

POWER FUNCTIONS

Monomial, and, more generally, Laurent monomial functions are specific examples of a much larger class of functions called **power functions**, as defined below.

Definition 1. Let a and p be nonzero real numbers. A **power function** is either a constant function or a function of the form $f(x) = ax^p$.

Definition ?? broadens our scope of functions to include non-integer exponents such as $f(x) = 2x^{4/3}$, $g(t) = t^{0.4}$ and $h(w) = w^{\sqrt{2}}$. Our primary aim in this section is to ascribe meaning to these quantities.

1 Rational Number Exponents

The road to real number exponents starts by defining rational number exponents.

Definition 2. Let r be a rational number where in lowest terms $r = \frac{m}{n}$ where m is an integer and n is a natural number.¹ If $n = 1$, then $x^r = x^m$. If $n > 1$, then

$$x^r = x^{\frac{m}{n}} = (\sqrt[n]{x})^m = \sqrt[n]{x^m},$$

whenever $(\sqrt[n]{x})^m$ is defined.²

There are quite a few items worthy of note which are consequences of Definition ???. First off, if m is an integer, then $x^{\frac{m}{n}} = x^m$ so expressions like $x^{\frac{3}{1}}$ are synonymous with x^3 , as we would expect.³ Second, the definition of $x^{\frac{m}{n}}$ can be taken as just $(\sqrt[n]{x})^m$ and shown to be equal to $\sqrt[n]{x^m}$ (or vice-versa) courtesy of properties of radicals. We state both in Definition ??? to allow for the reader to choose whichever form is more convenient in a given situation. The critical point to remember is no matter which representation you choose, keep in mind the restrictions if n is even, $x \geq 0$ and if $m < 0$, $x \neq 0$.

Moreover, per this definition, $x^{\frac{1}{n}} = \sqrt[n]{x^1} = \sqrt[n]{x}$, so we may rewrite principal roots as exponents: $\sqrt{x} = x^{\frac{1}{2}}$ and $\sqrt[5]{x} = x^{\frac{1}{5}}$. This makes sense from an algebraic standpoint since per Theorem ??, $(\sqrt[n]{x})^n = x$. Hence if we were to assign an exponent notation to $\sqrt[n]{x}$, say $\sqrt[n]{x} = x^r$, then $(\sqrt[n]{x})^n = (x^r)^n = x$. If the properties of exponents are to hold, then, necessarily, $(x^r)^n = x^m = x = x^1$, so $rn = 1$ or $r = \frac{1}{n}$. While this argument helps *motivate* the notation, as we shall see shortly, great care must be exercised in applying exponent properties in these cases. The long and short of this is that root functions as defined in Section ?? are all members of the 'power functions' family.

Another important item worthy of note in Definition ??? is that it is absolutely essential we express the rational number r in *lowest terms* before applying the root-power definition. For example, consider $x^{0.4}$. Expressing r in lowest terms, we get: $r = 0.4 = \frac{4}{10} = \frac{2}{5}$. Hence, $x^{0.4} = x^{2/5} = (\sqrt[5]{x})^2$ or $\sqrt[5]{x^2}$, either of which is defined for all real numbers x . In contrast, consider the equivalence $r = 0.4 = \frac{4}{10}$. Here, the expression $(\sqrt[10]{x})^4$ is defined only for $x \geq 0$ owing to the presence of the even indexed root, $\sqrt[10]{x}$. Hence, $(\sqrt[10]{x})^4 \neq x^{\frac{4}{10}} = x^{\frac{2}{5}}$ unless $x \geq 0$. On the other hand, the expression $\sqrt[10]{x^4}$ is defined for all numbers, x , since $x^4 \geq 0$ for all x . In fact, it can be shown that $\sqrt[10]{x^4} = \sqrt[5]{x^2}$ for all real numbers. This means $\sqrt[10]{x^4} = \sqrt[5]{x^2} = x^{\frac{2}{5}} = x^{\frac{4}{10}}$. So, to review, in general we have: $x^{\frac{4}{10}} = \sqrt[10]{x^4}$, but $x^{\frac{4}{10}} \neq (\sqrt[10]{x})^4$ unless $x \geq 0$. Once again the easiest way to avoid confusion here is to *reduce* the exponent to *lowest terms* before converting it to root-power notation.

Likewise, we have to be careful about the properties of exponents when it comes to rational exponents. Consider, for instance, the product rule for integer exponents: $x^m x^n = x^{m+n}$. Consider $f(x) = x^{\frac{1}{2}} x^{\frac{1}{2}}$ and $g(x) = x^{\frac{1}{2} + \frac{1}{2}}$. In the first case, $f(x) = x^{\frac{1}{2}} x^{\frac{1}{2}} = \sqrt{x} \sqrt{x} = (\sqrt{x})^2 = x$ only for $x \geq 0$. In the second case, $g(x) = x^{\frac{1}{2} + \frac{1}{2}} = x^1 = x^2 = x^1 = x$ for all real numbers x . Even though $f(x) = g(x)$ for $x \geq 0$, f and g are *different functions* since they have *different domains*.

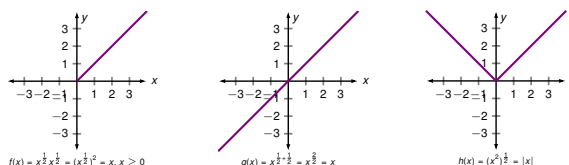
¹ Recall 'lowest terms' means m and n have no common factors other than 1.

² That is, if n is even, $x \geq 0$ and if $m < 0$, $x \neq 0$.

³ Either $n = 1$ is a special case in Definition ??? or we need to define what is meant by $\sqrt[n]{x}$. The authors chose the former.

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Similarly, the power rule for integer exponents: $(x^n)^m = x^{nm}$ does not hold in general for rational exponents. To see this, consider the three functions: $f(x) = (x^{\frac{1}{2}})^2$, $g(x) = x^{\frac{2}{2}}$, and $h(x) = (x^2)^{\frac{1}{2}}$. In the first case, $f(x) = (x^{\frac{1}{2}})^2 = (\sqrt{x})^2 = x$ for $x \geq 0$ only (this is the same function f above.) In the second case, the rational number $r = \frac{2}{2} = 1$, so $g(x) = x^{\frac{2}{2}} = x^1 = x^1 = x$ for *all* real numbers, x (this is the same function g from above.) In the last case, $h(x) = (x^2)^{\frac{1}{2}} = \sqrt{x^2} = |x|$ for all real numbers, x . Once again, despite $f(x) = g(x) = h(x)$ for all $x \geq 0$, f , g and h are *three different functions*. We graph f , g , and h below.



In general, the properties of integer exponents *do not extend* to rational exponents *unless* the bases involved represent non-negative real numbers *or* the roots involved are *odd*. We have the following:

Theorem 1. Let r and s be rational numbers. The following properties hold provided none of the computations results in division by 0 and **either** r and s have odd denominators **or** $x \geq 0$ **and** $y \geq 0$:

- **Product Rules:** $x^r x^s = x^{r+s}$ and $(xy)^r = x^r y^r$.
- **Quotient Rules:** $\frac{x^r}{x^s} = x^{r-s}$ and $\left(\frac{x}{y}\right)^r = \frac{x^r}{y^r}$
- **Power Rule:** $(x^r)^s = x^{rs}$

Next, we turn our attention to the graphs of $f(x) = x^r = x^{\frac{m}{n}}$ for varying values of m and n . When n is even, the domain is restricted owing to the presence of the even indexed root to $[0, \infty)$. The range is likewise $[0, \infty)$, a fact leave to the reader. All of the functions below are increasing on their domains, and it turns out this is always the case provided $r > 0$. There is, however, is a difference in *how* the functions are increasing - and this is the concept of *concavity*. As with many concepts we've encountered so far in the text, concavity is most precisely defined using Calculus terminology, but we can nevertheless get a sense of concavity geometrically. For us, a curve is **concave up** over an interval if it resembles a portion of a '⌒' shape. Similarly, a curve is called **concave down** over an interval if resembles part of a '⌒' shape. When $0 < r < 1$, the graphs of $f(x) = x^r$ resemble the left half of ⌒ and so are concave down; when $r > 1$, the graphs resemble the right half of a '⌒' and are hence described as 'concave up.'

Desmos link: <https://www.desmos.com/calculator/fni0liyn5n>

Below we graph several examples of $f(x) = x^r = x^{\frac{m}{n}}$ where n is odd. Here, the domain is $(-\infty, \infty)$ since the index on the root here is odd. Note that when m is even, the graphs appear to be symmetric about the y -axis and the range looks to be $[0, \infty)$. When m is odd, the graphs appear to be symmetric about the origin with range $(-\infty, \infty)$. We leave verification of these facts to the reader. Note here also that for $x \geq 0$, the graphs are down for $0 < r < 1$ and concave up for $r > 1$.

Desmos link: <https://www.desmos.com/calculator/gyv8qfxhjs>

When $r < 0$, we have variables appear in the denominator which open the opportunities for vertical and horizontal asymptotes. Below are graphed two examples

Desmos link: <https://www.desmos.com/calculator/t2bfjpr8ar>

Unsurprisingly, Theorem ??, which, as stated, applied to root functions, generalizes to all rational powers.

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Theorem 2. For real numbers a , b , h , and k and rational number r with $a, b, r \neq 0$, the graph of $F(x) = a(bx - h)^r + k$ can be obtained from the graph of $f(x) = x^r$ by performing the following operations, in sequence:

1. add h to each of the x -coordinates of the points on the graph of f . This results in a horizontal shift to the right if $h > 0$ or left if $h < 0$.

NOTE: This transforms the graph of $y = x^r$ to $y = (x - h)^r$.

2. divide the x -coordinates of the points on the graph obtained in Step 1 by b . This results in a horizontal scaling, but may also include a reflection about the y -axis if $b < 0$.

NOTE: This transforms the graph of $y = (x - h)^r$ to $y = (bx - h)^r$.

3. multiply the y -coordinates of the points on the graph obtained in Step 2 by a . This results in a vertical scaling, but may also include a reflection about the x -axis if $a < 0$.

NOTE: This transforms the graph of $y = (bx - h)^r$ to $y = a(bx - h)^r$.

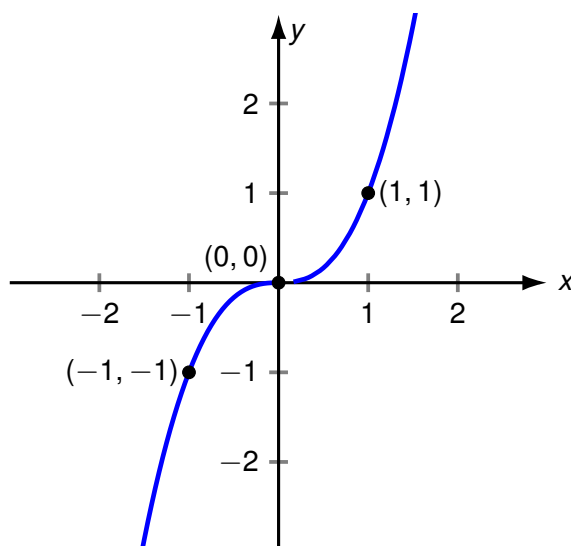
4. add k to each of the y -coordinates of the points on the graph obtained in Step 3. This results in a vertical shift up if $k > 0$ or down if $k < 0$.

NOTE: This transforms the graph of $y = a(bx - h)^r$ to $y = a(bx - h)^r + k$.

The proof of Theorem ?? is identical to that of Theorem ??, and we suggest the reader work through the details. We give Theorem ?? a test run in the following example.

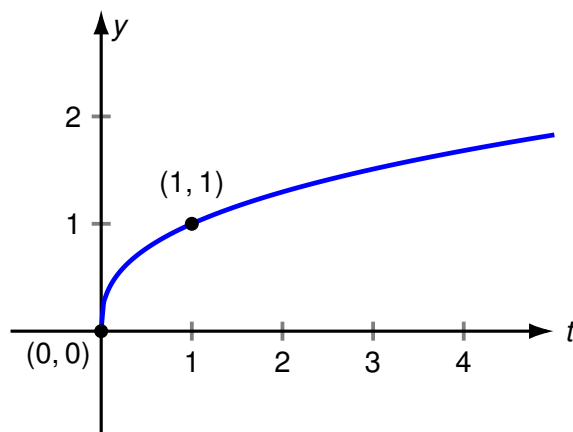
Example 1. Use the given graphs of f and g below long with Theorem ?? to graph F and G . State the domain and range of F and G using interval notation.

1. Graph $F(x) = (2x - 1)^{2.6}$.



2. Graph $G(t) = 1 - 2(t + 3)^{\frac{3}{5}}$.

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**Solution.**

1. The expression $F(x) = (2x - 1)^{2.6}$ is given to us in the form prescribed by Theorem ??, and we identify $r = 2.6$, $a = 1$, $b = 2$, $h = 1$, and $k = 0$. Even though the graph of $f(x) = x^{2.6}$ is given to us, it's worth taking a moment to reinforce some concepts. Since, in lowest terms, $2.6 = \frac{26}{10} = \frac{13}{5}$, it makes sense the domain and range of $f(x) = x^{2.6}$ are both all real numbers and the graph is symmetric about the origin.⁴ Moreover, since $2.6 > 1$, the concavity matches what we would expect, too. We proceed as we have several times in the past, beginning with the horizontal shift.

GeoGebra link: <https://www.geogebra.org/m/eazjuzbg>

We get the domain and range here are both $(-\infty, \infty)$.

2. We first need to rewrite $G(t) = 1 - 2(t + 3)^{\frac{3}{8}}$ in the form required by Theorem ??: $G(t) = -2(t + 3)^{\frac{3}{8}} + 1$. We identify $r = \frac{3}{8}$, $a = -2$, $b = 1$, $h = -3$, and $k = 1$. Since $\frac{3}{8}$ is in lowest terms and has an even denominator, it makes sense the domain and range of $g(t) = t^{\frac{3}{8}}$ is $[0, \infty)$, since the root here, 8 is even. Also, since $0 < \frac{3}{8} < 1$, the graph of $y = t^{\frac{3}{8}}$ is concave down, as we would expect. As usual, we start with the horizontal shift.

GeoGebra link: <https://www.geogebra.org/m/hr42fgrf>

From the graph, we get the domain is $[-3, \infty)$ and the range is $(-\infty, 1]$.

We now turn our attention to more complicated functions involving rational exponents.

Example 2. For the following functions:

- Analytically:
 - find the domain.
 - find the axis intercepts.
 - analyze the end behavior.
- Graph the function with help from a graphing utility and determine:

⁴The domain is all real numbers since the denominator (root) 5 is odd; the range is all real numbers since the numerator (power) 13 is odd. Since both power and root are odd, the function itself is an odd function, hence the symmetry about the origin.

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- the range.
- the local extrema, if they exist.
- intervals of increase.
- intervals of decrease.
- Construct a sign diagram for each function using the intercepts and graph.

1. $f(x) = 3x^2(x^3 - 8)^{-\frac{2}{3}}$

2. $g(t) = \frac{(t^2 - 4)^{\frac{3}{2}}}{t^2 - 36}$

Solution.

1. We first note that, owing to the negative exponent, the quantity $(x^3 - 8)^{\frac{2}{3}}$ is in the denominator, alerting us to a potential domain issue. Rewriting $(x^3 - 8)^{\frac{2}{3}}$ we set about solving $\sqrt[3]{(x^3 - 8)^2} = 0$. Cubing both sides and extracting square roots gives $x^3 - 8 = 0$ or $x = 2$. Hence, $x = 2$ is excluded from the domain.⁵ Since the root involved here is odd (3), the only issue we have is with the denominator, hence our domain is $\{x \in \mathbb{R} \mid x \neq 2\}$ or $(-\infty, 2) \cup (2, \infty)$.

While not required to do so, we analyze the behavior of f near $x = 2$. As $x \rightarrow 2^-$, $3x^2 \approx 12$ and $x^3 - 8 \approx \text{small } (-)$. Hence, $(x^3 - 8)^{\frac{2}{3}} = \sqrt[3]{(x^3 - 8)^2} \approx \sqrt[3]{(\text{small } (-))^2} \approx \sqrt[3]{\text{small } (+)} \approx \text{small } (+)$. As such, $f(x) \approx \frac{12}{\text{small } (+)} \approx \text{big } (+)$. We conclude $\lim_{x \rightarrow 2^-} f(x) = \infty$. As $x \rightarrow 2^+$, $3x^2 \approx 12$ and $x^3 - 8 \approx \text{small } (+)$, and we likewise get $\lim_{x \rightarrow 2^+} f(x) = \infty$. Since the unbounded behavior agrees from both directions as $x \rightarrow 2$, we write $\lim_{x \rightarrow 2} f(x) = \infty$. Our analysis suggests $x = 2$ is a vertical asymptote to the graph.

To find the x -intercepts, we set $f(x) = 3x^2(x^3 - 8)^{-\frac{2}{3}} = 0$, so that $3x^2 = 0$ or $x = 0$. We get $(0, 0)$ is our only x - (and y -)intercept.

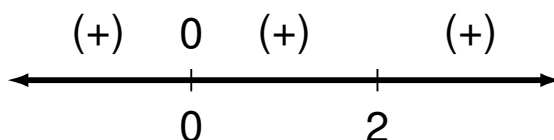
For end behavior, note that in the denominator the x^3 term dominates so as $x \rightarrow -\infty$ and $x \rightarrow \infty$,

$$f(x) = 3x^2(x^3 - 8)^{-\frac{2}{3}} = \frac{3x^2}{(x^3 - 8)^{\frac{2}{3}}} \approx \frac{3x^2}{(x^3)^{\frac{2}{3}}} = \frac{3x^2}{\sqrt[3]{(x^3)^2}} = \frac{3x^2}{\sqrt[3]{x^6}} = \frac{3x^2}{x^2} = 3.$$

This suggests $\lim_{x \rightarrow -\infty} f(x) = 3$ and $\lim_{x \rightarrow \infty} f(x) = 3$ so $y = 3$ is a horizontal asymptote to the graph.

Graphing $y = f(x)$ below on the right bears out our analysis regarding zeros and asymptotes. The range appears to be $[0, \infty)$, with the graph of $y = f(x)$ crossing its horizontal asymptote between $x = 1$ and $x = 2$. We see we have a single local minimum at $(0, 0)$ with f is decreasing on $(-\infty, 0]$ and $(2, \infty)$ and increasing on $[0, 2)$.

For the sign diagram, we note that f has only one zero, $x = 0$ and is undefined at $x = 2$. For all x values between these two numbers, $f(x) > 0$ or $(+)$. Our sign diagram for $f(x)$ is below on the right.



Desmos link: <https://www.desmos.com/calculator/nto7sdaomz>

2. To find the domain of $g(t) = \frac{(t^2 - 4)^{\frac{3}{2}}}{t^2 - 36}$, we have two issues to address: the denominator and an even (square) root. Solving $t^2 - 36 = 0$ gives two excluded values, $t = \pm 6$. For the numerator, we may rewrite $(t^2 - 4)^{\frac{3}{2}} = (\sqrt{t^2 - 4})^3$, so we require $t^2 - 4 \geq 0$, or $t^2 \geq 4$. Extracting square roots, we have

⁵In general if $u^p = 0$ where $p > 0$, then $u = 0$.

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$\sqrt{t^2} \geq \sqrt{4}$ or $|t| \geq 2$ which means $t \leq -2$ or $t \geq 2$. Taking into account our excluded values $t = \pm 6$, we get the domain of g is $(-\infty, -6) \cup (-6, -2] \cup [2, 6) \cup (6, \infty)$.

Looking near $t = -6$, we note that as $t \rightarrow -6$, $(t^2 - 4)^{\frac{3}{2}} \approx 32^{\frac{3}{2}} = 32^{1.5}$, a positive number. As $t \rightarrow -6^-$, $t^2 - 36 \approx \text{small } (+)$, so $g(t) \approx \frac{32^{1.5}}{\text{small}(+)} \approx \text{big } (+)$. This suggests $\lim_{t \rightarrow -6^-} g(t) = \infty$. On the other hand,

as $t \rightarrow -6^+$, $t^2 - 36 \approx \text{small } (-)$, so $g(t) \approx \frac{32^{1.5}}{\text{small}(-)} \approx \text{big } (-)$, suggesting $\lim_{t \rightarrow -6^+} g(t) = -\infty$. Similarly, we find as $\lim_{t \rightarrow 6^-} g(t) = -\infty$ and as $\lim_{t \rightarrow 6^+} g(t) = \infty$. This suggests we have two vertical asymptotes to the graph of $y = g(t)$: $t = -6$ and $t = 6$.

To find the t -intercepts, we set $g(t) = 0$ and solve $(t^2 - 4)^{\frac{3}{2}} = 0$. This reduces to $t^2 - 4 = 0$ or $t = \pm 2$. As these are (just barely!) in the domain of g , we have two t -intercepts, $(-2, 0)$ and $(2, 0)$. The graph of g has no y -intercepts, since 0 is not in the domain of g , so $g(0)$ is undefined.

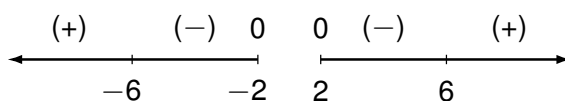
Regarding end behavior, as $t \rightarrow -\infty$ and $t \rightarrow \infty$, the t^2 in both numerator and denominator dominate the constant terms, so we have

$$g(t) = \frac{(t^2 - 4)^{\frac{3}{2}}}{t^2 - 36} \approx \frac{(t^2)^{\frac{3}{2}}}{t^2} = \frac{(\sqrt{t^2})^3}{t^2} = \frac{|t|^3}{t^2} = \frac{|t||t|^2}{t^2} = \frac{|t|t^2}{t^2} = |t|.$$

This suggests that as $t \rightarrow -\infty$ and $t \rightarrow \infty$, the graph of $y = g(t)$ resembles $y = |t|$. Hence, $\lim_{t \rightarrow -\infty} g(t) = \infty$ and $\lim_{t \rightarrow \infty} g(t) = \infty$. Using the piecewise definition of $|t|$, we have that as $t \rightarrow -\infty$, $g(t) \approx -t$ and as $t \rightarrow \infty$, $g(t) \approx t$. In other words, the graph of $y = g(t)$ has two slant asymptotes with slopes ± 1 .

Graphing $y = g(t)$ below on the left verifies our analysis. From the graph, the range appears to be $(-\infty, 0] \cup [14.697, \infty)$. The points $(-10, 14.697)$ and $(10, 14.697)$ are local minimums. g appears to be decreasing on $(-\infty, -10]$, $[2, 6)$, and $(6, 10]$. Likewise, g is increasing on $[-10, -6)$, $(-6, -2]$ and $[10, \infty)$. The graph of $y = g(t)$ certainly appears to be symmetric about the y -axis. We leave it to the reader to show g is, indeed, an even function.

For the sign diagram for $g(t)$, we note that g has zeros $t = \pm 2$ and is undefined at $t = \pm 6$. Moreover, there is a gap in the domain for all values in the interval $(-2, 2)$, so we excise that portion of the real number line for our discussion. We find $g(t) > 0$ or $(+)$ on the intervals $(-\infty, -6)$ and $(6, \infty)$ while $g(t) < 0$ or $(-)$ on $(-6, -2)$ and $(2, 6)$. Our sign diagram for $g(t)$ is below on the right.



Desmos link: <https://www.desmos.com/calculator/m2hzmagth8>



2 Real Number Exponents

We wish now to extend the concept of 'exponent' from rational to all real numbers which means we need to discuss how to interpret an irrational exponent. Once again, the notions presented here are best discussed using the language of Calculus or Analysis, but we nevertheless do what we can with the notions we have.

Consider the wildly famous irrational number π . The number π is defined geometrically as the ratio of the circumference of a circle to that circle's diameter.⁶ The reason we use the *symbol* ' π ' instead of any numerical expression is that π is an irrational number, and, as such, its decimal representation neither terminates nor repeats. Hence we *approximate* π as $\pi \approx 3.14$ or $\pi \approx 3.14159265$. No matter how many digits we write, however, what we have is a *rational number* approximation of π .

The good news is we can approximate π to any desired accuracy using rational numbers by taking enough digits, so while we'll never 'reach' the *exact* value of π with rational numbers, we can get as *close* as we like. Using rational numbers. That being said, we assume π exists on the real number line, despite the fact the list of digits to pinpoint its location is, in some sense, infinite.

⁶This works for each and every circle, by the way, regardless of how large or small the circle is!

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We take this tack when defining the value of a number raised to an irrational exponent. Consider, for instance, 2^π . We can compute $2^3 = 8$, $2^{3.1} = 2^{\frac{31}{10}} = \sqrt[10]{2^{31}} \approx 8.574$, $2^{3.14} = 2^{\frac{314}{100}} = 2^{\frac{157}{50}} = \sqrt[50]{2^{157}} \approx 8.8512$, and so on, so one way to *define* 2^π as the unique real number we obtain as the exponents ‘approach’ π .

It is with this understanding that we present the notion of a ‘power function,’ as described in Definition ??: $f(x) = ax^p$ where a and p are nonzero real number parameters. Here the exponent p is open to any (nonzero) real number. Because of how we define real number exponents, if p is irrational, then $x \geq 0$ to avoid having negatives under even-indexed roots as we go through the approximation process.⁷

In general, real number exponents inherit their properties from rational number exponents. For instance, Theorem ?? also holds for all real number exponents and the graphs of power functions inherit their behavior from graphs of rational exponent functions. More specifically, the graphs of functions of the form $f(x) = x^p$ where $p > 0$ all contain the points $(0, 0)$ and $(1, 1)$. Moreover, these functions are increasing and their graphs are concave down if $0 < p < 1$ and concave up if $p > 1$.

Desmos link: <https://www.desmos.com/calculator/sjuhbgu78>

Theorem ?? generalizes to real number power functions, so, for instance to graph $F(x) = (x - 2)^\pi$, one need only start with $y = x^\pi$ and shift horizontally two units to the right. (See the Exercises.)

We close this section with an application to economics. According to the [US Census](#), Table 2, the share of money income (2014-2015) is given in the table below on the left. From these data, we can create a cumulative distribution, $y = L(x)$ called the **Lorenz Curve**.

The number $L(x)$ gives the percentage of the total national income earned by the bottom x percent of wage earners, ranked from lowest income to highest income. Since the population here is separated into ‘quintiles,’ each data point corresponds to 20% of the population. So, for example, $L(20)$ is the percentage of money income earned by the lowest 20% of wage earners. In this case, we see $L(20) = 3.1$. The number $L(40)$ is the percentage of the money income earned by the bottom 40% of wage earners - so this includes not only the money from the Second Quintile, but also the Lowest Quintile: $L(40) = 8.2 + L(20) = 8.2 + 3.1 = 11.3$. Likewise, $L(60)$ is the total income share of the bottom 60% of wage earners which includes the income from the Middle, Second, and Lowest Quintiles: $L(60) = 14.3 + L(40) = 14.3 + (8.2 + 3.1) = 25.6$.

Continuing in this manner, we get $L(80) = 48.8$ and $L(100) = 100$, which is what we would expect: 100% of the income is earned by 100% of the population. We summarize these findings below on the right.

Portion of Population	Percent of Money Income	percent wage earners, x	percent income, $L(x)$
Lowest Quintile	3.1	20	3.1
Second Quintile	8.2	40	11.3
Middle Quintile	14.3	60	25.6
Fourth Quintile	23.2	80	48.8
Highest Quintile	51.2	100	100

Example 3.

1. Find power function to model the Lorenz Curve: $L(x) = ax^p$. Comment on the goodness of fit.
2. Find and interpret $L(90)$.

Solution.

1. Using [desmos](#), we get $L(x) = 0.00027901x^{2.7738}$ with $R^2 = 0.993$, indicating a pretty good fit.

Desmos link: <https://www.desmos.com/calculator/nqjv192ytp>

2. We compute $L(90) = 0.00027901(90)^{2.7738} \approx 73.5$ meaning that the bottom 90% of the wage earners brought home 73.5% of the total income. Said differently, the top 10% of wage earners made over 25% of the total national income. ■

⁷or $x > 0$ if p is negative.