Exercises for Summation

Question 1 *In Exercises **??** - **??**, find the value of each sum using Definition **??**.*

Problem 1.1 $\sum_{g=4}^{9}(5g+3)$

213

Problem 1.2 $\sum_{k=3}^{8}\frac{1}{k}$

$\frac{341}{280}$

Problem 1.3 $\sum_{j=0}^{5}2^j$

63

Problem 1.4 $\sum_{k=0}^{2}(3k-5)x^k$

$-5-2x+x^2$

Problem 1.5 $\sum_{i=1}^{4}\frac{1}{4}(i^2+1)$

$\frac{17}{2}$

Problem 1.6 $\sum_{n=1}^{100}(-1)^n$

0

Problem 1.7 $\sum_{n=1}^{5}\frac{(n+1)!}{n!}$

20

Problem 1.8 $\sum_{j=1}^{3}\frac{5!}{j!\,(5-j)!}$

25

Question 2 *In Exercises **??** - **??**, rewrite the sum using summation notation.*

Problem 2.1 $8+11+14+17+20$

$\sum_{k=1}^{5}(3k+5)$

Problem 2.2 $1-2+3-4+5-6+7-8$

$\sum_{k=1}^{8}(-1)^{k-1}k$

Problem 2.3 $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$

$\sum_{k=1}^{4} (-1)^{k-1} \frac{x^{2k-1}}{2k-1}$

Problem 2.4 $1 + 2 + 4 + \cdots + 2^{29}$

$\sum_{k=1}^{30} 2^{k-1}$

Problem 2.5 $2 + \frac{3}{2} + \frac{4}{3} + \frac{5}{4} + \frac{6}{5}$

$\sum_{k=1}^{5} \frac{k+1}{k}$

Problem 2.6 $-\ln(3) + \ln(4) - \ln(5) + \cdots + \ln(20)$

$\sum_{k=3}^{20} (-1)^k \ln(k)$

Problem 2.7 $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36}$

$\sum_{k=1}^{6} \frac{(-1)^{k-1}}{k^2}$

Problem 2.8 $\frac{1}{2}(x-5) + \frac{1}{4}(x-5)^2 + \frac{1}{6}(x-5)^3 + \frac{1}{8}(x-5)^4$

$\sum_{k=1}^{4} \frac{1}{2k}(x-5)^k$

Question 3 *In Exercises* **??** - **??**, *use the formulas in Equation* **??** *to find the sum.*

Problem 3.1 $\sum_{n=1}^{10} 5n + 3$

305

Problem 3.2 $\sum_{n=1}^{20} 2n - 1$

400

Problem 3.3 $\sum_{k=0}^{15} 3 - k$

-72

Problem 3.4 $\sum_{n=1}^{10} \left(\frac{1}{2}\right)^n$

$\frac{1023}{1024}$

Problem 3.5 $\sum_{n=1}^{5} \left(\frac{3}{2}\right)^n$

$\frac{633}{32}$

Problem 3.6 $\sum_{k=0}^{5} 2\left(\frac{1}{4}\right)^k$

$\frac{1365}{512}$

Problem 3.7 $1 + 4 + 7 + ... + 295$

14652

Problem 3.8 $4 + 2 + 0 - 2 - ... - 146$

-5396

Problem 3.9 $1 + 3 + 9 + ... + 2187$

3280

Problem 3.10 $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{256}$

$\frac{255}{256}$

Problem 3.11 $3 - \frac{3}{2} + \frac{3}{4} - \frac{3}{8} + - \cdots + \frac{3}{256}$

$\frac{513}{256}$

Problem 3.12 $\sum_{n=1}^{10} -2n + \left(\frac{5}{3}\right)^n$

$\frac{17771050}{59049}$

Question 4 *In Exercises* **??** *-* **??***, use Theorem* **??** *to find the sum of the given geometric series.*[1]

Problem 4.1 $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$

2

Problem 4.2 $\sum_{n=0}^{\infty} \frac{(-1)^n 3^{n-1}}{4^n}$

$\frac{4}{21}$

Problem 4.3 $\sum_{m=2}^{\infty} \frac{3}{2^{m-1}}$

3

[1] Remember, when in doubt ...

Problem 4.4 $\sum_{k=0}^{\infty} x^k$, $|x| < 1$

$\frac{1}{1-x}$

Question 5 *In Exercises **??** - **??**, use Theorem **??** to express each repeating decimal as a fraction of integers.*

Problem 5.1 $0.\overline{7}$

$\frac{7}{9}$

Problem 5.2 $0.\overline{13}$

$\frac{13}{99}$

Problem 5.3 $10.\overline{159}$

$\frac{3383}{333}$

Problem 5.4 $-5.8\overline{67}$

$-\frac{5809}{990}$

Question 6 *In Exercises **??** - **??**, use Equation **??** to compute the future value of the annuity with the given terms. In all cases, assume the payment is made monthly, the interest rate given is the annual rate, and interest is compounded monthly.*

Problem 6.1 *payments are $300, interest rate is 2.5%, term is 17 years.*

$76,163.67

Problem 6.2 *payments are $50, interest rate is 1.0%, term is 30 years.*

$20,981.40

Problem 6.3 *payments are $100, interest rate is 2.0%, term is 20 years*

$29,479.69

Problem 6.4 *payments are $100, interest rate is 2.0%, term is 25 years*

$38,882.12

Problem 6.5 *payments are $100, interest rate is 2.0%, term is 30 years*

49,272.55

Problem 6.6 *payments are $100, interest rate is 2.0%, term is 35 years*

60,754.80

Problem 7 *Suppose an ordinary annuity offers an annual interest rate of* 2%, *compounded monthly, for 30 years. What should the monthly payment be to have* $100,000 *at the end of the term?*

For $100,000, *the monthly payment is* $\approx$ $202.95.

Question 8 *In this exercise, use Theorem **??** to represent* $f(x) = \dfrac{1}{x^2+4}$ *as a series.*

Problem 8.1 *Show that* $f(x) = \dfrac{\frac{1}{4}}{1-\left(-\frac{x^2}{4}\right)}$.

Problem 8.2 *Use the formula in Theorem **??**:* $\dfrac{a}{1-r} = \displaystyle\sum_{k=1}^{\infty} ar^{k-1}$ *to write* $f(x)$ *as an infinite series.*

$$f(x) = \frac{\frac{1}{4}}{1-\left(-\frac{x^2}{4}\right)} = \sum_{k=1}^{\infty} \frac{1}{4}\left(-\frac{x^2}{4}\right)^{k-1} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\, x^{2k-2}}{4^k}$$

Problem 8.3 *Graph* $y = f(x)$ *along with some partial sums of the series. What do you notice?*

No matter how many terms are added, the graph of the series seems to only account for a portion of the graph of $y = f(x)$*. This is due to the fact that geometric series converge only when the ratio* $|r| < 1$*. In this case,* $r = -\dfrac{x^2}{4}$ *so* $|r| < 1$ *corresponds to the interval* $(-2, 2)$*.*

Question 9 *Using Example **??** as a guide, find and simplify formula for the right endpoint sum,* RS_n*, for each of the functions below on the specified interval. Find* $\lim\limits_{n\to\infty} RS_n$ *to find the area between the graph of f and the x-axis.*

Problem 9.1 $f(x) = 4 - x$ *over the interval* $[0, 4]$.

$RS_n = 8 - \dfrac{8}{n}$*; Area is 8 units*^{2}

Problem 9.2 $f(x) = 3x^2$ *over the interval* $[1, 3]$.

$RS_n = 26 + \dfrac{24}{n} + \dfrac{4}{n^2}$*; Area is 26 units*^{2}

Problem 9.3 $f(x) = 12 - x - x^2$ *over the interval* $[0, 3]$.

$RS_n = \dfrac{45}{2} - \dfrac{18}{n} - \dfrac{9}{2n^2}$*; Area is* $\dfrac{45}{2}$ *units*^{2}

Problem 10 *Prove the properties listed in Theorem **??**.*

Problem 11 *Show that the formula for the future value of an annuity due is*

$$A = P(1+i)\left[\frac{(1+i)^{nt}-1}{i}\right]$$

Question 12 *Discuss with your classmates what goes wrong when trying to find the following sums.*[2]

Problem 12.1 $\displaystyle\sum_{k=1}^{\infty} 2^{k-1}$

Problem 12.2 $\displaystyle\sum_{k=1}^{\infty} (1.0001)^{k-1}$

[2] When in doubt . . .

Problem 12.3 $\sum_{k=1}^{\infty}(-1)^{k-1}$

Question 13 *In this exercise, we walk through the proof of Cauchy's Bound, Theorem* **??** *in Section* **??***.*
Let $f(x) = a_n x^n + a_n - 1x^n - 1 + ... + a_1 x + a_0$ *be a polynomial of degree* n *and let* Z *be the largest zero of* f *in absolute value and let* M *be the largest of the numbers:* $\frac{|a_0|}{|a_n|}, \frac{|a_1|}{|a_n|}, \ldots, \frac{|a_{n-1}|}{|a_n|}$.

Problem 13.1 *Since* $P(Z) = 0$*, solve for* Z^n*:* $Z^n = \frac{a_n - 1}{a_n} Z^n - 1 + ... + \frac{a_1}{a_n} Z + \frac{a_0}{a_n}$.

Problem 13.2 *If* $-1 \leq Z \leq 1$*, then Cauchy's Bound is immediately satisfied since* Z *would automatically lie in the interval* $[-(M+1), M+1]$*. So we assume* $|Z| > 1$*. Under the assumption* $|Z| > 1$*. explain why*[3]

$$\begin{aligned} |Z|^n &= \left|\frac{a_n - 1}{a_n} Z^n - 1 + ... + \frac{a_1}{a_n} Z + \frac{a_0}{a_n}\right| \\ &\leq \frac{|a_{n-1}|}{|a_n|} |Z|^{n-1} + ... + \frac{|a_1|}{|a_n|} |Z| + \frac{|a_0|}{|a_n|} \end{aligned}$$

Problem 13.3 *Use the definition of* M *along with the Geometric Sum Formula, Equation* **??** *to show:*

$$|Z|^n \leq M\left(|Z|^{n-1} + ... + |Z| + 1\right) = M\frac{1 - |Z|^n}{1 - |Z|} = M\frac{|Z|^n - 1}{|Z| - 1}$$

Problem 13.4 *Now use the fact that* $|Z| > 1$ *to rearrange the above inequality to get:*

$$|Z| - 1 \leq M\frac{|Z|^n - 1}{|Z|^n} = M\left(1 - \frac{1}{|Z|^n}\right)$$

Problem 13.5 *Use the fact that* $1 - \frac{1}{|Z|^n} < 1$ *to get:*

$$|Z| - 1 \leq M\left(1 - \frac{1}{|Z|^n}\right) < M(1) = M$$

Problem 13.6 *From* $|Z| - 1 < M$*, we get* $|Z| < M + 1$*. Hence,* Z *lies in the interval* $[-(M+1), M+1]$*.*

[3] Feel free to use the Triangle Inequality, as needed. See Exercise **??** in Section **??**.