Equations and Inequalities involving Power Functions

In this section, we set about solving equations and inequalities involving power functions. Our first example demonstrates the usual sorts of strategies to employ when solving equations.

Example 1. Solve the following equations analytically and verify your answers using a graphing utility.

1. $(7-x)^{\frac{3}{2}} = 8$
2. $(2t-1)^{\frac{2}{3}} - 4 = 0$
3. $(x+3)^{0.5} = 2(7-x)^{0.5} + 1$
4. $2t^{\frac{2}{3}} + 5t^{\frac{1}{3}} = 3$
5. $2(3x-1)^{-0.5} = 3x(3x-1)^{-1.5}$
6. $6(9-t^2)^{\frac{1}{3}} = 4t^2(9-t^2)^{-\frac{2}{3}}$

Solution.

1. One way to proceed to solve $(7-x)^{\frac{3}{2}} = 8$ is to use Definition **??** to rewrite $(7-x)^{\frac{3}{2}}$ as either $(\sqrt{7-x})^3$ or $\sqrt{(7-x)^3}$. We opt for the former since, thinking ahead, 8 is a perfect cube:

$$\begin{array}{rcll} (7-x)^{\frac{3}{2}} & = & 8 & \\ (\sqrt{7-x})^3 & = & 8 & \text{use } a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m} \\ \sqrt[3]{(\sqrt{7-x})^3} & = & \sqrt[3]{8} & \text{extract cube roots} \\ \sqrt{7-x} & = & 2 & \sqrt[3]{u^3} = u \end{array}$$

From $\sqrt{7-x} = 2$, we square both sides and obtain $7-x = 4$, so $x = 3$. We verify our answer analytically by substituting $x = 3$ into the original equation and it checks.

Geometrically, we are looking for where the graph of $f(x) = (7-x)^{\frac{3}{2}}$ intersects the graph of $g(x) = 8$. While we could sketch both curves by hand and gauge the reasonableness of the result,[1] we are instructed to use a graphing utility. Using desmos, we see the graphs intersect at one point, $(3, 8)$, thereby checking our solution $x = 3$.

Desmos link: `https://www.desmos.com/calculator/dj9jqftbzt`

2. Proceeding similarly to the above, to solve $(2t-1)^{\frac{2}{3}} - 4 = 0$, we rewrite $(2t-1)^{\frac{2}{3}}$ as $(\sqrt[3]{2t-1})^2$ and solve:

$$\begin{array}{rcll} (2t-1)^{\frac{2}{3}} - 4 & = & 0 & \\ (\sqrt[3]{2t-1})^2 - 4 & = & 0 & \text{rewrite using the defintion of rational exponents} \\ (\sqrt[3]{2t-1})^2 & = & 4 & \text{isolate the variable term} \\ \sqrt{(\sqrt[3]{2t-1})^2} & = & \sqrt{4} & \text{extract square roots} \\ |\sqrt[3]{2t-1}| & = & 2 & \sqrt{u^2} = |u| \\ \sqrt[3]{2t-1} & = & \pm 2 & \text{for } c > 0,\ |u| = c \text{ is equivalent to } u = \pm c. \end{array}$$

From $\sqrt[3]{2t-1} = 2$ we cube both sides and obtain $2t-1 = 8$, so $t = \frac{9}{2} = 4.5$. Similarly, from $\sqrt[3]{2t-1} = -2$, we cube both sides and obtain $2t-1 = -8$, so $t = -\frac{7}{2} = -3.5$. Both of these solutions check in the given equation.

In this case we are looking for where the graph of $f(t) = (2t-1)^{\frac{2}{3}} - 4$ intersects the graph of $g(t) = 0$ - i.e., the t-intercepts of the graph of g. We find these are $(-3.5, 0)$ and $(4.5, 0)$, as verified below using desmos.

Desmos link: `https://www.desmos.com/calculator/ewrprzbizv`

[1] consider this an exercise!

3. Since $0.5 = \frac{1}{2}$, we may rewrite $(x+3)^{0.5} = 2(7-x)^{0.5}+1$ as $(x+3)^{\frac{1}{2}} = 2(7-x)^{\frac{1}{2}}+1$. Using Definition **??**, we then have $\sqrt{x+3} = 2\sqrt{7-x}+1$. Since one of the square roots is already isolated, we can rid ourselves of it by squaring both sides.

$$\begin{array}{rcll} \sqrt{x+3} & = & 2\sqrt{7-x}+1 & \\ (\sqrt{x+3})^2 & = & (2\sqrt{7-x}+1)^2 & \text{square both sides} \\ x+3 & = & (2\sqrt{7-x})^2+2(2\sqrt{7-x})(1)+1 & (\sqrt{u})^2 = u \text{ and } (a+b)^2 = a^2+2ab+b^2 \\ x+3 & = & 4(7-x)+4\sqrt{7-x}+1 & (ab)^2 = a^2b^2 \text{ and, again, } (\sqrt{u})^2 = u \\ x+3 & = & 28-4x+4\sqrt{7-x}+1 & \\ 5x-26 & = & 4\sqrt{7-x} & \text{isolate } \sqrt{7-x} \end{array}$$

We square both sides *again* and get $(5x-26)^2 = (4\sqrt{7-x})^2$ which reduces to $25x^2-260x+676 = 16(7-x)$. At last, we have a quadratic equation which we can solve by setting to zero and factoring. We get $25x^2-244x+564 = 0$, so $(x-6)(25x-94) = 0$ so $x = 6$ or $x = \frac{94}{25} = 3.76$. When we go to check these answers, we find $x = 6$ does check, but $x = 3.76$ does not. Hence, $x = 3.76$ is an 'extraneous' solution.[2]

We graph both $f(x) = \sqrt{x+3}$ and $g(x) = 2\sqrt{7-x}+1$ below using desmos and confirm there is only one intersection point, $(6,3)$.

Desmos link: `https://www.desmos.com/calculator/8p6vpfcgug`

4. While we *could* approach solving $2t^{\frac{2}{3}}+5t^{\frac{1}{3}} = 3$ as the previous example, we would encounter cubing binomials[3] which we would prefer to avoid. Instead, we take a step back and notice there are three terms here with the exponent on one term, $t^{\frac{2}{3}}$ exactly twice the exponent on another term, $t^{\frac{1}{3}}$. We have ourselves a 'quadratic in disguise.'[4] To help us see the forest for the trees, we let $u = t^{\frac{1}{3}}$ so that $u^2 = t^{\frac{2}{3}}$. (Note that since root here, 3, is odd, we can use the properties of exponents stated in Theorem **??**.) Hence, in terms of u, the equation $2t^{\frac{2}{3}}+5t^{\frac{1}{3}} = 3$ becomes the quadratic $2u^2+5u = 3$, or $2u^2+5u-3 = 0$. Factoring gives $(2u-1)(u+3) = 0$ so $u = t^{\frac{1}{3}} = \frac{1}{2}$ or $u = t^{\frac{1}{3}} = -3$. Since $t^{\frac{1}{3}} = \sqrt[3]{t}$, we solve both equations by cubing both sides to get $t = \frac{1}{8} = 0.125$ and $t = -27$. Both of these solutions check in our original equation.

Using desmos, we graph $f(t) = 2t^{\frac{2}{3}}+5t^{\frac{1}{3}}$ and $g(t) = 3$. We see the two graphs intersect at $(0.125, 3)$, and, after some adjustment, we find the other intersection point at $(-27, 3)$.

Desmos link: `https://www.desmos.com/calculator/pvhx6w0fep`

5. Next we are to solve $2(3x-1)^{-0.5} = 3x(3x-1)^{-1.5}$ which, when written without negative exponents is: $\frac{2}{(3x-1)^{0.5}} = \frac{3x}{(3x-1)^{1.5}}$. Since the rational exponents here are $0.5 = \frac{1}{2}$ and $1.5 = \frac{3}{2}$, both involve an *even* indexed root (the square root in this case!) which means $3x-1 \geq 0$. Moreover, since the $3x-1$ resides in the *denominator* $3x-1 \neq 0$ so our equation is really valid only for values of x where $3x-1 > 0$ or $x > \frac{1}{3}$. Hence, we clear denominators and can apply Theorem **??**:

[2] We invite the reader to see at which point in our machinations $x = 3.76$ *does* check. Knowing a solution is extraneous is one thing; understanding *how* it came about is another.

[3] that is, expanding things like $(a+b)^3$.

[4] See Section **??** or, more recently, Example **??** in Section **??**.

$$\begin{aligned}
\frac{2}{(3x-1)^{0.5}} &= \frac{3x}{(3x-1)^{1.5}} \\
\left[\frac{2}{(3x-1)^{0.5}}\right]\cdot(3x-1)^{1.5} &= \left[\frac{3x}{(3x-1)^{1.5}}\right]\cdot(3x-1)^{1.5} \\
2\cdot\frac{(3x-1)^{1.5}}{(3x-1)^{0.5}} &= 3x \\
2(3x-1)^{1.5-0.5} &= 3x && \text{we can subtract exponents since } 3x-1>0. \\
2(3x-1)^{1} &= 3x
\end{aligned}$$

We get $6x-2=3x$, or $x=\frac{2}{3}$. Since $x=\frac{2}{3}>\frac{1}{3}$, we keep it and, sure enough, it checks in our original equation. Graphically we see $f(x)=2(3x-1)^{-0.5}$ intersects $g(x)=3x(3x-1)^{-1.5}$ at the point $(0.6667,2)$ which is the desmos's way of representing $\left(\frac{2}{3},2\right)$.

Desmos link: `https://www.desmos.com/calculator/ltnevjtebq`

6. Our last equation to solve is $6(9-t^2)^{\frac{1}{3}}=4t^2(9-t^2)^{-\frac{2}{3}}$, which, when rewritten without negative exponents is: $6(9-t^2)^{\frac{1}{3}}=\frac{4t^2}{(9-t^2)^{\frac{2}{3}}}$. Again, the root here (3) is odd, so we can use the exponent properties listed in Theorem **??**. We begin by clearing denominators:

$$\begin{aligned}
6(9-t^2)^{\frac{1}{3}} &= \frac{4t^2}{(9-t^2)^{\frac{2}{3}}} \\
6(9-t^2)^{\frac{1}{3}}\cdot(9-t^2)^{\frac{2}{3}} &= \left[\frac{4t^2}{(9-t^2)^{\frac{2}{3}}}\right]\cdot(9-t^2)^{\frac{2}{3}} \\
6(9-t^2)^{\frac{1}{3}+\frac{2}{3}} &= 4t^2 && \text{we can add exponents since the root here 3 is odd.} \\
6(9-t^2)^{1} &= 4t^2
\end{aligned}$$

We get $54-6t^2=4t^2$ or $10t^2=54$. As fraction $t^2=\frac{54}{10}=\frac{27}{5}$ so $t=\pm\sqrt{\frac{27}{5}}=\pm3\sqrt{155}$. While not the easiest to check analytically, both of these solutions do work in the original equation. Graphing $f(t)=6(9-t^2)^{\frac{1}{3}}$ and $g(t)=4t^2(9-t^2)^{-\frac{2}{3}}$ below using desmos, we see the graphs intersect when $t\approx\pm2.324$ which are decimal approximations of our exact answers.

Desmos link: `https://www.desmos.com/calculator/3li5pqrdxu`

Note that Example **??**, there are several ways to correctly solve each equation, and we endeavored to demonstrate a variety of methods. For example, for number **??**, instead of converting $(7-x)^{\frac{3}{2}}$ to a radical equation, we could use Theorem **??**. Since the root here (2) is even, we know $7-x\geq0$ or $x\leq7$. Hence we may apply exponent properties:

$$\begin{aligned}
(7-x)^{\frac{3}{2}} &= 8 \\
\left[(7-x)^{\frac{3}{2}}\right]^{\frac{2}{3}} &= 8^{\frac{2}{3}} && \text{raise both sides to the } \tfrac{2}{3} \text{ power} \\
(7-x)^{\frac{3}{2}\cdot\frac{2}{3}} &= 4 && \text{we can multiply exponents since the root here, 3, is odd} \\
(7-x)^{1} &= 4
\end{aligned}$$

from which we get $x=3$. If we try this same approach to solve number **??**, however, we encounter difficulty. From $(2t-1)^{\frac{2}{3}}-4=0$, we get $(2t-1)^{\frac{2}{3}}=4$.

$$\begin{array}{rcll} (2t-1)^{\frac{2}{3}} & = & 4 & \\ \left[(2t-1)^{\frac{2}{3}}\right]^{\frac{3}{2}} & = & 4^{\frac{3}{2}} & \text{raise both sides to the } \dfrac{3}{2} \text{ power} \end{array}$$

Since the root here (3) is odd, we have no restriction on $2t-1$ but the exponent $\dfrac{3}{2}$ has an even denominator. Hence, Theorem **??** *does not apply*. That is,

$$\left[(2t-1)^{\frac{2}{3}}\right]^{\frac{3}{2}} \neq (2t-1)^{\frac{2}{3}\cdot\frac{3}{2}} = (2t-1)^1 = (2t-1).$$

Note that if we weren't careful, we'd have $2t-1 = 4^{\frac{3}{2}} = 8$ which gives $t = \dfrac{9}{2} = 4.5$ only. We'd have missed the solution $t = -3.5$. Truth be told, you *can* simplify $\left[(2t-1)^{\frac{2}{3}}\right]^{\frac{3}{2}}$ - just not using Theorem **??**. We leave it as an exercise to show $\left[(2t-1)^{\frac{2}{3}}\right]^{\frac{3}{2}} = |2t-1|$ and, more generally, $\left(x^{\frac{2}{3}}\right)^{\frac{3}{2}} = |x|$.

Our next example is an application of the Cobb Douglas production model of an economy. The Cobb-Douglas model states that the yearly total dollar value of the production output in an economy is a function of *two* variables: labor (the total number of hours worked in a year) and capital (the total dollar value of the physical goods required for manufacturing.) The equation relating the production output level P, labor L and capital K takes the form $P = aL^bK^{1-b}$ where $0 < b < 1$; that is, the production level varies jointly with some power of the labor and capital.

Example 2. In their original paper *A Theory of Production*[5] Cobb and Douglas modeled the output of the US Economy (using 1899 as a baseline) using the formula $P = 1.01L^{0.75}K^{0.25}$ where P, L, and K were percentages of the 1899 figures for total production, labor, and capital, respectively.

1. For 1910, the recorded labor and capital figures for the US Economy are 144% and 208% of the 1899 figures, respectively. Find P using these figures and interpret your answer.
2. The recorded production value figure for 1920 is 231% of the 1899 figure. Use this to write K as a function of L, $K = f(L)$. Find and interpret $f(193)$.
3. Graph $K = f(L)$ and interpret the behavior as $L \to 0^+$ and $L \to \infty$.

Solution.

1. In this case, $P = 1.01L^{0.75}K^{0.25} = 1.01(144)^{0.75}(208)^{0.25} \approx 159$ which means the dollar value of the total US Production in 1920 was approximately 159% of what it was in 1899.[6]
2. We are given $P = 231 = 1.01L^{0.75}K^{0.25}$, so to write K as a function of L, we need to solve this equation for K. Since L and K are positive by definition, we can employ properties of exponents:

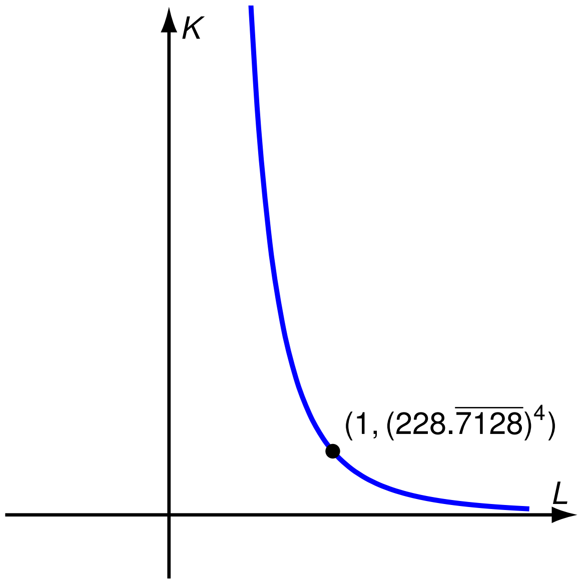
$$\begin{array}{rcll} 231 & = & 1.01L^{0.75}K^{0.25} & \\ \dfrac{231}{1.01L^{0.75}} & = & \dfrac{1.01L^{0.75}K^{0.25}}{1.01L^{0.75}} & L > 0, \text{ hence } L^{0.75} \neq 0. \\ K^{0.25} & = & 228.\overline{7128}L^{-0.75} & \text{rewrite} \\ \left(K^{0.25}\right)^{\frac{1}{0.25}} & = & \left(228.\overline{7128}L^{-0.75}\right)^{\frac{1}{0.25}} & \\ K^{\frac{0.25}{0.25}} & = & (228.\overline{7128})^{\frac{1}{0.25}}L^{-\frac{0.75}{0.25}} & \text{for a power of a power, multiply exponents} \\ K & = & (228.\overline{7128})^4L^{-3} & \text{simplify} \end{array}$$

[5]available here.

[6]This answer is remarkably accurate. Note: all the dollar values here are recorded in '1880' dollars, per the source article.

Hence, $K = f(L) = (228.\overline{7128})^4 L^{-3}$ where $L > 0$. We find $f(193) = (228.\overline{7128})^4 (193)^{-3} \approx 381$ meaning that in order to maintain a production level of 231% of 1889 with a labor level at 193% of 1889, the required capital is 381% that of 1889.[7]

3. The function $f(L)$ is a Laurent Monomial (see Section **??**) with $n = 3$ and $a = (228.\overline{7128})^4$ and is graphed below on the applied domain, $L > 0$.



We see that as $L \to 0^+$, $f(L) \to \infty$. This means that in order to maintain the given production level, as the available labor diminishes, the capital requirement becomes unbounded. As $L \to \infty$, we have $f(L) \to 0$ meaning that as the available labor increases, the need for capital diminishes. The graph of f is called an 'isoquant' - meaning 'same quantity.' In this context, the graph displays all combinations of labor and capital, (L, K) which result in the *same* production level, in this case, 231% of what was produced in 1889.

■

Next, we move on to solving inequalities with power functions. As we've seen with other types of non-linear inequalities,[8] an invaluable tool for us is the Sign Diagram.

Steps for Constructing a Sign Diagram for an Algebraic Function

Suppose f is an algebraic function.

1. Place any values excluded from the domain of f on the number line. You may wish to use a dashed line for a vertical asymptote and an open circle for a hole in the graph.
2. Find the zeros of f and place them on the number line with the number 0 above them.
3. Choose a test value in each of the intervals determined in steps 1 and 2.
4. Determine and record the sign of $f(x)$ for each test value in step 3.

As you may recall, since sign diagrams compare functions to 0, the first step in solving inequalities using a sign diagram is to gather all the nonzero terms one one side of the inequality. We demonstrate this technique in the following example.

[7]The actual recorded figure is 407.
[8]see Sections **??**, **??**, and **??**

Example 3.

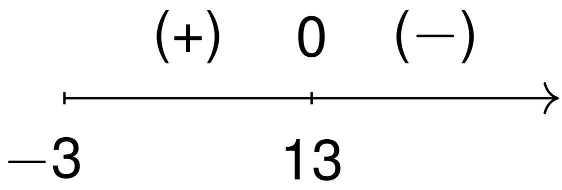
Solve the following inequalities. Check your answers graphically with a calculator.

1. $2 - \sqrt[4]{x+3} \geq 0$
2. $t^{2/3} < t^{4/3} - 6$
3. $3(2-x)^{\frac{1}{3}} \leq x(2-x)^{-\frac{2}{3}}$
4. $(t-4)^{\frac{2}{3}} \geq -\dfrac{2t}{3(t-4)^{\frac{1}{3}}}$

Solution.

1. To solve $2 - \sqrt[4]{x+3} \geq 0$, it is tempting to rewrite this inequality as $2 \geq \sqrt[4]{x+3}$ and rid ourselves of the fourth root by raising both sides of this inequality to the fourth power. While this technique works *sometimes*, it doesn't work *all* the time since raising both sides of an inequality to the fourth (more generally, to an even) power does not necessarily preserve inequalities .[9] For that reason, we solve this inequality using a sign diagram since this technique will *always* produce a correct solution.

   We already have all the nonzero terms on one side of the inequality, so we let $r(x) = 2 - \sqrt[4]{x+3}$ and proceed to make a sign diagram. Owing to the presence of the fourth root, we know $x + 3 \geq 0$ or $x \geq -3$. Hence, we only concern ourselves with the portion of the number line representing $[3, \infty)$. Next, we find the zeros of r by solving $r(x) = 2 - \sqrt[4]{x+3} = 0$. We get $\sqrt[4]{x+3} = 2$, so $x + 3 = 16$ and we get $x = 13$. We find this solution checks in our original equation,[10] and proceed to construct the sign diagram below.



   Since we are looking for where $r(x) = 2 - \sqrt[4]{x+3} \geq 0$, we are looking for the zeros of r along with the intervals over which $r(x)$ is (+). We record our answer as $[-3, 13]$. Using desmos, we graph $y = 2 - \sqrt[4]{x+3}$, and we can see that, indeed, the graph is above the x-axis ($y = 0$) from $[-3, 13)$ and meets the x-axis at $x = 13$, verifying our answer.

   Desmos link: `https://www.desmos.com/calculator/tdsuunihkt`

2. To solve $t^{\frac{2}{3}} < t^{\frac{4}{3}} - 6$, we first rewrite as $t^{\frac{4}{3}} - t^{\frac{2}{3}} - 6 > 0$. We set $r(t) = t^{\frac{4}{3}} - t^{\frac{2}{3}} - 6$ and note that since the denominators in the exponents are 3, they correspond to cube roots, which means the domain of r is $(-\infty, \infty)$. To find the zeros for the sign diagram, we set $r(t) = 0$ and attempt to solve $t^{\frac{4}{3}} - t^{\frac{2}{3}} - 6 = 0$. Since there are three terms, and the exponent on one of the variable terms, $t^{\frac{4}{3}}$, is exactly twice that of the other, $t^{\frac{2}{3}}$, we have ourselves a 'quadratic in disguise.' If we let $u = t^{\frac{2}{3}}$, then $u^2 = t^{\frac{4}{3}}$, so in terms of u, we have $u^2 - u - 6 = 0$. Solving we get $u = -2$ or $u = 3$, hence $t^{\frac{2}{3}} = -2$ or $t^{\frac{2}{3}} = 3$. In root-power notation, these are $\sqrt[3]{t^2} = -2$ or $\sqrt[3]{t^2} = 3$. Cubing both sides of these equations results in $t^2 = -8$, which admits no real solution, or $t^2 = 27$, which gives $t = \pm 3\sqrt{3}$. Using these zeros, we construct the sign diagram below.

[9] For instance, $-2 \leq 1$ but $(-2)^4 \geq (1)^2$. We invite the reader to see what goes wrong if attempting to solve either of the following inequalities using this method: $-2 \geq \sqrt[4]{x+3}$, which has no solution, or $-2 \leq \sqrt[4]{x+3}$, whose solution is $[-3, \infty)$.

[10] Recall that raising both sides to an even power could produce extraneous solutions, so it is important we check here.

$$\begin{array}{ccccc} (+) & 0 & (-) & 0 & (+) \\ \longleftarrow & | & \longrightarrow & | & \longrightarrow \\ & -3\sqrt{3} & & 3\sqrt{3} & \end{array}$$

We find $r(t) = t^{\frac{4}{3}} - t^{\frac{2}{3}} - 6 > 0$ on $\left(-\infty, -3\sqrt{3}\right) \cup \left(3\sqrt{3}, \infty\right)$. To check our answer graphically, we set $f(t) = t^{\frac{2}{3}}$ and $g(t) = t^{\frac{4}{3}} - 6$. The solution to $t^{\frac{2}{3}} < t^{\frac{4}{3}} - 6$ corresponds to the inequality $f(t) < g(t)$, which means we are looking for the t values for which the graph of f is *below* the graph of g. Using desmos, we see the graph of $y = f(t) = t^{\frac{2}{3}}$ is below the graph of $y = g(t) = t^{\frac{4}{3}} - 6$ for $t < -5.196$ and again for $t > 5.196$, which are desmos's approximations to $\pm 3\sqrt{3}$.

Desmos link: `https://www.desmos.com/calculator/n6gohzoqb0`

3. To solve $3(2-x)^{\frac{1}{3}} \leq x(2-x)^{-\frac{2}{3}}$, we first gather all the nonzero terms to one side and obtain $3(2-x)^{\frac{1}{3}} - x(2-x)^{-\frac{2}{3}} \leq 0$. Setting $r(x) = 3(2-x)^{\frac{1}{3}} - x(2-x)^{-\frac{2}{3}}$, we note since the denominators of the rational exponents are odd, we have no domain concerns owing to even indexed roots. However, the negative exponent on the second term indicates a denominator.

   Rewriting $r(x)$ with positive exponents, we obtain

$$r(x) = 3(2-x)^{\frac{1}{3}} - \frac{x}{(2-x)^{\frac{2}{3}}}$$

   Setting the denominator equal to zero we get $(2-x)^{\frac{2}{3}} = 0$, which reduces to $2 - x = 0$, or $x = 2$. Hence, the domain of r is $(-\infty, 2) \cup (2, \infty)$.

   To find the zeros of r, we set $r(x) = 0$, so we set about solving

$$3(2-x)^{\frac{1}{3}} - \frac{x}{(2-x)^{\frac{2}{3}}} = 0.$$

   Clearing denominators, we get $3(2-x)^{\frac{1}{3}}(2-x)^{\frac{2}{3}} - x = 0$. Since the denominators of the exponents are odd, we may use Theorem **??** to simplify this to $3(2-x)^1 - x = 0$, and obtain $6 - 4x = 0$ or $x = \dfrac{3}{2}$. In order for us to be able to more easily determine the sign of $r(x)$ at the test values, we rewrite $r(x)$ as a single term.[11] There are two schools of thought on how to proceed, so we demonstrate both.

   - *Factoring Approach.* From $r(x) = 3(2-x)^{\frac{1}{3}} - x(2-x)^{-\frac{2}{3}}$, we note that the quantity $(2-x)$ is common to both terms. When we factor out common factors, we factor out the quantity with the *smaller* exponent. In this case, since $-\dfrac{2}{3} < \dfrac{1}{3}$, we factor $(2-x)^{-\frac{2}{3}}$ from both quantities. While it may seem odd to do so, we need to factor $(2-x)^{-\frac{2}{3}}$ *from* $(2-x)^{\frac{1}{3}}$, which results in subtracting the exponent $-\dfrac{2}{3}$ from $\dfrac{1}{3}$. We proceed using the usual properties of exponents.

$$\begin{array}{rcl} r(x) & = & 3(2-x)^{\frac{1}{3}} - x(2-x)^{-\frac{2}{3}} \\ & = & (2-x)^{-\frac{2}{3}} \left[3(2-x)^{\frac{1}{3}-\left(-\frac{2}{3}\right)} - x\right] \\ & = & (2-x)^{-\frac{2}{3}} \left[3(2-x)^{\frac{3}{3}} - x\right] \\ & = & (2-x)^{-\frac{2}{3}} \left[3(2-x)^{1} - x\right] \\ & = & (2-x)^{-\frac{2}{3}} \left(6 - 4x\right) \\ & = & (2-x)^{-\frac{2}{3}} \left(6 - 4x\right) \end{array}$$

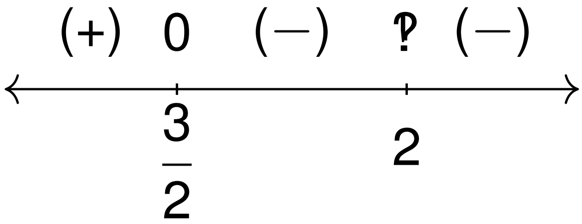
     Written without negative exponents, we have $r(x) = \dfrac{6-4x}{(2-x)^{\frac{2}{3}}}$.

[11] This also gives us a chance to review some good intermediate algebra!

- *Common Denominator Approach.* We rewrite

$$
\begin{aligned}
r(x) &= 3(2-x)^{\frac{1}{3}} - x(2-x)^{-\frac{2}{3}} \\
&= 3(2-x)^{\frac{1}{3}} - \frac{x}{(2-x)^{\frac{2}{3}}} \\
&= \frac{3(2-x)^{\frac{1}{3}}(2-x)^{\frac{2}{3}}}{(2-x)^{\frac{2}{3}}} - \frac{x}{(2-x)^{\frac{2}{3}}} && \text{common denominator} \\
&= \frac{3(2-x)^{\frac{1}{3}+\frac{2}{3}}}{(2-x)^{\frac{2}{3}}} - \frac{x}{(2-x)^{\frac{2}{3}}} && \text{root here, 3, is odd, so add exponents} \\
&= \frac{3(2-x)^{\frac{3}{3}}}{(2-x)^{\frac{2}{3}}} - \frac{x}{(2-x)^{\frac{2}{3}}} \\
&= \frac{3(2-x)^{1}}{(2-x)^{\frac{2}{3}}} - \frac{x}{(2-x)^{\frac{2}{3}}} \\
&= \frac{3(2-x)-x}{(2-x)^{\frac{2}{3}}} \\
&= \frac{6-4x}{(2-x)^{\frac{2}{3}}}
\end{aligned}
$$

Using either approach, we end up with the same, simpler, expression for $r(x)$ and we use that to create our sign diagram as shown below.



We find $r(x) \leq 0$ on $\left[\frac{3}{2}, 2\right) \cup (2, \infty)$. To check, we graph $f(x) = 3(2-x)^{\frac{1}{3}}$ and $g(x) = x(2-x)^{-\frac{2}{3}}$ using desmos. We confirm that the graphs intersect at $x = \frac{3}{2}$ and the graph of $y = f(x) = 3(2-x)^{\frac{1}{3}}$ is below the graph of $y = g(x) = x(2-x)^{-\frac{2}{3}}$ for $x > \frac{3}{2}$, with the exception of $x = 2$ where it appears the graph of g has a vertical asymptote.

Desmos link: `https://www.desmos.com/calculator/k9art9xq0i`

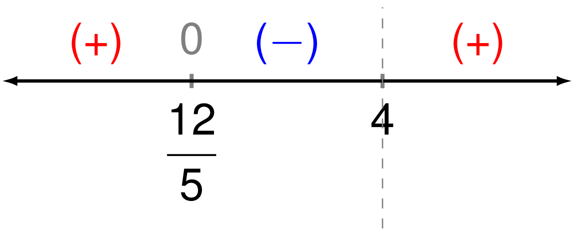
4. While it may be tempting to begin solving our last inequality by clearing denominators, owing to the odd root, the quantity $3(t-4)^{\frac{1}{3}}$ can be both positive and negative for different values of t. This means that if we chose to multiply both sides of our inequality by this quantity, we have no guarantee if the inequality would be preserved. Hence we proceed as usual by gathering all the nonzero terms to one side, and, with the ultimate goal of creating a sign diagram, get common denominators.

$$
\begin{aligned}
(t-4)^{\frac{2}{3}} &\geq -\frac{2t}{3(t-4)^{\frac{1}{3}}} && \\
(t-4)^{\frac{2}{3}} + \frac{2t}{3(t-4)^{\frac{1}{3}}} &\geq 0 && \\
\frac{(t-4)^{\frac{2}{3}} \cdot 3(t-4)^{\frac{1}{3}}}{3(t-4)^{\frac{1}{3}}} + \frac{2t}{3(t-4)^{\frac{1}{3}}} &\geq 0 && \text{common denominator} \\
\frac{3(t-4)^{\frac{2}{3}+\frac{1}{3}}}{3(t-4)^{\frac{1}{3}}} + \frac{2t}{3(t-4)^{\frac{1}{3}}} &\geq 0 && \text{root is odd so add exponents} \\
\frac{3(t-4)^{1}}{3(t-4)^{\frac{1}{3}}} + \frac{2t}{3(t-4)^{\frac{1}{3}}} &\geq 0 && \\
\frac{3(t-4)+2t}{3(t-4)^{\frac{1}{3}}} &\geq 0 && \\
\frac{5t-12}{3(t-4)^{\frac{1}{3}}} &\geq 0 &&
\end{aligned}
$$

We identify $r(t)$ as the left hand side of the inequality and see right away we must exclude $t = 4$ from the domain owing to the quantity $(t-4)$ in the denominator. As we have already mentioned, the root here (3) is odd, so we have no domain issues stemming from that. To find the zeros of r, we set $r(t) = 0$ which quickly reduces to solving $5t - 12 = 0$. We get $t = \frac{12}{5}$. Our sign diagram is below.

Figure 1: Sign diagram

Alt text: Sign diagram showing the intervals on which the function $r(x)$ is positive or negative. Dashed vertical line at $x = 2$ indicates a vertical asymptote. The function is positive on $(-\infty, \frac{12}{5})$, negative on $(\frac{12}{5}, 4)$, and positive on $(4, \infty)$. The function has a zero at $x = \frac{12}{5}$.



From the sign diagram, we find $r(t) \geq 0$ on $\left(-\infty, \frac{12}{5}\right] \cup (4, \infty)$. Graphing $f(t) = (t-4)^{\frac{2}{3}}$ and $g(t) = -\frac{2t}{3(t-4)^{\frac{1}{3}}}$ using desmos, we see the graph of $y = f(t) = (t-4)^{\frac{2}{3}}$ is above the graph of $y = g(t) = -\frac{2t}{3(t-4)^{\frac{1}{3}}}$ for $t < 2.4$ and again for $t > 4$, with an intersection point at $t = 2.4 = \frac{12}{5}$.

Desmos link: `https://www.desmos.com/calculator/ipnyncdfbb`

■

Note that in Example **??** number **??**, since $(2-x)^{\frac{2}{3}}$ is always positive for $x \neq 2$ (owing to the squared exponent), we *could* have short-cut the sign diagram, choosing to clear denominators instead:

$$\begin{array}{rcl}
3(2-x)^{\frac{1}{3}} & \leq & x(2-x)^{-\frac{2}{3}} \\
3(2-x)^{\frac{1}{3}} & \leq & \dfrac{x}{(2-x)^{\frac{2}{3}}} \\
\left[3(2-x)^{\frac{1}{3}}\right]\left[(2-x)^{\frac{2}{3}}\right] & \leq & \dfrac{x}{(2-x)^{\frac{2}{3}}}\left[(2-x)^{\frac{2}{3}}\right] \quad \text{provided } x \neq 2 \\
3(2-x)^{\frac{1}{3}}(2-x)^{\frac{2}{3}} & \leq & x \\
3(2-x)^{\frac{1}{3}+\frac{2}{3}} & \leq & x \\
3(2-x) & \leq & x
\end{array}$$

Hence, we get $6-3x \leq x$ or $x \geq \frac{3}{2}$, provided $x \neq 2$. This matches our solution $\left[\frac{3}{2}, 2\right) \cup (2, \infty)$. If, on the other hand, we tried this same manipulation with number **??**, we would clear denominators, assuming $t \neq 4$ to obtain $3(t-4) \geq -2t$ or $t \geq \frac{12}{5}$ which is *not* the correct solution. The moral of the story is the more you understand, the less you need to rely on memorized processes and the more efficient your solution methodologies can become. The sign diagram algorithm is a fail-safe method, but, in some cases, may be far from the most efficient one. It's always best to understand the *why* of a procedure as much as the *how*.