infty\right)$

  For the following inequalities, remember $\sqrt[n]{x^n} = |x|$ *if n is even:*

- $x^4 \leq 16$

  $[-2, 2]$

- $6 - t^6 < -58$

  $(-\infty, -2) \cup (2, \infty)$

- $\dfrac{(2z+1)^4}{3} \geq 27$

  $(-\infty, -2] \cup [1, \infty)$

Question 5 *For each function in Exercises* **??** - **??** *below*

1. *Analytically:*
   - *find the domain.*
   - *find the axis intercepts.*
   - *analyze the end behavior.*
2. *Graph the function with help from a graphing utility and determine:*
   - *the range.*
   - *the local extrema, if they exist.*
   - *intervals of increase/decrease.*
   - *any 'unusual steepness' or 'local' verticality.*
   - *vertical asymptotes.*
   - *horizontal / slant asymptotes.*
3. *Construct a sign diagram for each function using the intercepts and graph.*
4. *Comment on any observed symmetry.*

Problem 5.1 $f(x) = \sqrt{1 - x^2}$

$f(x) = \sqrt{1 - x^2}$
Domain: $[-1, 1]$
Intercepts: $(-1, 0)$, $(1, 0)$
Range: $[0, 1]$
Local maximum: $(0, 1)$
Increasing: $[-1, 0]$, *Decreasing:* $[0, 1]$
Unusual steepness[1] *at* $x = -1$ *and* $x = 1$
Sign Diagram:

0 (+) 0
-1 1

[1] You may need to zoom in to see this.

Graph:

Graph of $f(x) = \sqrt{1 - x^2}$

Note: f is even.

Problem 5.2 $f(x) = \sqrt{x^2 - 1}$

$f(x) = \sqrt{x^2 - 1}$
Domain: $(-\infty, -1] \cup [1, \infty)$
Intercepts: $(-1, 0)$, $(1, 0)$
$\lim_{x \to -\infty} f(x) = \infty$
[2] $\lim_{x \to \infty} f(x) = \infty$
Range: $[0, \infty)$
Increasing: $[1, \infty)$*, Decreasing:* $(-\infty, -1]$
Unusual steepness[3] *at* $x = -1$ *and* $x = 1$
Sign Diagram:

$$\begin{array}{ccccc} (+) & 0 & & 0 & (+) \\ \longleftarrow & \vdash & & \vdash & \longrightarrow \\ & -1 & & 1 & \end{array}$$

Graph:

Graph of $f(x) = \sqrt{x^2 - 1}$

Note: f is even.

Problem 5.3 $g(t) = t\sqrt{1 - t^2}$ $g(t) = t\sqrt{1 - t^2}$
Domain: $[-1, 1]$
Intercepts: $(-1, 0)$, $(0, 0)$, $(1, 0)$
Range: $\approx [-0.5, 0.5]$
Local minimum $\approx (-0.707, -0.5)$
Local maximum: $\approx (0.707, 0.5)$
Increasing: $\approx [-0.707, 0.707]$
Decreasing: $\approx [-1, -0.707]$, $[0.707, 1]$
Unusual steepness at $t = -1$ *and* $t = 1$
Sign Diagram:

$$\begin{array}{ccccc} 0 & (-) & 0 & (+) & 0 \\ \vdash & \text{---} & + & \text{---} & \dashv \\ -1 & & 0 & & 1 \end{array}$$

Graph:

Graph of $g(t) = t\sqrt{1 - t^2}$

Note: g is odd.

[2] Using Calculus, one can show $y = -x$ and $y = -x$ are slant asymptotes to the graph.
[3] You may need to zoom in to see this.

Problem 5.4 $g(t) = t\sqrt{t^2 - 1}$

$g(t) = t\sqrt{t^2 - 1}$
Domain: $(-\infty, -1] \cup [1, \infty)$
Intercepts: $(-1, 0)$, $(1, 0)$
$\lim\limits_{t \to -\infty} g(t) = -\infty$
$\lim\limits_{t \to \infty} g(t) = \infty$
Range: $(-\infty, \infty)$
Increasing: $(-\infty, -1]$, $[1, \infty)$
Unusual steepness at $t = -1$ *and* $t = 1$
Sign Diagram:

$$\begin{array}{cccc} (-) & 0 & 0 & (+) \\ \longleftarrow\!\!\!\longrightarrow\!| & & |\!\longrightarrow & \\ & -1 & 1 & \end{array}$$

Graph:

Graph of $g(t) = t\sqrt{t^2 - 1}$

Note: g is odd.

Problem 5.5 $f(x) = \sqrt[4]{\dfrac{16x}{x^2 - 9}}$

$f(x) = \sqrt[4]{\dfrac{16x}{x^2 - 9}}$
Domain: $(-3, 0] \cup (3, \infty)$
Intercept: $(0, 0)$
Range: $[0, \infty)$
Decreasing: $(-3, 0]$, $(3, \infty)$
Unusual steepness at $x = 0$
Vertical asymptotes: $x = -3$ *and* $x = 3$
Horizontal asymptote: $y = 0$
Sign Diagram:

$$\begin{array}{ccccc} ? & (+) & 0 & ? & (+) \\ |\!\longrightarrow & & | & |\!\longrightarrow & \\ -3 & & 0 & 3 & \end{array}$$

Graph:

Graph of $f(x) = \sqrt[4]{\dfrac{16x}{x^2 - 9}}$

Problem 5.6 $f(x) = \dfrac{5x}{\sqrt[3]{x^3 + 8}}$

$f(x) = \dfrac{5x}{\sqrt[3]{x^3 + 8}}$
Domain: $(-\infty, -2) \cup (-2, \infty)$
Intercept: $(0, 0)$
Range: $(-\infty, 5) \cup (5, \infty)$

Increasing: $(-\infty, -2)$, $(-2, \infty)$
Vertical asymptote $x = -2$
Horizontal asymptote $y = 5$
Sign Diagram:

$$\begin{array}{ccccccccc} & (+) & ? & (-) & 0 & (+) & \\ \longleftarrow & & \vert & & \vert & & \longrightarrow \\ & & -2 & & 0 & & \end{array}$$

Graph:

Graph of $f(x) = \dfrac{5x}{\sqrt[3]{x^3 + 8}}$

Problem 5.7 $g(t) = \sqrt{t(t+5)(t-4)}$ $g(t) = \sqrt{t(t+5)(t-4)}$
Domain: $[-5, 0] \cup [4, \infty)$
Intercepts $(-5, 0)$, $(0, 0)$, $(4, 0)$
$\lim_{t\to\infty} g(t) = \infty$
Range: $[0, \infty)$
Local maximum $\approx (-2.937, 6.483)$
Increasing: $\approx [-5, -2.937]$, $[4, \infty)$
Decreasing: $\approx [-2.937, 0]$
Unusual steepness at $t = -5$, $t = 0$ *and* $t = 4$
Sign Diagram:

$$\begin{array}{ccccccc} 0 & (+) & 0 & & 0 & (+) & \\ \vert & \text{———} & \vert & & \vert & \text{———} & \longrightarrow \\ -5 & & 0 & & 4 & & \end{array}$$

Graph:

Graph of $g(t) = \sqrt{t(t+5)(t-4)}$

Problem 5.8 $g(t) = \sqrt[3]{t^3 + 3t^2 - 6t - 8}$

$g(t) = \sqrt[3]{t^3 + 3t^2 - 6t - 8}$
Domain: $(-\infty, \infty)$
Intercepts: $(-4, 0)$, $(-1, 0)$, $(0, -2)$, $(2, 0)$
[4] $\lim_{t\to-\infty} g(t) = -\infty$
$\lim_{t\to\infty} g(t) = \infty$
Range: $(-\infty, \infty)$
Local maximum: $\approx (-2.732, 2.182)$
Local minimum: $\approx (0.732, -2.182)$
Increasing: $\approx (-\infty, -2.732]$, $[0.732, \infty)$
Decreasing: $\approx [-2.732, 0.732]$
Unusual steepness at $t = -4$, $t = -1$ *and* $t = 2$

Sign Diagram:

[4] Using Calculus it can be shown that $y = t + 1$ is a slant asymptote of this graph.

$$\begin{array}{ccccccccc} (-) & 0 & (+) & 0 & (-) & 0 & (+) \\ \hline & -4 & & -1 & & 2 & \end{array}$$

Graph:

Graph of $g(t) = \sqrt[3]{t^3 + 3t^2 - 6t - 8}$

Problem 6 *Rework Example **??** so that the outpost is 10 miles from Route 117 and the nearest junction box is 30 miles down the road for the post.* $C(x) = 15x + 20\sqrt{100 + (30 - x)^2}$, $0 \leq x \leq 30$. *The calculator gives the absolute minimum at approximately* $(18.66, 582.29)$. *This means to minimize the cost, approximately 18.66 miles of cable should be run along Route 117 before turning off the road and heading towards the outpost. The minimum cost to run the cable is approximately* \$582.29.

Problem 7 *The volume V of a right cylindrical cone depends on the radius of its base r and its height h and is given by the formula* $V = \frac{1}{3}\pi r^2 h$. *The surface area S of a right cylindrical cone also depends on r and h according to the formula* $S = \pi r\sqrt{r^2 + h^2}$. *In the following problems, suppose a cone is to have a volume of 100 cubic centimeters.*

- *Use the formula for volume to find the height as a function of r, $h(r)$.*

  $h(r) = \dfrac{300}{\pi r^2}$, $r > 0$.

- *Use the formula for surface area along with your answer to **??** to find the surface area as a function of r, $S(r)$.*

  $S(r) = \pi r\sqrt{r^2 + \left(\dfrac{300}{\pi r^2}\right)^2} = \dfrac{\sqrt{\pi^2 r^6 + 90000}}{r}$, $r > 0$

- *Use your calculator to find the values of r and h which minimize the surface area. What is the minimum surface area? Round your answers to two decimal places.*

  The calculator gives the absolute minimum at the point $\approx (4.07, 90.23)$. *This means the radius should be (approximately) 4.07 centimeters and the height should be 5.76 centimeters to give a minimum surface area of 90.23 square centimeters.*

Problem 8 *The period of a pendulum in seconds is given by*

$$T = 2\pi\sqrt{\frac{L}{g}}$$

(for small displacements) where L is the length of the pendulum in meters and $g = 9.8$ *meters per second per second is the acceleration due to gravity. My Seth-Thomas antique schoolhouse clock needs* $T = \frac{1}{2}$ *second and I can adjust the length of the pendulum via a small dial on the bottom of the bob. At what length should I set the pendulum?*

$9.8\left(\dfrac{1}{4\pi}\right)^2 \approx 0.062$ *meters or* 6.2 *centimeters*

Problem 9 *According to Einstein's Theory of Special Relativity, the observed mass of an object is a function of how fast the object is traveling. Specifically, if m_r is the mass of the object at rest, v is the speed of the object and c is the speed of light, then the observed mass of the object $m(v)$ is given by:*

$$m(v) = \frac{m_r}{\sqrt{1 - \frac{v^2}{c^2}}}$$

- *Find the applied domain of the function.*

  $[0, c)$

- *Compute $m(.1c)$, $m(.5c)$, $m(.9c)$ and $m(.999c)$.* $m(.1c) = \frac{m_r}{\sqrt{.99}} \approx 1.005m_r$, $m(.5c) = \frac{m_r}{\sqrt{.75}} \approx 1.155m_r$, $m(.9c) = \frac{m_r}{\sqrt{.19}} \approx 2.294m_r$, $m(.999c) = \frac{m_r}{\sqrt{.0.001999}} \approx 22.366m_r$.

- *Find $\lim_{v \to c^-} m(v)$.* $\lim_{v \to c^-} m(x) \to \infty$*; as the object's velocity approaches the speed of light, mass becomes infinite.*

- *How slowly must the object be traveling so that the observed mass is no greater than 100 times its mass at rest? If the object is traveling no faster than approximately* 0.99995 *times the speed of light, then its observed mass will be no greater than* $100m_r$.

Problem 10 *Find the inverse of* $k(x) = \frac{2x}{\sqrt{x^2 - 1}}$.

$k^{-1}(x) = \frac{x}{\sqrt{x^2 - 4}}$