

EXERCISES FOR INDUCTION

Question 1 In Exercises ?? - ??, prove each assertion using the Principle of Mathematical Induction.

Problem 1.1 $\sum_{j=1}^n j^2 = \frac{n(n+1)(2n+1)}{6}$

Let $P(n)$ be the sentence $\sum_{j=1}^n j^2 = \frac{n(n+1)(2n+1)}{6}$. For the base case, $n = 1$, we get

$$\begin{aligned} \sum_{j=1}^1 j^2 & \stackrel{?}{=} \frac{(1)(1+1)(2(1)+1)}{6} \\ 1^2 & = 1 \checkmark \end{aligned}$$

We now assume $P(k)$ is true and use it to show $P(k+1)$ is true. We have

$$\begin{aligned} \sum_{j=1}^{k+1} j^2 & \stackrel{?}{=} \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \\ \sum_{j=1}^k j^2 + (k+1)^2 & \stackrel{?}{=} \frac{(k+1)(k+2)(2k+3)}{6} \\ \underbrace{\frac{k(k+1)(2k+1)}{6}}_{\text{Using } P(k)} + (k+1)^2 & \stackrel{?}{=} \frac{(k+1)(k+2)(2k+3)}{6} \\ \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} & \stackrel{?}{=} \frac{(k+1)(k+2)(2k+3)}{6} \\ \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} & \stackrel{?}{=} \frac{(k+1)(k+2)(2k+3)}{6} \\ \frac{(k+1)(k(2k+1) + 6(k+1))}{6} & \stackrel{?}{=} \frac{(k+1)(k+2)(2k+3)}{6} \\ \frac{(k+1)(2k^2 + 7k + 6)}{6} & \stackrel{?}{=} \frac{(k+1)(k+2)(2k+3)}{6} \\ \frac{(k+1)(k+2)(2k+3)}{6} & = \frac{(k+1)(k+2)(2k+3)}{6} \checkmark \end{aligned}$$

By induction, $\sum_{j=1}^n j^2 = \frac{n(n+1)(2n+1)}{6}$ is true for all natural numbers $n \geq 1$.

Problem 1.2 $\sum_{j=1}^n j^3 = \frac{n^2(n+1)^2}{4}$

Problem 1.3 $2^n > 500n$ for $n > 12$

Problem 1.4 $3^n \geq n^3$ for $n \geq 4$

Let $P(n)$ be the sentence $3^n > n^3$. Our base case is $n = 4$ and we check $3^4 = 81$ and $4^3 = 64$ so that $3^4 > 4^3$ as required. We now assume $P(k)$ is true, that is $3^k > k^3$, and try to show $P(k+1)$ is true. We note that $3^{k+1} = 3 \cdot 3^k > 3k^3$ and so we are done if we can show $3k^3 > (k+1)^3$ for $k \geq 4$. We can solve the inequality $3x^3 > (x+1)^3$ using the techniques of Section ??, and doing so gives us $x > \frac{1}{\sqrt[3]{3}-1} \approx 2.26$. Hence, for $k \geq 4$, $3^{k+1} = 3 \cdot 3^k > 3k^3 > (k+1)^3$ so that $3^{k+1} > (k+1)^3$. By induction, $3^n > n^3$ is true for all natural numbers $n \geq 4$.

EXERCISES FOR INDUCTION

Problem 1.5 Use the Product Rule for Absolute Value to show $|x^n| = |x|^n$ for all real numbers x and all natural numbers $n \geq 1$

Let $P(n)$ be the sentence $\log(x^n) = n \log(x)$. For the duration of this argument, we assume $x > 0$. The base case $P(1)$ amounts checking that $\log(x^1) = 1 \log(x)$ which is clearly true. Next we assume $P(k)$ is true, that is $\log(x^k) = k \log(x)$ and try to show $P(k+1)$ is true. Using the Product Rule for Logarithms along with the induction hypothesis, we get

$$\log(x^{k+1}) = \log(x^k \cdot x) = \log(x^k) + \log(x) = k \log(x) + \log(x) = (k+1) \log(x)$$

Hence, $\log(x^{k+1}) = (k+1) \log(x)$. By induction $\log(x^n) = n \log(x)$ is true for all $x > 0$ and all natural numbers $n \geq 1$.

Problem 1.6 Use the Product Rule for Logarithms to show $\log(x^n) = n \log(x)$ for all real numbers $x > 0$ and all natural numbers $n \geq 1$.

Problem 1.7
$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix} \text{ for } n \geq 1.$$

Problem 2 Prove Equations ?? and ?? for the case of geometric sequences. That is:

1. For the sequence $a_1 = a$, $a_{n+1} = ra_n$, $n \geq 1$, prove $a_n = ar^{n-1}$, $n \geq 1$.

2. $\sum_{j=1}^n ar^{j-1} = a \left(\frac{1-r^n}{1-r} \right)$, if $r \neq 1$, $\sum_{j=1}^n ar^{j-1} = na$, if $r = 1$.

Problem 3 Prove that the determinant of a lower triangular matrix is the product of the entries on the main diagonal. (See Exercise ?? in Section ??.) Use this result to then show $\det(I_n) = 1$ where I_n is the $n \times n$ identity matrix.

Let A be an $n \times n$ lower triangular matrix. We proceed to prove the $\det(A)$ is the product of the entries along the main diagonal by inducting on n . For $n = 1$, $A = [a]$ and $\det(A) = a$, so the result is (trivially) true. Next suppose the result is true for $k \times k$ lower triangular matrices. Let A be a $(k+1) \times (k+1)$ lower triangular matrix. Expanding $\det(A)$ along the first row, we have

$$\det(A) = \sum_{p=1}^n a_{1p} C_{1p}$$

Since $a_{1p} = 0$ for $2 \leq p \leq k+1$, this simplifies $\det(A) = a_{11} C_{11}$. By definition, we know that $C_{11} = (-1)^{1+1} \det(A_{11}) = \det(A_{11})$ where A_{11} is $k \times k$ matrix obtained by deleting the first row and first column of A . Since A is lower triangular, so is A_{11} and, as such, the induction hypothesis applies to A_{11} . In other words, $\det(A_{11})$ is the product of the entries along A_{11} 's main diagonal. Now, the entries on the main diagonal of A_{11} are the entries $a_{22}, a_{33}, \dots, a_{(k+1)(k+1)}$ from the main diagonal of A . Hence,

$$\det(A) = a_{11} \det(A_{11}) = a_{11} (a_{22} a_{33} \cdots a_{(k+1)(k+1)}) = a_{11} a_{22} a_{33} \cdots a_{(k+1)(k+1)}$$

We have $\det(A)$ is the product of the entries along its main diagonal. This shows $P(k+1)$ is true, and, hence, by induction, the result holds for all $n \times n$ upper triangular matrices. The $n \times n$ identity matrix I_n is a lower triangular matrix whose main diagonal consists of all 1's. Hence, $\det(I_n) = 1$, as required.

Problem 4 Prove the Power Rule for Limits (see Theorem ?? in Section ??): $\lim_{x \rightarrow a} [f(x)]^n = \left[\lim_{x \rightarrow a} f(x) \right]^n = L^n$, where n is any natural number.

Problem 5 Discuss the classic 'paradox' [All Horses are the Same Color](#) problem with your classmates.