SUMMATION NOTATION

In Section **??**, we showed how the formula for compound interest is a geometric sequence. In retirement planning, it is seldom the case that an investor deposits a set amount of money into an account and waits for it to grow. Usually, additional payments of principal are made at regular intervals and the value of the investment grows accordingly. This kind of investment is called an *annuity* and will be discussed in later in this section once we have developed more mathematical machinery that enables us to *add* sequences.

In the previous section, we introduced sequences. Each of the numbers in the sequence is called a 'term' which implies these numbers are meant to be added. To that end, we introduce the following notation which is used to describe the sum of (some of the) terms of a sequence.

Definition 1. Summation Notation: Given a sequence $\{a_n\}_{n=k}^{\infty}$ and numbers m and p satisfying $k \leq m \leq p$, the summation from m to p of the sequence $\{a_n\}$ is written

$$\sum_{n=m}^{p} a_n = a_m + a_{m+1} + \ldots + a_p$$

The variable n is called the **index of summation**. The number m is called the **lower limit of summation** while the number p is called the **upper limit of summation**.

In English, Definition 1 is simply defining a short-hand notation for adding up the terms of the sequence $\{a_n\}_{n=k}^{\infty}$ from a_m through a_p. The symbol Σ is the capital Greek letter sigma and is shorthand for 'sum'. The lower and upper limits of the summation tells us which term to start with and which term to end with, respectively. For example, using the sequence $a_n = 2n - 1$ for $n \geq 1$, we can write $a_3 + a_4 + a_5 + a_6$ as

$$\begin{aligned} \sum_{n=3}^{6}(2n-1) &= (2(3)-1)+(2(4)-1)+(2(5)-1)+(2(6)-1) \\ &= 5+7+9+11 \\ &= 32 \end{aligned}$$

The index variable is considered a 'dummy variable' in the sense that it may be changed to any letter without affecting the value of the summation. For instance,

$$\sum_{n=3}^{6}(2n-1) = \sum_{k=3}^{6}(2k-1) = \sum_{j=3}^{6}(2j-1)$$

One place you may encounter summation notation is in mathematical definitions. For example, summation notation allows us to define polynomials as functions of the form

$$f(x) = \sum_{k=0}^{n} a_k x^k$$

for real numbers a_k, $k = 0, 1, \ldots n$. The reader is invited to compare this with what is given in Definition **??**. Summation notation is particularly useful when talking about matrix operations. For example, we can write the product of the ith row R_i of a matrix $A = [a_{ij}]_{m\times n}$ and the j^{th} column C_j of a matrix $B = [b_{ij}]_{n\times r}$ as

$$Ri \cdot Cj = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Again, the reader is encouraged to write out the sum and compare it to Definition **??**. Our next example gives us practice with this new notation.

Example 1.

1. Find the following sums.

(i) $\displaystyle\sum_{k=1}^{4} \frac{13}{100^k}$ (ii) $\displaystyle\sum_{n=0}^{4} \frac{n!}{2}$ (iii) $\displaystyle\sum_{n=1}^{5} \frac{(-1)^{n+1}}{n}(x-1)^n$

2. Write the following sums using summation notation.

(i) $1 + 3 + 5 + ... + 117$

(ii) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - ... + \frac{1}{117}$

(iii) $0.9 + 0.09 + 0.009 + ... 0.\underbrace{0\cdots0}_{n-1 \text{ zeros}}9$

Solution.

1. (i) We substitute $k = 1$ into the formula $\frac{13}{100^k}$ and add successive terms until we reach $k = 4$.

$$\begin{aligned}\sum_{k=1}^{4} \frac{13}{100^k} &= \frac{13}{100^1} + \frac{13}{100^2} + \frac{13}{100^3} + \frac{13}{100^4} \\ &= 0.13 + 0.0013 + 0.000013 + 0.00000013 \\ &= 0.13131313\end{aligned}$$

(ii) Proceeding as in (a), we replace every occurrence of n with the values 0 through 4. We recall the factorials, $n!$ as defined in number Example **??**, number **??** and get:

$$\begin{aligned}\sum_{n=0}^{4} \frac{n!}{2} &= \frac{0!}{2} + \frac{1!}{2} + \frac{2!}{2} + \frac{3!}{2} = \frac{4!}{2} \\ &= \frac{1}{2} + \frac{1}{2} + \frac{2\cdot 1}{2} + \frac{3\cdot 2\cdot 1}{2} + \frac{4\cdot 3\cdot 2\cdot 1}{2} \\ &= \frac{1}{2} + \frac{1}{2} + 1 + 3 + 12 \\ &= 17\end{aligned}$$

(iii) We proceed as before, replacing the index n, but *not* the variable x, with the values 1 through 5 and adding the resulting terms.

$$\begin{aligned}\sum_{n=1}^{5} \frac{(-1)^{n+1}}{n}(x-1)^n &= \frac{(-1)^{1+1}}{1}(x-1)^1 + \frac{(-1)^{2+1}}{2}(x-1)^2 + \frac{(-1)^{3+1}}{3}(x-1)^3 \\ &\quad + \frac{(-1)^{1+4}}{4}(x-1)^4 + \frac{(-1)^{1+5}}{5}(x-1)^5 \\ &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5}\end{aligned}$$

2. The key to writing these sums with summation notation is to find the pattern of the terms. To that end, we make good use of the techniques presented in Section **??**.

(i) The terms of the sum 1, 3, 5, etc., form an arithmetic sequence with first term $a = 1$ and common difference $d = 2$. Using Equation **??**, we get $a_n = 1 + (n-1)2 = 2n - 1$, $n \geq 1$.

At this stage, we have the formula for the terms, namely $2n - 1$, and the lower limit of the summation, $n = 1$. To finish the problem, we need to determine the upper limit of the summation. In other words, we need to determine which value of n produces the term 117. Setting $a_n = 117$, we get $2n - 1 = 117$ or $n = 59$. Our final answer is

$$1 + 3 + 5 + ... + 117 = \sum_{n=1}^{59}(2n-1)$$

(ii) We rewrite all of the terms as fractions, the subtraction as addition, and associate the negatives '$-$' with the numerators to get

$$\frac{1}{1}+\frac{-1}{2}+\frac{1}{3}+\frac{-1}{4}+\ldots+\frac{1}{117}$$

The numerators, 1, -1, etc. can be described by the geometric sequence[1] $C_n = (-1)^{n-1}$ for $n \geq 1$, while the denominators are given by the arithmetic sequence[2] $D_n = n$ for $n \geq 1$. Hence, we get the formula $a_n = \dfrac{(-1)^{n-1}}{n}$ for our terms, and we find the lower and upper limits of summation to be $n = 1$ and $n = 117$, respectively. Thus

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+-\ldots+\frac{1}{117} \quad = \quad \sum_{n=1}^{117}\frac{(-1)^{n-1}}{n}$$

(iii) Thanks to Example **??**, we know that one formula for the n^{th} term is $a_n = \dfrac{9}{10^n}$ for $n \geq 1$. This gives us a formula for the summation as well as a lower limit of summation.

To determine the upper limit of summation, we note that to produce the $n-1$ zeros to the right of the decimal point before the 9, we need a denominator of 10^n. Hence, n is the upper limit of summation.

Since n is used in the limits of the summation, we need to choose a different letter for the index of summation.[3] We choose k and get

$$0.9+0.09+0.009+\ldots 0.\underbrace{0\cdots0}_{n-1\text{ zeros}}9 \quad = \quad \sum_{k=1}^{n}\frac{9}{10^k}$$

■

The following theorem presents some general properties of summation notation.

Theorem 1. Properties of Summation Notation: Suppose $\{a_n\}$ and $\{b_n\}$ are sequences so that the following sums are defined.

- **Sum and Difference Property:** $\displaystyle\sum_{n=m}^{p}(a_n \pm b_n) = \sum_{n=m}^{p}a_n \pm \sum_{n=m}^{p}b_n$
- **Distributive Property:** $\displaystyle\sum_{n=m}^{p}c\,a_n = c\sum_{n=m}^{p}a_n$, for any real number c.
- **Additive Index Property:** $\displaystyle\sum_{n=m}^{j}a_n + \sum_{n=j+1}^{p}a_n = \sum_{n=m}^{p}a_n$, for any natural number $m \leq j < j+1 \leq p$.
- **Re-indexing:** $\displaystyle\sum_{n=m}^{p}a_n = \sum_{n=m+r}^{p+r}a_{n-r}$, for any integer r.

There is much to be learned by thinking about why the properties hold, so we leave the proof of these properties to the reader.[4]

Example 2.

[1] This is indeed a geometric sequence with first term $a = 1$ and common ratio $r = -1$.
[2] It is an arithmetic sequence with first term $a = 1$ and common difference $d = 1$.
[3] To see why, try writing the summation using 'n' as the index.
[4] To get started, remember the mantra "When in doubt, write it out!"

1. If $\sum_{n=2}^{50}(a_n - 3b_n) = 17$ and $\sum_{n=2}^{50} a_n = 10$, find $\sum_{n=2}^{50} b_n$.
2. If $\sum_{n=1}^{20} a_n = -3$ and $\sum_{n=1}^{21} a_n = 7$, find a_{21}.
3. Rewrite the sum so the index starts at 0: $\sum_{n=2}^{437} n(n-1)x^{n-2}$

Solution.

1. Using the Sum and Difference Property along with the Distributive Property of Theorem 1, we get:

$$\sum_{n=2}^{50}(a_n - 3b_n) = \sum_{n=2}^{50} a_n - \sum_{n=2}^{50} 3b_n = \sum_{n=2}^{50} a_n - 3\sum_{n=2}^{50} b_n$$

Hence, $\sum_{n=2}^{50} a_n - 3\sum_{n=2}^{50} b_n = 17$. If $\sum_{n=2}^{50} a_n = 10$, then $10 - 3\sum_{n=2}^{50} b_n = 17$ so $\sum_{n=2}^{50} b_n = -\frac{7}{3}$.

2. There are at least two ways to approach this problem. By definition, $\sum_{n=1}^{21} a_n = a_1 + a_2 + ... + a_{21}$. That is, we add up the first 21 terms of the sequence a_n. Similarly, $\sum_{n=1}^{20} a_n = a_1 + a_2 + ... + a_{20}$ means we add up the first 20 terms of the sequence. Hence, $a_{21} = \sum_{n=1}^{21} a_n - \sum_{n=1}^{20} a_n = 7 - (-3) = 10$.

Alternatively, we can use the Additive Index Property:

$$\sum_{n=1}^{21} a_n = \sum_{n=1}^{20} a_n + \sum_{n=21}^{21} a_n = \sum_{n=1}^{20} a_n + a_{21},$$

which gives $a_{21} = \sum_{n=1}^{21} a_n - \sum_{n=1}^{20} a_n = 7 - (-3) = 10$ as well.

3. To re-index $\sum_{n=2}^{437} n(n-1)x^{n-2}$ so n starts at 0, we follow the formula in Theorem 2 with $r = -2$:

$$\sum_{n=2}^{437} n(n-1)x^{n-2} = \sum_{n=2+(-2)}^{437+(-2)} (n-(-2))(n-(-2)-1)x^{n-(-2)-2} = \sum_{n=0}^{435}(n+2)(n+1)x^n.$$

We leave it to the reader to check by writing out the first few, and last few, terms.

Alternatively, to better see *why* the re-indexing works in this way, we can introduce a new counter, k. We want this new counter to start at $k = 0$ whereas the current counter starts at $n = 2$, so we want $k = n - 2$. When $n = 2$, $k = 0$, as required, and when $n = 437$, $k = 435$.

Moreover, $n = k + 2$, so substituting this into the sum, we get

$$\sum_{n=2}^{437} n(n-1)x^{n-2} = \sum_{k=0}^{435}(k+2)((k+2)-1)x^{(k+2)-2} = \sum_{k=0}^{435}(k+2)(k+1)x^k,$$

which is the same sum we had before, just with a different dummy variable. ■

We now turn our attention to the sums involving arithmetic and geometric sequences. Given an arithmetic sequence $a_k = a + (k-1)d$ for $k \geq 1$, we let S denote the sum of the first n terms. To derive a formula for S, we write it out in two different ways

$$\begin{array}{rclclclclcl} S & = & a & + & (a+d) & + & \ldots & + & (a+(n-2)d) & + & (a+(n-1)d) \\ S & = & (a+(n-1)d) & + & (a+(n-2)d) & + & \ldots & + & (a+d) & + & a \end{array}$$

If we add these two equations and combine the terms which are aligned vertically, we get

$$2S = (2a+(n-1)d) + (2a+(n-1)d) + \ldots + (2a+(n-1)d) + (2a+(n-1)d)$$

The right hand side of this equation contains n terms, all of which are equal to $(2a+(n-1)d)$ so we get $2S = n(2a+(n-1)d)$. Dividing both sides of this equation by 2, we obtain the formula

$$S = \frac{n}{2}(2a+(n-1)d)$$

If we rewrite the quantity $2a+(n-1)d$ as $a+(a+(n-1)d) = a_1 + a_n$, we get the formula

$$S = n\left(\frac{a_1 + a_n}{2}\right)$$

A helpful way to remember this last formula is to recognize that we have expressed the sum as the product of the number of terms n and the *average* of the first and n^{th} terms.

To derive the formula for the geometric sum, we start with a geometric sequence $a_k = ar^{k-1}$, $k \geq 1$, and let S once again denote the sum of the first n terms. Comparing S and rS, we get

$$\begin{array}{rclclclclclclcl} S & = & a & + & ar & + & ar^2 & + & \ldots & + & ar^{n-2} & + & ar^{n-1} & & \\ rS & = & & & ar & + & ar^2 & + & \ldots & + & ar^{n-2} & + & ar^{n-1} & + & ar^n \end{array}$$

Subtracting the second equation from the first forces all of the terms except a and ar^n to cancel out and we get $S - rS = a - ar^n$. Factoring, we get $S(1-r) = a\left(1 - r^n\right)$. Assuming $r \neq 1$, we can divide both sides by the quantity $(1-r)$ to obtain

$$S = a\left(\frac{1-r^n}{1-r}\right)$$

If we distribute a through the numerator, we get $a - ar^n = a_1 - a_{n+1}$ which yields the formula

$$S = \frac{a_1 - a_{n+1}}{1-r}$$

In the case when $r = 1$, we get the formula

$$S = \underbrace{a + a + \ldots + a}_{n \text{ times}} = n\,a$$

Our results are summarized below.[5]

Formula 1. Sums of Arithmetic and Geometric Sequences:

[5] Alternatively, we can use Exercise **??** in Section **??**.

- The sum S of the first n terms of an arithmetic sequence $a_k = a + (k-1)d$ for $k \geq 1$ is

$$S = \sum_{k=1}^{n} a_k = n\left(\frac{a_1 + a_n}{2}\right) = \frac{n}{2}(2a + (n-1)d)$$

- The sum S of the first n terms of a geometric sequence $a_k = ar^{k-1}$ for $k \geq 1$ is

  1. $S = \displaystyle\sum_{k=1}^{n} a_k = \frac{a_1 - a_{n+1}}{1-r} = a\left(\frac{1-r^n}{1-r}\right)$, if $r \neq 1$.
  2. $S = \displaystyle\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} a = na$, if $r = 1$.

While we have made an honest effort to derive the formulas in Equation 0.1, formal proofs require the machinery in Section **??**.

Example 3.

1. (i) Find the sum: $1 + 3 + 5 + ... + 117$

   (ii) Find a formula for the sum $\displaystyle\sum_{k=1}^{n} k$.

2. The classic wheat and chessboard problem asks the following question. Given a chessboard with its squares numbered 1 to 64, suppose on the first square was placed one grain of wheat, the second square, two grains, the third square, four grains, and so on, each square receiving twice the number of grains as its predecessor. How many total grains of wheat would end up on the chessboard?

Solution.

1. (i) Recognizing the terms of $1 + 3 + 5 + ... + 117$ as 1, 3, 5, and so on, we see we have an arithmetic sequence with $a = 1$ and $d = 2$. Using Equation **??**, we get a formula for the terms $a_n = 1 + 2(n-1) = 2n - 1$ for $n \geq 1$. In order to use the formula in Equation 0.1, we need to determine the number of terms being added, n. Setting $2n - 1 = 117$, we find $n = 59$. Feeding in all of our data into Equation 0.1, we get $1 + 3 + 5 + ... + 117 = 59\left(\frac{1+117}{2}\right) = 3481$.

   (ii) Applying the adage 'when in doubt, write it out,' we have $\displaystyle\sum_{k=1}^{n} k = 1 + 2 + 3 + 4 + ... + n$. We see the terms here form an arithmetic sequence with $a = d = 1$. Moreover, we are adding exactly n terms, so Equation 0.1 gives $\displaystyle\sum_{k=1}^{n} k = 1 + 2 + 3 + 4 + ... + n = \frac{n(n+1)}{2}$.
   As a side note, the special case: $1 + 2 + 3 + ... + 100$ was allegedly given to Carl Friedrich Gauss while he was in elementary school. Instead of computing the sum in a brute force method, he arrived at the answer by grouping $1 + 99 = 100$, $2 + 98 = 100$, etc. so that he had 50 groups of 100 with 50 left over for a total of 5050. This is the exact same methodology we used to prove the sum of the arithmetic sequence formula in Equation 0.1.

2. Since we are *doubling* the number of grains of wheat as we move from one square to the next, a geometric sequence with $r = 2$ describes the number of grains on each individual square.

   Since we start with one grain on the first square, the number of grains on the kth square is $a_k = (1)(2)^{k-1} = 2^{k-1}$ for $k \geq 1$.

   Adding up the number of grains on each square gives:

$$1 + 2 + ... + 2^{64-1} = 1 + 2 + ... 2^{63} = \frac{1 - 2^{64}}{1 - 2} = 2^{64} - 1 \approx 1.8 \times 10^{19},$$

in accordance with Equation 0.1. (The weight of these grains would total approximately 2.6×10^{15} pounds which is approximately 15 times the entire biomass of the planet.) ■

An important application of the geometric sum formula is the investment plan called an *annuity*. Annuities differ from the kind of investments we studied in Section **??** in that payments are deposited into the account on an on-going basis, and this complicates the mathematics a little.[6]

Suppose you have an account with annual interest rate r which is compounded n times per year. We let $i = \frac{r}{n}$ denote the interest rate per period. Suppose we wish to make ongoing deposits of P dollars at the *end* of each compounding period. Let A_k denote the amount in the account after k compounding periods.

Then $A_1 = P$, because we have made our first deposit at the *end* of the first compounding period and no interest has been earned. During the second compounding period, we earn interest on A_1 so that our initial investment has grown to $A_1(1 + i) = P(1 + i)$ in accordance with Equation **??**. Adding our second payment at the end of the second period, we get

$$A_2 = A_1(1 + i) + P = P(1 + i) + P = P(1 + i)\left(1 + \frac{1}{1 + i}\right)$$

The reason for factoring out the $P(1 + i)$ will become apparent in short order. During the third compounding period, we earn interest on A_2 which then grows to $A_2(1 + i)$. We add our third payment at the end of the third compounding period to obtain

$$A_3 = A_2(1 + i) + P = P(1 + i)\left(1 + \frac{1}{1 + i}\right)(1 + i) + P = P(1 + i)^2\left(1 + \frac{1}{1 + i} + \frac{1}{(1 + i)^2}\right)$$

During the fourth compounding period, A_3 grows to $A_3(1 + i)$, and when we add the fourth payment, we factor out $P(1 + i)^3$ to get

$$A_4 = P(1 + i)^3\left(1 + \frac{1}{1 + i} + \frac{1}{(1 + i)^2} + \frac{1}{(1 + i)^3}\right)$$

This pattern continues so that at the end of the kth compounding, we get

$$A_k = P(1 + i)^{k-1}\left(1 + \frac{1}{1 + i} + \frac{1}{(1 + i)^2} + ... + \frac{1}{(1 + i)^{k-1}}\right)$$

The sum in the parentheses above is the sum of the first k terms of a geometric sequence with $a = 1$ and $r = \frac{1}{1 + i}$. Using Equation 0.1, we get

$$1 + \frac{1}{1 + i} + \frac{1}{(1 + i)^2} + ... + \frac{1}{(1 + i)^{k-1}} = 1\left(\frac{1 - \frac{1}{(1+i)^k}}{1 - \frac{1}{1+i}}\right) = \frac{(1 + i)\left(1 - (1 + i)^{-k}\right)}{i}$$

Hence, we get

$$A_k = P(1 + i)^{k-1}\left(\frac{(1 + i)\left(1 - (1 + i)^{-k}\right)}{i}\right) = \frac{P\left((1 + i)^k - 1\right)}{i}$$

If we let t be the number of years this investment strategy is followed, then $k = nt$, and we get the formula for the future value of an *ordinary annuity*.

[6]The reader may wish to re-read the discussion on compound interest in Section **??** before proceeding.

Formula 2. Future Value of an Ordinary Annuity: Suppose an annuity offers an annual interest rate r compounded n times per year. Let $i = \frac{r}{n}$ be the interest rate per compounding period. If a deposit P is made at the end of each compounding period, the amount A in the account after t years is given by

$$A = \frac{P\left((1+i)^{nt} - 1\right)}{i}$$

The reader is encouraged to substitute $i = \frac{r}{n}$ into Equation 0.2 and simplify. Some familiar equations arise which are cause for pause and meditation. One last note: if the deposit P is made a the *beginning* of the compounding period instead of at the end, the annuity is called an *annuity-due*. We leave the derivation of the formula for the future value of an annuity-due as an exercise for the reader.

Example 4. An ordinary annuity offers a 6% annual interest rate, compounded monthly.

1. If monthly payments of \$50 are made, find the value of the annuity in 30 years.
2. How many years will it take for the annuity to grow to \$100,000?

Solution.

1. We have $r = 0.06$ and $n = 12$ so that $i = \frac{r}{n} = \frac{0.06}{12} = 0.005$. With $P = 50$ and $t = 30$,

$$A = \frac{50\left((1+0.005)^{(12)(30)} - 1\right)}{0.005} \approx 50225.75$$

   Our final answer is \$50,225.75.

2. To find how long it will take for the annuity to grow to \$100,000, we set $A = 100000$ and solve for t. We isolate the exponential and take natural logs of both sides of the equation.

$$\begin{aligned} 100000 &= \frac{50\left((1+0.005)^{12t} - 1\right)}{0.005} \\ 10 &= (1.005)^{12t} - 1 \\ (1.005)^{12t} &= 11 \\ \ln\left((1.005)^{12t}\right) &= \ln(11) \\ 12t\ln(1.005) &= \ln(11) \\ t &= \frac{\ln(11)}{12\ln(1.005)} \approx 40.06 \end{aligned}$$

   This means that it takes just over 40 years for the investment to grow to \$100,000. Comparing this with our answer to part 1, we see that in just 10 additional years, the value of the annuity nearly doubles. This is a lesson worth remembering. ■

1 Geometric Series

As defined in Section **??**, sequences are an *infinite* list of numbers. So far in this section, we have concerned ourselves with adding only *finitely* many terms. In Calculus, *infinite* sums, called **series** are studied at great length. While we do not have the mathematical machinery to embark upon an exhaustive study here, we can nevertheless focus our attention on what is arguably one of the most prevalent and useful types of series, **geometric series**.

As a motivating example, consider the number $0.\overline{9}$. We can write this number as

$$0.\overline{9} = 0.9999... = 0.9 + 0.09 + 0.009 + 0.0009 + ...$$

From Example 1, we know we can write the sum of the first n of these terms as

$$0.\underbrace{9\cdots9}_{n\text{ nines}} = .9 + 0.09 + 0.009 + ...0.\underbrace{0\cdots0}_{n-1\text{ zeros}}9 = \sum_{k=1}^{n}\frac{9}{10^k}$$

Using Equation 0.1, we have

$$\sum_{k=1}^{n}\frac{9}{10^k} = \sum_{k=1}^{n}\frac{9}{10}\left(\frac{1}{10^{k-1}}\right) = \sum_{k=1}^{n}\frac{9}{10}\left(\frac{1}{10}\right)^{k-1} = \frac{9}{10}\left(\frac{1-\frac{1}{10^n}}{1-\frac{1}{10}}\right) = 1 - \frac{1}{10^n}$$

It stands to reason that we should define $0.\overline{9} = \lim_{n\to\infty}\left(1 - \frac{1}{10^n}\right)$. Passing to a continuous variable along with our knowledge of exponential functions gives $\lim_{n\to\infty}\left(1-\frac{1}{10^n}\right) = \lim_{x\to\infty}\left(1-\frac{1}{10^x}\right) = 1 - 0 = 1$.

We have just argued that $0.\overline{9} = 1$, which may shock some readers.[7]

Note that in this manner, any non-terminating decimal can be thought of as an infinite sum whose denominators are the powers of 10, so the phenomenon of adding up infinitely many terms and arriving at a finite number is not as foreign of a concept as it may appear. We have the following theorem.

Theorem 2. Geometric Series: Given the sequence $a_k = ar^{k-1}$ for $k \geq 1$, where $|r| < 1$,

$$a + ar + ar^2 + ... = \sum_{k=1}^{\infty} ar^{k-1} = \lim_{n\to\infty}\sum_{k=1}^{n} ar^{k-1} = \frac{a}{1-r}$$

If $|r| \geq 1$, the sum $a + ar + ar^2 + ...$ does not exist.

The justification of the result in Theorem 2 comes from taking the formula in Equation 0.1 for the sum of the first n terms of a geometric sequence and taking the limit as $n \to \infty$.

Assuming $|r| < 1$ means $-1 < r < 1$, so per Theorem **??**, $\lim_{n\to\infty} r^n = 0$. Using this fact along with the Limit Properties listed in Theorem **??**:

$$\lim_{n\to\infty}\sum_{k=1}^{n} ar^{k-1} = \lim_{n\to\infty} a\left(\frac{1-r^n}{1-r}\right) = \frac{a}{1-r}$$

We'll explore what goes wrong when $|r| \geq 1$ in some of the Exercises. For now, we put this theorem to good use in the following example.

Example 5.

1. Find the sum: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...$
2. Represent $4.2\overline{17}$ as a fraction in lowest terms.

Solution.

[7] To make this more palatable, it is usually accepted that $0.\overline{3} = \frac{1}{3}$ so that $0.\overline{9} = 3\left(0.\overline{3}\right) = 3\left(\frac{1}{3}\right) = 1$.

1. We recognize $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ as a geometric series with $a = r = \frac{1}{2}$. Using Theorem 2, we get

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1.$$

   The interested reader is invited to research this sum as it relates to Zeno's Dichotomy Paradox.

2. To use Theorem 2 as it applies to the repeating decimal $4.2\overline{17}$, we first need to rewrite this decimal in terms of a geometric series. Expanding $4.2\overline{17} = 4.2 + 0.017 + 0.00017 + 0.0000017 + \ldots$, we see the series $0.017 + 0.00017 + 0.0000017 + \ldots$ is geometric with $a = 0.017$ and $r = 0.01$. Hence, we can apply Theorem 2 to that part of the decimal to get:

$$0.017 + 0.00017 + 0.0000017 + \ldots = \frac{0.017}{1 - 0.01} = \frac{\frac{17}{1000}}{\frac{99}{100}} = \frac{17}{990}$$

   Hence, $4.2\overline{17} = 4.2 + \frac{17}{990} = \frac{42}{10} + \frac{17}{990} = \frac{835}{198}$. ∎

We note that another popular method for converting repeating decimals to fractions goes something like this: let $x = 4.2\overline{17}$. Then, $100x = 421.7\overline{17}$. Hence, $99x = 100x - x = 421.7\overline{17} - 4.2\overline{17} = 417.5$. Hence, $x = \frac{417.5}{99} = \frac{835}{198}$. While this procedure results in the same (correct!) answer, the manipulations involved (such as the multiplication and subtraction) are actually using some of the properties listed in Theorem 1 extended to infinite sums via Theorem **??**.

2 Area

One of the (two) major geometric problems studied in Calculus is finding the area under a curve[8] (more specifically, the area between the graph of a function and the x-axis.)[9] In this section, we explore how summation notation is used to help better formulate this problem, and, as with our study of Geometric Series, sneak a peak into Calculus itself.

Suppose we wish to determine the area between the graph of a continuous function $y = f(x)$ over the interval $[a, b]$ and the x-axis as shown below.

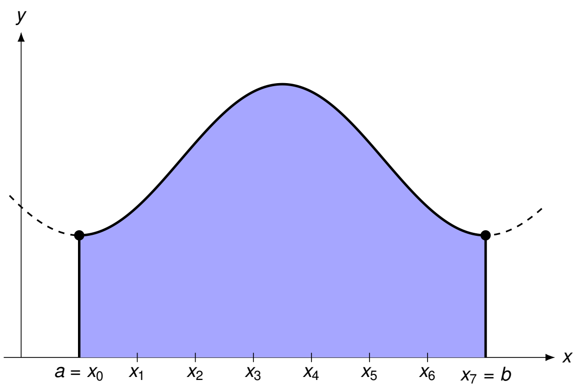

Figure 1: Area under the graph of $y = f(x)$.

[8]The other is the concept of tangent lines which we touch on (awesome pun!) in Section **??**.

[9]The area can actually represent a wide variety of things such as displacement, probability, or, as odd as it sounds, volume.

Since we don't know any area formulas for arbitrary regions, we stick to what we know - rectangles. To keep things simple, we divide $[a, b]$ into n equal pieces (subintervals), and use the right-endpoints of each piece to determine the height of the rectangles.[10] We let x_k represent the right endpoint of the kth subinterval, so the height of the kth rectangle is $f(x_k)$.

The width of the kth rectangle is the length of the kth subinterval. Since the interval itself is $b - a$ units long and we are dividing the interval into n *equal* pieces, each piece is $\dfrac{b-a}{n}$ units long. For brevity, we'll call this length 'Δx.' Below on the right is a depiction of RS_7, a 'right endpoint sum' using 7 (equally spaced) subintervals.[11]

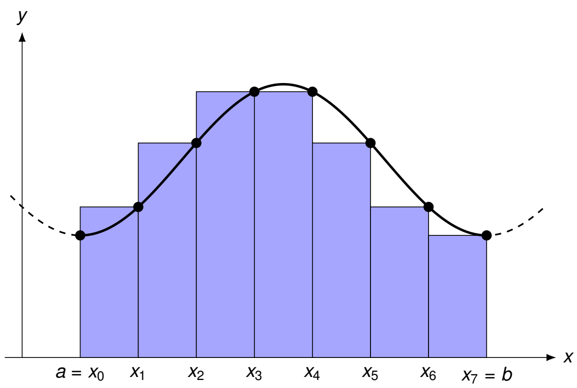

Figure 2: Visualizing RS_7, a 'right endpoint sum.'

The idea here is to approximate the area of the shaded region by the sum of the areas of the rectangles. In symbols:

$$\text{Area} \approx f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + ... + f(x_7)\Delta x = \sum_{k=1}^{7} f(x_k)\Delta x$$

Our ultimate goal is to find a formula for the area approximation as described above as a function of the number of rectangles n and look to see what happens as $n \to \infty$.

We first note that the right endpoints x_k, are terms in an arithmetic sequence: the first right endpoint, x_1 is Δx to the right of $a = x_0$, so $x_1 = x_0 + \Delta x$; the second right endpoint, x_2 is Δx units to the right of x_1, so $x_2 = x_1 + \Delta x$; the third right endpoint $x_3 = x_2 + \Delta x$ and so on. In general, $x_k = x_{k-1} + \Delta x$, proving the x_k are terms of an arithmetic sequence with common difference $d = \Delta x$. It follows that x_k, the kth right endpoint is $k\Delta x$ units to the right of $x_0 = a$, so that $x_k = a + k\Delta x$. We summarize the notation and formulas for right endpoint sums below.

Summary of Formulas for Right Endpoint Sums, RS_n

- Number of rectangles: n
- Width of each rectangle: $\Delta x = \dfrac{b-a}{n}$
- Right endpoint: $x_k = a + k\Delta x$
- Height of kth rectangle: $f(x_k)$

[10] In Calculus, you'll also use left endpoints and midpoints . . . .

[11] On intervals over which the function is *increasing*, we find the area of rectangles *overestimates* the area we want; on intervals over which the function is *decreasing*, we find the area of the rectangles *underestimates* the area we want.

- Area $\approx RS_n$ = the sum of the area of the rectangles $= \displaystyle\sum_{k=1}^{n} f(x_k)\Delta x_k$

Below we summarize some common summation formulas we'll need when actually computing these sums. Formal proofs of these require the machinery of Section **??** and are found there.

Summation Formulas

- $\displaystyle\sum_{k=1}^{n} c = cn$
- $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$
- $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
- $\displaystyle\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

It is high time for an example.

Example 6. Consider $f(x) = 4x - x^2$ over the interval $[0, 4]$.

1. Graph f over this interval and shade the area between the graph of f and the x-axis.
2. Compute RS_n for $n = 4$ and $n = 8$. Interpret your results graphically.
3. Find a formula for RS_n in terms of n and determine $\displaystyle\lim_{n\to\infty} RS_n$.

Solution.

1. The graph of $f(x) = 4x - x^2$ is a parabola with intercepts $(0, 0)$ and $(4, 0)$ with a vertex at $(2, 4)$. Using the GeoGebra interactive below, we can input $f(x) = 4x - x^2$ and set the bounds of the interval $[a, b] = [0, 4]$. GeoGebra numerically[12] determines the area to be (what appears to be) $10.\overline{6}$ square units.

   GeoGebra link: `https://www.geogebra.org/m/t5pspeyd`

2. To find RS_4, we begin by chopping up the interval $[0, 4]$ into 4 equal pieces so each subinterval has length $\Delta x = \frac{4}{4} = 1$ unit. Our right endpoints are: $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, and $x_4 = 4$. We find $f(1) = 3$, $f(2) = 4$, $f(3) = 3$, and $f(4) = 0$. Hence,

   $$RS_4 = \sum_{k=1}^{4} f(x_k)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x = (3)(1) + (4)(1) + 3(1) + (0)(1) = 10.$$

   The GeoGebra interactive below shows us what this sum represents geometrically. After inputting our function $f(x) = 4x - x^2$ and the bounds $[a, b] = [0, 4]$ we can adjust a slider to select the number of rectangles, n, used to calculate the right endpoint sum, RS_n. A different slider allows us to select each rectangle, in sequence, and computes the area of the selected rectangle. The value of RSn is the sum of the areas of these rectangles.

   GeoGebra link: `https://www.geogebra.org/m/cmxnab2q`

   To find RS_8, we divide the interval $[0, 4]$ into 8 equal pieces, so each has length $\Delta x = \frac{4}{8} = 0.5$ units. This produces the right endpoints: $x_1 = 0.5$, $x_2 = 1$, $x_3 = 1.5$, $x_4 = 2$, $x_5 = 2.5$, $x_6 = 3$, $x_7 = 3.5$, $x_8 = 4$. In addition to the function values we used to compute RS_4, we need $f(0.5) = 1.75$, $f(1.5) = 3.75$, $f(2.5) = 3.75$, and $f(3.5) = 1.75$. Hence,

   $$\begin{aligned} RS_8 &= \sum_{k=1}^{8} f(x_k)\Delta x \\ &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \\ &\quad + f(x_5)\Delta x + f(x_6)\Delta x + f(x_7)\Delta x + f(x_8)\Delta x \\ &= (1.75)(0.5) + (3)(0.5) + (3.75)(0.5) + (4)(0.5) \\ &\quad + (3.75)(0.5) + (3)(0.5) + (1.75)(0.5) + (0)(0.5) \\ &= 10.5 \end{aligned}$$

   Hence, the area under the graph f is approximately 10.5 square units as approximated by the sum of the rectangles above on the right. (Again, since $f(x_8) = f(4) = 0$, the eighth 'rectangle' has 0 height.) Once again, the GeoGebra interactive above is very useful in visulizing this sum by adjusting the slider to $n = 8$.

3. To find a formula for RS_n, we imagine dividing the interval $[0, 4]$ into n equal pieces each of length $\Delta x = \frac{4}{n}$. We have n right endpoints, $x_1, x_2, \ldots x_n$ where $x_k = 0 + k\Delta x = \frac{4k}{n}$. Since $f(x) = 4x - x^2$,

   $$f(x_k) = 4x_k - x_k^2 = 4\left(\frac{4k}{n}\right) - \left(\frac{4k}{n}\right)^2 = \frac{16k}{n} - \frac{16k^2}{n^2}.$$

   Hence,

[12] Curious as to how does GeoGebra does this? Research Quadrature Rules.

$$
\begin{aligned}
RS_n &= \sum_{k=1}^{n} f(x_k)\Delta x \\
&= \sum_{k=1}^{n} \left[\frac{16k}{n} - \frac{16k^2}{n^2}\right]\left(\frac{4}{n}\right) \\
&= \sum_{k=1}^{n} \left[\frac{64k}{n^2} - \frac{64k^2}{n^3}\right] && \text{Distribute the } \frac{4}{n}. \\
&= \sum_{k=1}^{n} \frac{64k}{n^2} - \sum_{k=1}^{n} \frac{64k^2}{n^3} && \text{Sum and Difference Property} \\
&= \frac{64}{n^2}\sum_{k=1}^{n} k - \frac{64}{n^3}\sum_{k=1}^{n} k^2 && \text{Distributive Property (n is a constant we can factor out)} \\
&= \frac{64}{n^2}\left(\frac{n(n+1)}{2}\right) - \frac{64}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) && \text{Summation Formulas} \\
&= \frac{32(n+1)}{n} - \frac{32(n+1)(2n+1)}{3n^2} = \frac{32n^2-32}{3n^2}
\end{aligned}
$$

Note we can partially check our answer at thus point by substituting $n = 4$ and $n = 8$ to our formula to RS_n to see if we recover our answers from above. We get $RS_4 = \dfrac{32(4)^2 - 32}{3(4^2)} = \dfrac{480}{48} = 10$ and $RS_8 = \dfrac{32(8)^2 - 32}{3(8)^2} = \dfrac{2016}{192} = 10.5$, as required.

To find $\lim\limits_{n\to\infty} RS_n = \lim\limits_{n\to\infty} \dfrac{32n^2-32}{3n^2}$, we compare the leading term of the numerator and denominator. As $n \to \infty$, $\dfrac{32n^2-32}{3n^2} \approx \dfrac{32n^2}{3n^2} = \dfrac{32}{3}$. Hence, $\lim\limits_{n\to\infty} RS_n = \dfrac{32}{3}$. Hence, as we use more and more rectangles,[13] the sum total of the area of those rectangles approaches $\dfrac{32}{3} = 10.\overline{6}$ square units, which matches with what GeoGebra had predicted in part 1. In Calculus, we more or less **define** the area under f to be $\dfrac{32}{3}$ square units. ■

The reader is encouraged to revisit the GeoGebra interactives showcased in Example 6 and input different functions, different bounds, and different numbers of rectangles for RS_n. For example, what happens when you have a function which dips below the x-axis? What does RS_n and $\lim\limits_{n\to\infty} RS_n$ calculate then?

It is worth noting that, as with other examples in the text, Example 6 is more or less lifted straight out of a Calculus lecture. That being said, the vast majority of the mechanics here involve precalculus notions.[14] In general, the machinations in Calculus amount to applying the limit concept to the mechanics of precalculus.

[13] even though they become skinnier and skinner and hence, *individually* have smaller and smaller areas . . .

[14] The only Calculus bit is the limit concept which I guess is precalculus now . . .