ROOT AND RADICAL FUNCTIONS

In Sections **??**, **??** and **??**, we studied constant, linear, absolute value,[1] and quadratic functions. Constant, linear and quadratic functions were specific examples of polynomial functions, which we studied in generality in Chapter **??**. Chapter **??** culminated with the Real Factorization Theorem, Theorem **??**, which says that all polynomial functions with real coefficients can be thought of as products of linear and quadratic functions. Our next step was to enlarge our field[2] of study to rational functions in Chapter **??**. Being quotients of polynomials, we can ultimately view this family of functions as being built up of linear and quadratic functions as well. So in some sense, Sections **??**, **??** and **??** along with Chapters **??** and **??** can be thought of as an exhaustive study of linear and quadratic[3] functions. We now turn our attention to functions involving radicals which cannot be written in terms of linear functions. For a more detailed review of the basics of roots and radicals, we refer the reader to Sections **??** and **??**.

1 Root Functions

As with polynomial functions and rational functions, we begin our study of functions involving radical with a special family of functions: the (principal) root functions.

Definition 1. Let $n \in \mathbb{N}$ with $n \geq 2$. The n **th (principal) root function** is the function $f(x) = \sqrt[n]{x}$.

NOTE: If n is even, the domain of f is $[0, \infty)$; if n is odd, the domain of f is $(-\infty, \infty)$.

The domain restriction for even indexed roots means that, once again, we are restricting our attention to *real* numbers.[4] We graph a few members of the root function family below using Desmos. As with the monomial, and, more generally, the Laurent monomial functions, the behavior of the root functions depends primarily on whether the root is even or odd.

As we select the graphs for indices $n = 2$, $n = 4$, and $n = 6$, we observe that in addition to each function having the common domain $[0, \infty)$, the graphs all share the points $(0, 0)$ and $(1, 1)$. As the index n increases, the functions become 'steeper' near the y-axis and 'flatter' as $x \to \infty$. These characteristics are common to all functions of the form $f(x) = \sqrt[n]{x}$.

While the graphs appear to 'flatten out' as $x \to \infty$, it turns out that $f(x) \to \infty$ as $x \to \infty$. First note that if $c \geq 0$ is a real number, then $f(c^n) = \sqrt[n]{c^n} = c$ so c is in the range of f. This shows the range of f is $[0, \infty)$. Next note that f is increasing: that is, if $a < b$, then $f(a) = \sqrt[n]{a} < \sqrt[n]{b} = f(b)$. (This property is useful in solving certain types of polynomial inequalities.[5]) Hence, we get that $\lim_{x\to\infty} f(x) = \infty$.

Desmos link: `https://www.desmos.com/calculator/twightirsz`

The functions $f(x) = \sqrt[n]{x}$ for odd natural numbers $n \geq 3$ also follow a predictable trend - steepening near $x = 0$ and flattening as $x \to -\infty$ and $x \to -\infty$. The range for these functions is $(-\infty, \infty)$ since if c is any real number, $f(c^n) = \sqrt[n]{c^n} = c$, so c is in the range of f. Like the even indexed roots, the odd indexed roots are also increasing. This gives us that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.

Moreover, these graphs appear to be symmetric about the origin. Sure enough, when n is odd, $f(-x) = \sqrt[n]{-x} = -\sqrt[n]{x} = -f(x)$ so f is an odd function.

Desmos link: `https://www.desmos.com/calculator/6cw2sm1l0h`

At this point, you're probably expecting a theorem like Theorems **??**, **??**, **??**, **??** - that is, a theorem which tells us how to obtain the graph of $F(x) = a\sqrt[n]{x-h} + k$ from the graph of $f(x) = \sqrt[n]{x}$ - and you would not

[1] These were introduced, as you may recall, as piecewise-defined linear functions.

[2] This is a really bad math pun.

[3] If we broaden our concept of functions to allow for complex valued coefficients, the Complex Factorization Theorem, Theorem **??**, tells us every function we have studied thus far is a combination of linear functions.

[4] Although we discussed imaginary numbers in Section **??**, we restrict our attention to real numbers in this section. See the epilogue on page **??** for more details.

[5] See Exercise **??**.

be wrong. Here, however, we need to add an extra parameter 'b' to the recipe and discuss functions of the form $F(x) = a\sqrt[n]{bx-h}+k$. The reason is that, with all of the previous function families, we were always able to factor out the coefficient of x. We list some examples of this below, and invite the reader to revisit other examples in the text:

- $F(x) = |6-2x| = |-2x+6| = |-2(x+3)| = |-2||x+3| = 2|x+3|$.
- $F(x) = (2x-1)^2+1 = \left[2\left(x-\frac{1}{2}\right)\right]^2+1 = (2)^2\left(x-\frac{1}{2}\right)^2+1 = 4\left(x-\frac{1}{2}\right)^2+1$
- $F(x) = \dfrac{2}{(1-x)^3}-5 = \dfrac{2}{[(-1)(x-1)]^3}-5 = \dfrac{2}{(-1)^3(x-1)^3}-5 = \dfrac{2}{-(x-1)^3}-5 = \dfrac{-2}{(x-1)^3}-5$.

For a function like $F(x) = \sqrt{4x-12}+1 = \sqrt{4(x-6)}+1 = \sqrt{4}\sqrt{x-3}+1 = 2\sqrt{x-3}+1$, this approach works fine. However, if the coefficient of x is *negative*, for example, $F(x) = \sqrt{1-x} = \sqrt{(-1)(x-1)}$ we get stuck the product rule for radicals doesn't extend to negative quantities when the index is even.[6] Hence we add an extra parameter which means we have an extra step. We state Theorem **??** below.

Theorem 1. For real numbers a, b, h, and k with $a, b \neq 0$, the graph of $F(x) = a\sqrt[n]{bx-h}+k$ can be obtained from the graph of $f(x) = \sqrt[n]{x}$ by performing the following operations, in sequence:

1. add h to each of the x-coordinates of the points on the graph of f. This results in a horizontal shift to the right if $h > 0$ or left if $h < 0$.

   NOTE: This transforms the graph of $y = \sqrt[n]{x}$ to $y = \sqrt[n]{x-h}$.
2. divide the x-coordinates of the points on the graph obtained in Step 1 by b. This results in a horizontal scaling, but may also include a reflection about the y-axis if $b < 0$.

   NOTE: This transforms the graph of $y = \sqrt[n]{x-h}$ to $y = \sqrt[n]{bx-h}$.
3. multiply the y-coordinates of the points on the graph obtained in Step 2 by a. This results in a vertical scaling, but may also include a reflection about the x-axis if $a < 0$.

   NOTE: This transforms the graph of $y = \sqrt[n]{bx-h}$ to $y = a\sqrt[n]{bx-h}$.
4. add k to each of the y-coordinates of the points on the graph obtained in Step 3. This results in a vertical shift up if $k > 0$ or down if $k < 0$.

   NOTE: This transforms the graph of $y = a\sqrt[n]{bx-h}$ to $y = a\sqrt[n]{bx-h}+k$.

Proof. As usual, we 'build' the graph of $F(x) = a\sqrt[n]{bx-h}+k$ starting with the graph of $f(x) = \sqrt[n]{x}$ one step at a time. First, we consider the graph of $F_1(x) = \sqrt{x-h}$. A generic point on the graph of F_1 looks like $(x, \sqrt[n]{x-h})$. Note that if n is odd, x can be any real number whereas if n is even $x-h \geq 0$ so $x \geq h$. If we let $c = x-h$, then $x = c+h$ and we can change (dummy) variables[7] and obtain a new representation of the point: $(c+h, \sqrt[n]{c})$. Note that if n is odd, x and c vary through all real numbers; if n is even, $x \geq h$ and, hence, $c \geq 0$. Since a generic point on the graph of $f(x) = \sqrt[n]{x}$ can be represented as $(c, \sqrt[n]{c})$ for applicable values of c, we see that we can obtain every point on the graph of F_1 by adding h to each x-coordinate of the graph of f, establishing step 1 of the theorem.

Proceeding to (the new!) step 2, a point on the graph of $F_2(x) = \sqrt[n]{bx-h}$ has the form $(x, \sqrt[n]{bx-h})$. If n is odd, as usual, x can vary through all real numbers. If n is even, we require $bx-h \geq 0$ or $bx \geq h$. If $b > 0$, this gives $x \geq \dfrac{h}{b}$. If, on the other hand, $b < 0$, the we have $x \leq \dfrac{h}{b}$. Let $c = bx$ and since by assumption $b \neq 0$, we have $x = \dfrac{c}{b}$. Once again, we change dummy variables from x to c and describe a generic point on the graph of F_2 as $\left(\dfrac{c}{b}, \sqrt[n]{c-h}\right)$. If n is odd, x and c can vary through all real numbers. If n is even and $b > 0$,

[6] Since, otherwise, $-1 = i^2 = i \cdot i = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$, a contradiction.

[7] again this is because every real number can be represented as both $x-h$ for some value x and as $c+h$ for some value c.

then $x \geq \frac{h}{b}$ and, hence, $c = bx \geq h$; if $b < 0$, then $x \leq \frac{h}{b}$ also gives $c = bx \geq h$. Since a generic point on the graph of F_1 can be represented as $(c, \sqrt{c-h})$ for applicable values of c, we see we can obtain every point on the graph of F_2 by dividing every x-coordinate on the graph of F_1 by b, as per step 2 of the theorem.

The proof of steps 3 and 4 of Theorem **??** are identical to the proof of Theorem **??** (just with $\sqrt[n]{\cdot}$ instead of $(\cdot)^n$) so we invite the reader to work through the details on their own. ■

We demonstrate Theorem **??** in the following example.

Example 1. Theorem **??** to graph the following functions. Label at least three points on the graph. State the domain and range using interval notation.

1. $f(x) = 1 - 2\sqrt[3]{x+3}$
2. $g(t) = \dfrac{\sqrt{1-2t}}{4}$

Solution.

1. We begin by rewriting the expression for $f(x)$ in the form prescribed Theorem **??**: $f(x) = -2\sqrt[3]{x+3} + 1$. We identify $n = 3$, $a = -2$, $b = 1$, $h = -3$ and $k = 1$. GeoGebra helps us visualize the steps prescribed in Theorem **??** to transform the graph of $y = \sqrt[3]{x}$ to the graph of $y = f(x)$. We track the points $(-1, 1)$, $(0, 0)$ and $(1, 1)$ through each step.

   Step 1: add -3 to each of the x-coordinates of each of the points on the graph of $y = \sqrt[3]{x}$:

   Since $b = 1$, we can proceed to Step 3 (since dividing a real by 1 just results in the same real number.)

   Step 3: multiply each of the y-coordinates of each point on the graph of $y = \sqrt[3]{x+3}$ by -2:

   Step 4: add 1 to y-coordinates of each point on the graph of $y = -2\sqrt[3]{x+3}$:

   GeoGebra link: https://www.geogebra.org/m/gfjgcbsp

   We get the domain and range of f are $(-\infty, \infty)$.

2. For $g(t) = \dfrac{\sqrt{1-2t}}{4} = \dfrac{1}{4}\sqrt{-2t+1}$, we identify $n = 2$, $a = \frac{1}{4}$, $b = -2$, $h = -1$ and $k = 0$. Once again, GeoGebra helps us visualize the the steps taken to transform the graph of $y = \sqrt{t}$ to the graph of $y = g(t)$. Since we are asked to label *three* points on the graph, we track $(4, 2)$ along with $(0, 0)$ and $(1, 1)$.[8]

   Step 1: add -1 to each of the t-coordinates of each of the points on the graph of $y = \sqrt{t}$:

   Step 2: divide each of the t-coordinates of each of the points on the graph of $y = \sqrt{t+1}$ by -2:

   Step 3: multiply each of the y-coordinates of each of the points on the graph of $y = \sqrt{-2t+1}$ by $\frac{1}{4}$:

   GeoGebra link: https://www.geogebra.org/m/q5zycxxw

   We get the domain is $\left(-\infty, \frac{1}{2}\right]$ and the range is $[0, \infty)$. ■

2 Other Functions involving Radicals

Now that we have some practice with basic root functions, we turn our attention to more general functions involving radicals. In general, Calculus is the best tool with which to study these functions. Nevertheless, we will use what algebra we know in combination with a graphing utility to help us visualize these functions and preview concepts which are studied in greater depth in later courses. In the table below, we summarize some of the properties of radicals from elsewhere in this text (and Intermediate Algebra) we will be using in the coming examples.

[8] As $\sqrt{4} = 2$, we know $(4, 2)$ is on the graph of $y = \sqrt{t}$.

Theorem 2. Some Useful Properties of Radicals: Suppose $\sqrt[n]{x}$, $\sqrt[n]{a}$, and $\sqrt[n]{b}$ are real numbers.[9]

Simplifying n th powers and n th roots:[10]

- $\left(\sqrt[n]{x}\right)^n = x$.
- if n is odd, then $\sqrt[n]{x^n} = x$
- if n is even, then $\sqrt[n]{x^n} = |x|$.

Root Functions Preserve Inequality:[11] if $a \leq b$, then $\sqrt[n]{a} \leq \sqrt[n]{b}$.

Example 2. For the following functions:

- Analytically:
  - find the domain.
  - find the axis intercepts.
  - analyze the end behavior.

[9] i.e., if n is odd, x, a, and b can be any real numbers; if, on the other hand n is even, $x \geq 0$, $a \geq 0$, and $b \geq 0$.
[10] a.k.a., 'Inverse Properties.' See Section **??**.
[11] i.e., root functions are increasing.

- Graph the function with help from a graphing utility and determine:
  - the range.
  - the local extrema, if they exist.
  - intervals of increase.
  - intervals of decrease.
- Construct a sign diagram for each function using the intercepts and graph.[12]

1. $f(x) = 3x\sqrt[3]{2 - x}$
2. $g(t) = \sqrt[3]{\dfrac{8t}{t + 1}}$
3. $h(x) = \dfrac{3x}{\sqrt{x^2 + 1}}$
4. $r(t) = t^{-1}\sqrt{16t^4 - 1}$

Solution.

1. When looking for the domain, we have two thing to watch out for: denominators (which we must make sure aren't 0) and even indexed radicals (whose radicands we must ensure are nonnegative.) Looking at the expression for $f(x)$, we have no denominators nor do we have an even indexed radical, so we are confident the domain is all real numbers, $(-\infty, \infty)$.

   To find the x-intercepts, we find the zeros of f by solving $f(x) = 3x\sqrt[3]{2 - x} = 0$. Using the zero product property, we get $3x = 0$ or $\sqrt[3]{2 - x} = 0$. The former gives $x = 0$ and to solve the latter, we cube both sides and get $2 - x = 0$ or $x = 2$. Hence, the x-intercepts are $(0, 0)$ and $(2, 0)$. Since $(0, 0)$ is also on the y-axis and functions can can have at most one y-intercept, we know $(0, 0)$ is the only y-intercept.[13] That being said, we can quickly verify $f(0) = 3(0)\sqrt[3]{2 - 0} = 0$.

   To determine the end behavior, we first consider $f(x)$ as $x \to \infty$. Using 'number sense,' [14] we have $f(x) = 3x\sqrt[3]{2 - x} = 3x\sqrt[3]{-x + 2} \approx (\text{big }(+))\sqrt[3]{\text{big }(-)} = (\text{big }(+))(\text{big }(-)) = \text{big }(-)$, so $\lim\limits_{x\to\infty} f(x) = -\infty$. As $x \to -\infty$ we get $f(x) = 3x\sqrt[3]{-x + 2} \approx (\text{big }(-))\sqrt[3]{\text{big }(+)} = (\text{big }(-))(\text{big }(+)) = \text{big }(-)$, so $\lim\limits_{x\to-\infty} f(x) = -\infty$ as well.

   We graph f below using Desmos. From the graph, the range appears to be $(-\infty, 3.572]$ with a local maximum (which also happens to be *the* maximum) at $(1.5, 3.572)$. We also see f appears to be increasing on $(-\infty, 1.5)$ and decreasing on $(1.5, \infty)$. It is also worth noting that there appears to be 'unusual steepness' near the x-intercept $(2, 0)$. We invite the reader to zoom in on the graph near $(2, 0)$ to see that the function is 'locally vertical.'[15]

   Desmos link: `https://www.desmos.com/calculator/b8jy0qbed8`

   To create a sign diagram for $f(x)$, we note that the function has zeros $x = 0$ and $x = 2$. For $x < 0$, $f(x) < 0$ or $(-)$, for $0 < x < 2$, $f(x) > 0$ or $(+)$, and for $x > 2$, $f(x) < 0$ or $(-)$. The sign diagram for $f(x)$ is below.

$(-)$	0	$(+)$	0	$(-)$
	0		2	

[12] We'll revisit sign diagrams for these functions in Section **??** where we will use them to solve inequalities (surprised?)
[13] Why is this, again?
[14] remember this means we use the adjective 'big' here to mean large in *absolute value*
[15] Of course, the Vertical Line Test prohibits the graph from actually *being* a vertical line. This behavior is more precisely defined and more closely studied in Calculus.

2. The index of the radical in the expression for $g(t)$ is odd, so our only concern is the denominator. Setting $t+1=0$ gives $t=-1$, which we exclude, so our domain is $\{t \in \mathbb{R} \mid t \neq -1\}$ or using interval notation, $(-\infty,-1) \cup (-1,\infty)$.

   If we take the time to analyze the behavior of g near $t=-1$, we find that as $t \to -1^-$, $g(t) = \sqrt[3]{\dfrac{8t}{t+1}} \approx \sqrt[3]{\dfrac{-8}{\text{small }(-)}} \approx \sqrt[3]{\text{big}(+)} = \text{big }(+)$. That is, $\lim\limits_{t \to -1^-} g(t) = \infty$. Likewise, as $t \to -1^+$, $g(t) \approx \sqrt[3]{\dfrac{-8}{\text{small }(+)}} \approx \sqrt[3]{\text{big }(-)} = \text{big }(-)$. This suggests $\lim\limits_{t \to -1^+} g(t) = -\infty$. This behavior points to a vertical asymptote, $t=-1$.

   To find the t-intercepts of the graph of g, we find the zeros of g by setting $g(t) = \sqrt[3]{\dfrac{8t}{t+1}} = 0$. Cubing both sides and clearing denominators gives $8t=0$ or $t=0$. Hence our t-, and in this case, y- intercept is $(0,0)$.

   To determine the end behavior, we note that as $t \to -\infty$ or $t \to \infty$, $\dfrac{8t}{t+1} \approx \dfrac{8t}{t} = 8$. Since $g(t) = \sqrt[3]{\dfrac{8t}{t+1}}$ it stands to reason that $\lim\limits_{t \to -\infty} g(t) = \sqrt[3]{8} = 2$ and, likewise, $\lim\limits_{t \to \infty} g(t) = 2$ This suggests the graph of $y=g(t)$ has a horizontal asymptote at $y=2$.

   We graph $y=g(t)$ below using Desmos. The graph confirms our suspicions about the asymptotes $t=-1$ and $y=2$. Moreover, the range appears to be $(-\infty,2) \cup (2,\infty)$.

   Desmos link: `https://www.desmos.com/calculator/utcpj39vk8`

   We could check if the graph ever crosses its horizontal asymptote by attempting to solve $g(t) = \sqrt[3]{\dfrac{8t}{t+1}} = 2$. Cubing both sides and clearing denominators gives $8t = 8(t+1)$ which gives $0=8$, a contradiction. This proves 2 is not in the range, as we had suspected.

   Scanning the graph, there appears to be no local extrema, and, moreover, the graph suggests g is increasing on $(-\infty,-1)$ and again on $(-1,\infty)$. As with the previous example, the graph appears locally vertical near its intercept $(0,0)$.

   To create a sign diagram for $g(t)$, we note that the function is undefined when $t=-1$ (so we place a '?' above it) and has a zero $t=0$. When $t<-1$, $g(t)>0$ or (+), for $-1<t<0$, $g(t)<0$ or $(-)$, and for $t>0$, $g(t)>0$ or (+). The sign diagram for $g(t)$ is below.

   (+) ? (−) 0 (+)

   −1 0

3. The expression for $h(x) = \dfrac{3x}{\sqrt{x^2+1}}$ has both a denominator and an even-indexed radical, so we have to be extra cautious here. Fortunately for us, the quantity $x^2+1>0$ for all real numbers x. Not only does this mean $\sqrt{x^2+1}$ is always defined, it also tells us $\sqrt{x^2+1}>0$ for all x, too. This means the domain of h is all real numbers, $(-\infty,\infty)$.

   Solving for the zeros of h gives only $x=0$, and we find, once again, $(0,0)$ is both our lone x- and y-intercept.

   Moving on to end behavior, as $x \to -\infty$ or $x \to \infty$, the term x^2 is the dominant term in the radicand in the denominator. As such, $h(x) = \dfrac{3x}{\sqrt{x^2+1}} \approx \dfrac{3x}{\sqrt{x^2}} = \dfrac{3x}{|x|}$. As $x \to -\infty$, $|x| = -x$ (since $x<0$) and hence, $h(x) \approx \dfrac{3x}{-x} = -3$, so $\lim\limits_{x \to -\infty} h(x) = -3$. As $x \to \infty$, $|x| = x$ (since $x>0$), so $h(x) \approx \dfrac{3x}{x} = 3$, so $\lim\limits_{x \to \infty} h(x) = 3$.

This analysis suggests the graph of $y = h(x)$ has not one, but *two* horizontal asymptotes.[16] The graph of h below bears this out.

Desmos link: `https://www.desmos.com/calculator/gkloclketw`

From the graph, we see the range of h appears to be $(-3, 3)$. Attempting to solve $h(x) = \dfrac{3x}{\sqrt{x^2+1}} = -3$ or $h(x) = \dfrac{3x}{\sqrt{x^2+1}} = 3$ gives, in either case, $9x^2 = 9(x^2+1)$ which reduces to $0 = 9$, a contradiction. Hence, the graph of $y = h(x)$ never reaches its horizontal asymptotes. Moreover, h appears to be always increasing, with no local extrema or 'unusual' steepness. One last remark: it appears as if the graph of h is symmetric about the origin. We check $h(-x) = \dfrac{3(-x)}{\sqrt{(-x)^2+1}} = -\dfrac{3x}{\sqrt{x^2+1}} = -h(x)$ which verifies h is odd.

Since the domain of h is all real number and the only zero of h is $x = 0$, the sign diagram for $h(x)$ is fairly straight forward. For $x < 0$, $h(x) < 0$ or $(-)$ and for $x > 0$, $h(x) > 0$ or $(+)$. The sign diagram for $h(x)$ is below.

$$\begin{array}{ccc} (-) & 0 & (+) \\ \longleftarrow & \vert & \longrightarrow \\ & 0 & \end{array}$$

4. The first thing to note about the expression $r(t) = t^{-1}\sqrt{16t^4-1}$ is that $t^{-1} = \dfrac{1}{t}$. Hence, we must exclude $t = 0$ from the domain straight away. Next, we have an even-indexed radical expression: $\sqrt{16t^4-1}$. In order for this to return a real number, we require $16t^4 - 1 \geq 0$. Instead of using a sign diagram to solve this,[17] we opt instead to *carefully* use properties of radicals. Isolating t^4, we have $t^4 \geq \dfrac{1}{16}$. Since the root functions are increasing, we can apply the fourth root to both sides and preserve the inequality: $\sqrt[4]{t^4} \geq \sqrt[4]{\dfrac{1}{16}}$ which gives[18] $|t| \geq \dfrac{1}{2}$. Note that since $t = 0$ does *not* satisfy this inequality, restricting t in this manner takes care of *both* domain issues, so the domain is $\left(-\infty, -\dfrac{1}{2}\right] \cup \left[\dfrac{1}{2}, \infty\right)$.

   Next, we look for zeros. Setting $r(t) = t^{-1}\sqrt{16t^4-1} = \dfrac{\sqrt{16t^4-1}}{t} = 0$ gives $\sqrt{16t^4-1} = 0$. After squaring both sides, we get $16t^4 - 1 = 0$ or $t^4 = \dfrac{1}{16}$. Extracting fourth roots, we get $t = \pm\dfrac{1}{2}$. Both of these are (barely!) in the domain of r, so our t intercepts are $\left(-\dfrac{1}{2}, 0\right)$ and $\left(\dfrac{1}{2}, 0\right)$. Note, the graph of r has no y-intercept, since $r(0)$ is undefined ($t = 0$ is not in the domain of r).

   Concerning end behavior, we note the term $16t^4$ dominates the radicand $\sqrt{16t^4-1}$ as $t \to -\infty$ or $t \to \infty$, hence, $r(t) = \dfrac{\sqrt{16t^4-1}}{t} \approx \dfrac{\sqrt{16t^4}}{t} = \dfrac{4t^2}{t} = 4t$. This suggests the graph of $y = r(t)$ has a slant asymptote with slope 4.[19] At this point, we can at least write $\lim\limits_{t\to-\infty} r(t) = -\infty$. and $\lim\limits_{t\to\infty} r(t) = \infty$.

   We graph $y = r(t)$ below. We see the range appears to be all real numbers, $(-\infty, \infty)$. It appears as if r is increasing on $\left(-\infty, -\dfrac{1}{2}\right]$ and again on $\left[\dfrac{1}{2}, \infty\right)$. The graph does appear to be asymptotic to $y = 4t$,

[16]We warned you this was coming . . . see the discussion following Theorem **??** in Section **??**.

[17]See Section **??**

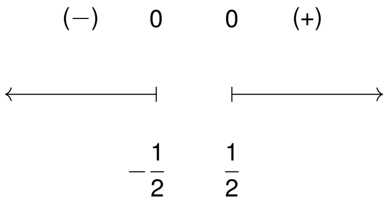
[18]Recall: $\sqrt[n]{x^n} = |x|$, not x, if n is even.

[19]Note: this analysis suggests the slant asymptote is $y = 4t + b$, but from this analysis, we cannot determine the value of b. As with slant asymptotes in Section **??**, we'd need to perform a more detailed analysis which we omit in this case owing to the complexity of the function. (You'll have the tools in Calculus, however!)

and it also appears to be symmetric about the origin. Sure enough, we find $r(-t) = \dfrac{\sqrt{16(-t)^4 - 1}}{-t} = -\dfrac{\sqrt{16t^4 - 1}}{t} = -r(t)$, proving r is an odd function.

Desmos link: `https://www.desmos.com/calculator/usuiezrrvm`

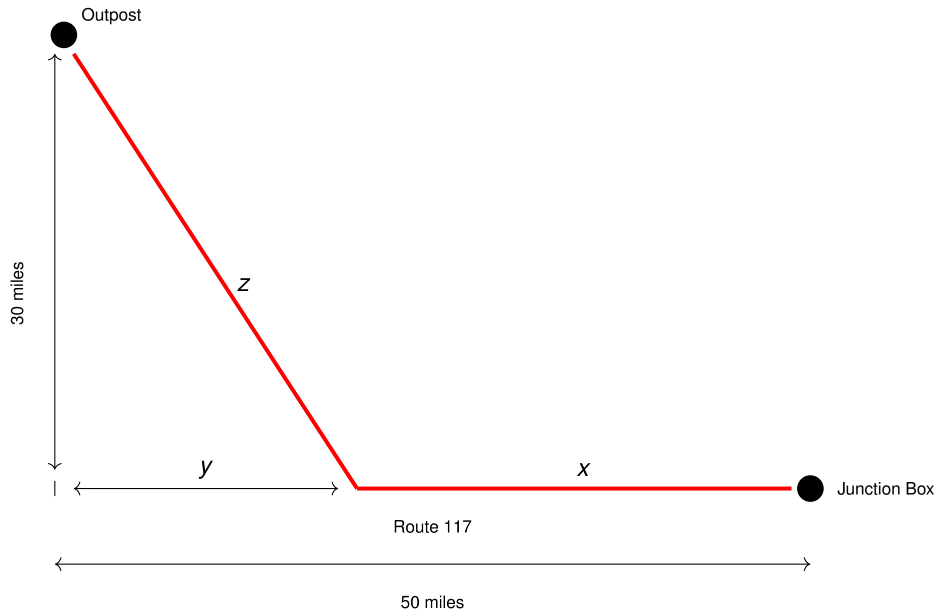
To construct the sign diagram for $r(t)$ we note r has two zeros, $t = \pm\frac{1}{2}$. For $t < \frac{1}{2}$, $r(t) < 0$ or $(-)$ and when $t > \frac{1}{2}$, $r(t) > 0$ or $(+)$. When $-\frac{1}{2} < t < \frac{1}{2}$, r is undefined so we have removed that segment from the diagram, as seen below.



■

We end this section with a classic application of root functions.

Example 3. Carl wishes to get high speed internet service installed in his remote Sasquatch observation post located 30 miles from Route 117. The nearest junction box is located 50 miles down the road from the post, as indicated in the diagram below. Suppose it costs \$15 per mile to run cable along the road and \$20 per mile to run cable off of the road.



1. Find an expression $C(x)$ which computes the cost of connecting the Junction Box to the Outpost as a function of x, the number of miles the cable is run along Route 117 before heading off road directly towards the Outpost. Determine a reasonable applied domain for the problem.

2. Use your calculator to graph $y = C(x)$ on its domain. What is the minimum cost? How far along Route 117 should the cable be run before turning off of the road?

Solution.

1. The cost is broken into two parts: the cost to run cable along Route 117 at \$15 per mile, and the cost to run it off road at \$20 per mile. Since x represents the miles of cable run along Route 117, the cost for that portion is $15x$. From the diagram, we see that the number of miles the cable is run off road is z, so the cost of that portion is $20z$. Hence, the total cost is $15x + 20z$.

   Our next goal is to determine z in terms of x. The diagram suggests we can use the Pythagorean Theorem to get $y^2 + 30^2 = z^2$. But we also see $x + y = 50$ so that $y = 50 - x$. Substituting $(50 - x)$ in for y we obtain $z^2 = (50 - x)^2 + 900$. Solving for z, we obtain $z = \pm\sqrt{(50 - x)^2 + 900}$. Since z represents a distance, we choose $z = \sqrt{(50 - x)^2 + 900}$.

   Hence, the cost as a function of x is given by $C(x) = 15x + 20\sqrt{(50 - x)^2 + 900}$. From the context of the problem, we have $0 \leq x \leq 50$.

2. As we adjust the slider for x on the GeoGebra interactive below, we can not only see the path and cost breakdown for the internet installation, we also see the corresponding point on the graph of $y = C(x)$:

   GeoGebra link: `https://www.geogebra.org/m/usjkbqza`

   We find our (local) minimum to be at the point $(15.98, 1146.86)$. Here the x-coordinate tells us that in order to minimize cost, we should run 15.98 miles of cable along Route 117 and then turn off of the road and head towards the outpost. The y-coordinate tells us that the minimum cost, in dollars, to do so is \$1146.86. The ability to stream live SasquatchCasts? Priceless.

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