ELLIPSES

In the definition of a circle, Definition **??**, we fixed a point called the **center** and considered all of the points which were a fixed distance r from that one point. For our next conic section, the ellipse, we fix two distinct points and a distance d to use in our definition.

Definition 1. Given two distinct points F_1 and F_2 in the plane and a fixed distance d, an **ellipse** is the set of all points (x, y) in the plane such that the sum of each of the distances from F_1 and F_2 to (x, y) is d. The points F_1 and F_2 are called the **foci**[1] of the ellipse.

In the GeoGebra interactive below, we fix two points, F_1 and F_2, to serve as foci and fix a distance $d = 4$ units. As we adjust the slider, we plot points whose distance to F_1 plus the distance to F_2 is exactly 4 units.

GeoGebra link: `https://www.geogebra.org/m/bcpfyra3`

We can replicate this interactive physically by taking a length of string and anchoring it to two points on a piece of paper. The curve traced out by taking a pencil and moving it so the string is always taut is an ellipse.

Each ellipse has an assortment of parameters associated with it which we sketch below.

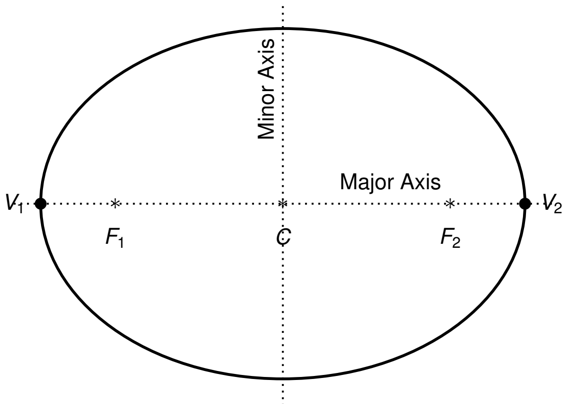


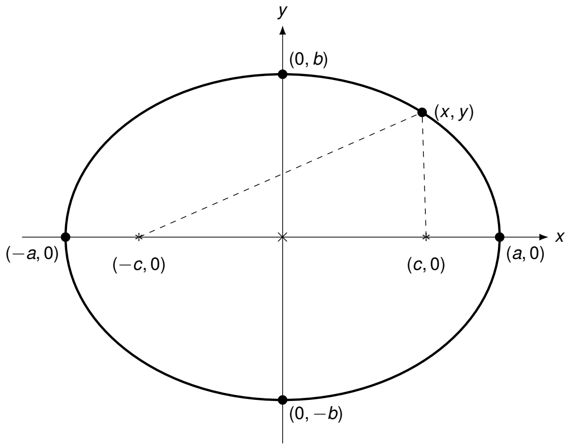
Figure 1: An ellipse with center C; foci F_1, F_2; and vertices V_1, V_2

As depicted above, the **center** of the ellipse is the midpoint of the line segment connecting the two foci. The **major axis** of the ellipse is the line segment connecting two opposite ends of the ellipse which also contains the center and foci. The **minor axis** of the ellipse is the line segment connecting two opposite ends of the ellipse which contains the center but is perpendicular to the major axis. The **vertices** of an ellipse are the points of the ellipse which lie on the major axis.

Notice that the center is also the midpoint of the major axis, hence it is the midpoint of the vertices. Also note that the major axis is the longer of the two axes through the center, hence the moniker 'major.' Likewise, the minor axis is the shorter of the two, whence the adjective 'minor.'

In order to derive the standard equation of an ellipse, we assume that the ellipse has its center at $(0, 0)$, its major axis along the x-axis, and has foci $(c, 0)$ and $(-c, 0)$ and vertices $(-a, 0)$ and $(a, 0)$. We will label the y-intercepts of the ellipse as $(0, b)$ and $(0, -b)$ (We assume a, b, and c are all positive numbers.)

[1] the plural of 'focus'



Note that since $(a, 0)$ is on the ellipse, it must satisfy the conditions of Definition **??**. That is, the distance from $(-c, 0)$ to $(a, 0)$ plus the distance from $(c, 0)$ to $(a, 0)$ must equal the fixed distance d. Since all of these points lie on the x-axis, we get

$$\begin{aligned}
\text{distance from } (-c, 0) \text{ to } (a, 0) + \text{distance from } (c, 0) \text{ to } (a, 0) &= d \\
(a + c) + (a - c) &= d \\
2a &= d
\end{aligned}$$

In other words, the fixed distance d mentioned in the definition of the ellipse is none other than the length of the major axis. We now use that fact $(0, b)$ is on the ellipse, along with the fact that $d = 2a$ to get

$$\begin{aligned}
\text{distance from } (-c, 0) \text{ to } (0, b) + \text{distance from } (c, 0) \text{ to } (0, b) &= 2a \\
\sqrt{(0 - (-c))^2 + (b - 0)^2} + \sqrt{(0 - c)^2 + (b - 0)^2} &= 2a \\
\sqrt{b^2 + c^2} + \sqrt{b^2 + c^2} &= 2a \\
2\sqrt{b^2 + c^2} &= 2a \\
\sqrt{b^2 + c^2} &= a
\end{aligned}$$

From this, we get $a^2 = b^2 + c^2$, or $b^2 = a^2 - c^2$, which will prove useful later. Now consider a point (x, y) on the ellipse. Applying Definition **??**, we get

$$\begin{aligned}
\text{distance from } (-c, 0) \text{ to } (x, y) + \text{distance from } (c, 0) \text{ to } (x, y) &= 2a \\
\sqrt{(x - (-c))^2 + (y - 0)^2} + \sqrt{(x - c)^2 + (y - 0)^2} &= 2a \\
\sqrt{(x + c)^2 + y^2} + \sqrt{(x - c)^2 + y^2} &= 2a
\end{aligned}$$

In order to make sense of this situation, we need to make good use of Intermediate Algebra.

$$
\begin{aligned}
\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2} &= 2a \\
\sqrt{(x+c)^2+y^2} &= 2a-\sqrt{(x-c)^2+y^2} \\
\left(\sqrt{(x+c)^2+y^2}\right)^2 &= \left(2a-\sqrt{(x-c)^2+y^2}\right)^2 \\
(x+c)^2+y^2 &= 4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2 \\
4a\sqrt{(x-c)^2+y^2} &= 4a^2+(x-c)^2-(x+c)^2 \\
4a\sqrt{(x-c)^2+y^2} &= 4a^2-4cx \\
a\sqrt{(x-c)^2+y^2} &= a^2-cx \\
\left(a\sqrt{(x-c)^2+y^2}\right)^2 &= \left(a^2-cx\right)^2 \\
a^2\left((x-c)^2+y^2\right) &= a^4-2a^2cx+c^2x^2 \\
a^2x^2-2a^2cx+a^2c^2+a^2y^2 &= a^4-2a^2cx+c^2x^2 \\
a^2x^2-c^2x^2+a^2y^2 &= a^4-a^2c^2 \\
\left(a^2-c^2\right)x^2+a^2y^2 &= a^2\left(a^2-c^2\right)
\end{aligned}
$$

We are nearly finished. Recall that $b^2 = a^2 - c^2$ so that

$$
\begin{aligned}
\left(a^2-c^2\right)x^2+a^2y^2 &= a^2\left(a^2-c^2\right) \\
b^2x^2+a^2y^2 &= a^2b^2 \\
\frac{x^2}{a^2}+\frac{y^2}{b^2} &= 1
\end{aligned}
$$

This equation is for an ellipse centered at the origin. To get the formula for the ellipse centered at (h, k), we could use the transformations from Section **??** or re-derive the equation using Definition **??** and the distance formula to obtain the formula below.

Formula 1. The Standard Equation of an Ellipse:

For positive unequal numbers a and b, the equation of an ellipse with center (h, k) is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

Some remarks about Equation **??** are in order. First note that the values a and b determine how far in the x and y directions, respectively, one counts from the center to arrive at points on the ellipse.

Also note that if $a > b$, then we have an ellipse whose major axis is horizontal, and hence, the foci lie to the left and right of the center. In this case, as we've seen in the derivation, the distance from the center to the focus, c, can be found by $c = \sqrt{a^2 - b^2}$.

If $b > a$, the roles of the major and minor axes are reversed, and the foci lie above and below the center. In this case, $c = \sqrt{b^2 - a^2}$. In either case, it's best to just remember that c is the distance from the center to each focus, and, formulaically, $c = \sqrt{\text{bigger denominator} - \text{smaller denominator}}$.

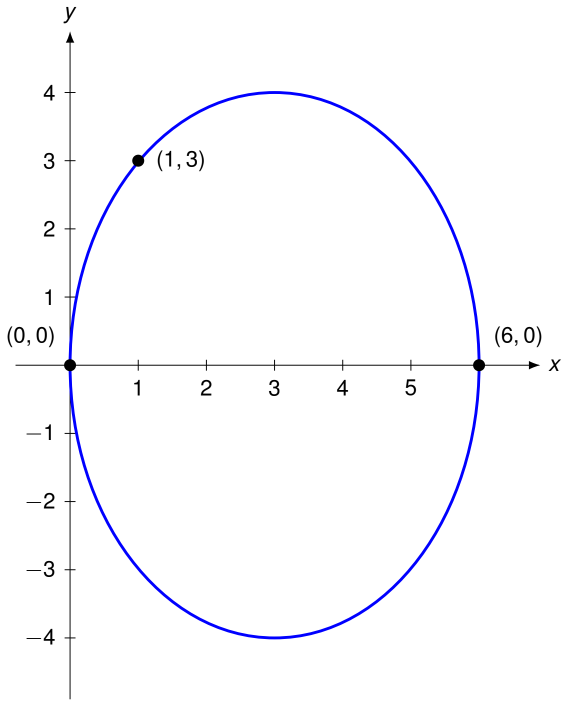
Finally, it is worth mentioning that if we compare Equation **??** with the alternate standard equation of the circle, Equation **??**, the only difference between the forms is that with a circle, the denominators are the same, and with an ellipse, they are different.

If we take a transformational approach, we can consider both Equations **??** and **??** as shifts and stretches of the Unit Circle $x^2 + y^2 = 1$ in Definition **??**. Replacing x with $(x - h)$ and y with $(y - k)$ causes the usual horizontal and vertical shifts. Replacing x with $\frac{x}{a}$ and y with $\frac{y}{b}$ causes the usual vertical and horizontal stretches.

In other words, it is perfectly fine to think of an ellipse as the deformation of a circle in which the circle is stretched farther in one direction than the other.

Example 1.

1. Graph each of the following equations below in the xy-plane. Find the center, the lines which contain the major and minor axes, the vertices, the endpoints of the minor axis, and the foci.
   (i) $25(x+1)^2 + 9(y-2)^2 = 225$.
   (ii) $x^2 + 4y^2 - 2x + 24y + 33 = 0$.
2. Graph $f(x) = 1 + 2\sqrt{-x^2 - 4x - 3}$
3. Find the standard form of the equation of an ellipse which satisfies the following characteristics:
   (i) the foci are at $(2,1)$ and $(4,1)$ and vertex $(0,1)$.
   (ii) the ellipse graphed below:



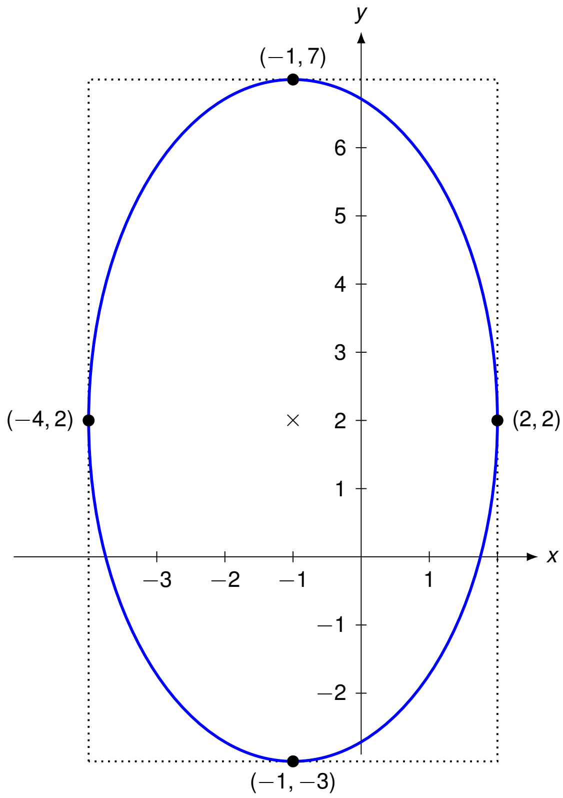
Solution.

1. (i) To put $25(x+1)^2 + 9(y-2)^2 = 225$ in the form prescribed by Equation **??**, we rewrite the quantity $(x+1)$ as $(x-(-1))$ and divide through by 225 to obtain an expression equal to 1:

$$25(x+1)^2 + 9(y-2)^2 = 225 \to \frac{(x-(-1))^2}{9} + \frac{(y-2)^2}{25} = 1 \leftrightarrow \frac{(x-(-1))^2}{(3)^2} + \frac{(y-2)^2}{(5)^2} = 1.$$

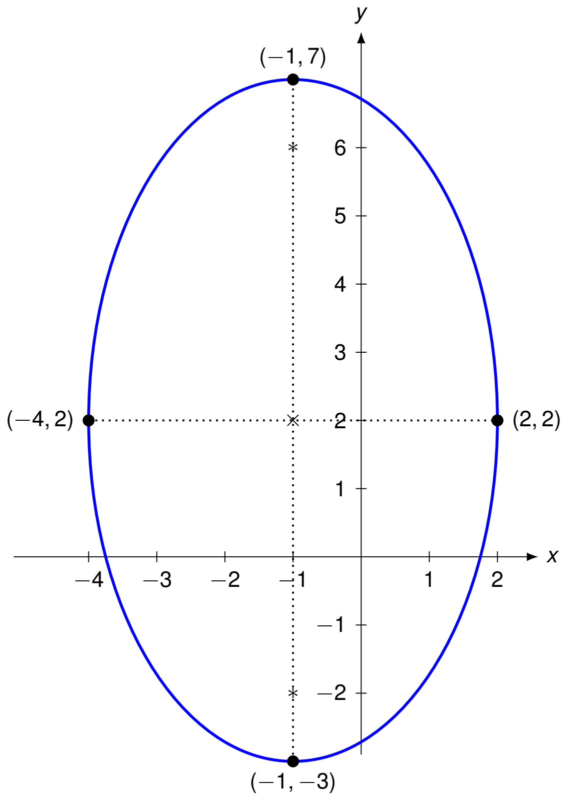
We identify $h = -1$ and $k = 2$, so the center of the ellipse is $(-1,2)$. We have $a = 3$ so we move 3 units left and right from the center to obtain two points on the ellipse: $(-1-3,2) = (-4,2)$ and $(-1+3,2) = (2,2)$. Likewise, since $b = 5$, we move up and down 5 units from the center to find two more points on the ellipse: $(-1,2+5) = (-1,7)$ and $(-1,2-5) = (-1,-3)$.

As an aid to sketching, we draw a rectangle matching this description, called a **guide rectangle**, and sketch the ellipse inside this rectangle as seen below.



Since we moved farther from the center in the y direction than in the x direction, the major axis will lie along the vertical line $x = -1$, while the minor axis lies along the horizontal line $y = 2$. The vertices are the points on the ellipse which lie along the major axis so in this case, they are the points $(-1, 7)$ and $(-1, -3)$, and the endpoints of the minor axis are $(-4, 2)$ and $(2, 2)$. (Notice these points are the four points we used to draw the guide rectangle.)

To find the foci, we find $c = \sqrt{25 - 9} = \sqrt{16} = 4$, which means the foci lie 4 units from the center. Since the major axis is vertical, the foci lie 4 units above and below the center, at $(-1, 2 - 4) = (-1, -2)$ and $(-1, 2 + 4) = (-1, 6)$. Our final graph appears below.



(ii) In the equation $x^2+4y^2-2x+24y+33=0$ we have a sum of two squares with unequal coefficients, it's a good bet we have an ellipse on our hands.[2]

In order to put this equation into the form stated in Equation **??**, we need to complete both squares and then divide, if necessary, to get the right-hand side equal to 1:

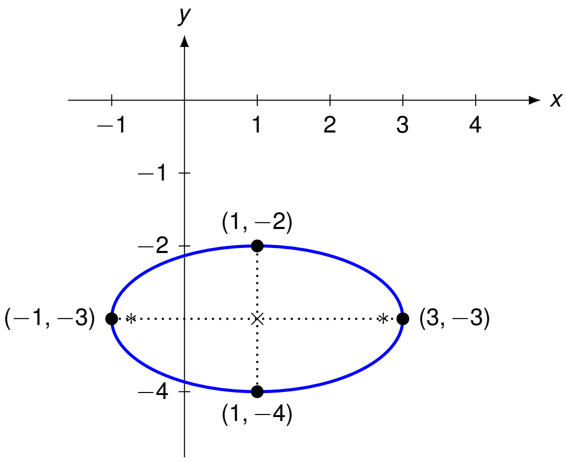
$$
\begin{array}{rcll}
x^2+4y^2-2x+24y+33 & = & 0 & \\
x^2-2x+4y^2+24y & = & -33 & \text{subtract 33 from both sides} \\
x^2-2x+4\left(y^2+6y\right) & = & -33 & \text{factor out leading coefficients} \\
\left(x^2-2x+\underline{1}\right)+4\left(y^2+6y+\underline{\underline{9}}\right) & = & -33+\underline{1}+4(\underline{\underline{9}}) & \text{complete the squares} \\
(x-1)^2+4(y+3)^2 & = & 4 & \text{factor} \\
\dfrac{(x-1)^2+4(y+3)^2}{4} & = & \dfrac{4}{4} & \text{divide through by 4} \\
\dfrac{(x-1)^2}{4}+(y+3)^2 & = & 1 & \\
\dfrac{(x-1)^2}{(2)^2}+\dfrac{(y-(-3))^2}{(1)^2} & = & 1 & \text{Rewrite in the desired form.}
\end{array}
$$

Now that this equation is in the standard form of Equation **??**, identify $h=1$ and $k=-3$ so our ellipse is centered at $(1,-3)$. With $a=2$, we move 2 units left and right from the center to get two points on the ellipse: $(1-2,-3)=(-1,-3)$ and $(1+2,-3)=(3,-3)$. Since $b=1$, we move 1 unit up and down from the center to obtain two additional points on the ellipse: $(1,-3+1)=(1,-2)$ and $(1,-3-1)=(1,-4)$.

Since we moved farther from the center in the x direction than in the y direction, the major axis will lie along the horizontal line $y=-3$ so the minor axis lies along the vertical line $x=1$. The vertices are the points on the ellipse which lie along the major axis so in this case, they are the points $(-1,-3)$ and $(3,-3)$, and the endpoints of the minor axis are $(1,-2)$ and $(1,-4)$.

[2] Recall the equation of a parabola has exactly *one* squared variable and the equation of a circle has two squared variables, but with *identical* coefficients.

To find the foci, we find $c = \sqrt{4-1} = \sqrt{3}$, which means the foci lie $\sqrt{3}$ units from the center. Since the major axis is horizontal, the foci lie $\sqrt{3}$ units to the left and right of the center, at $(1-\sqrt{3}, -3)$ and $(1+\sqrt{3}, -3)$. Plotting all of this information gives the graph below.



2. At first glance, it doesn't seem as if the function $f(x) = 1 + 2\sqrt{-x^2 - 4x - 3}$ will have *any* graph owing to the presence of the '$-$' signs which decorate *all* of the terms beneath the radical. However, since x is a *variable*, $-x^2 - 4x - 3$ is not necessarily negative.[3]

   Recall to graph the function $f(x) = 1+2\sqrt{-x^2-4x-3}$, we graph the equation $y = 1+2\sqrt{-x^2-4x-3}$. To make sense of this equation in the context of this chapter, we first isolate, then eliminate, the square root in order to obtain a quadratic equation in one or more variables.

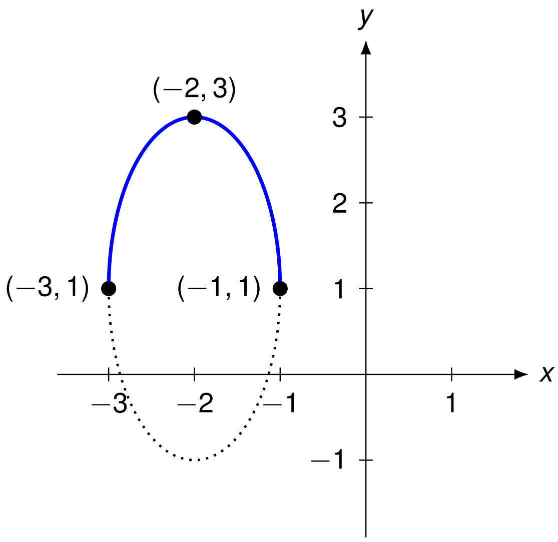
   From $y = 1+2\sqrt{-x^2-4x-3}$, we get $y-1 = 2\sqrt{-x^2-4x-3}$ so that $(y-1)^2 = (2\sqrt{-x^2-4x-3})^2$. Hence, $(y-1)^2 = 4(-x^2-4x-3)$ which is equivalent to $4x^2+16x+(y-1)^2 = -12$. At this point, we see we have the sum of two squared variables with unequal coefficients present which indicates an ellipse, so we complete the square on x and rewrite the equation so it fits Equation **??**:

$$\begin{array}{rcll}
4x^2 + 16x + (y-1)^2 & = & -12 & \\
4\left(x^2+4x\right) + (y-1)^2 & = & -12 & \text{factor out leading coefficient of } x^2 \\
4\left(x^2+4x+\underline{4}\right) + (y-1)^2 & = & -12 + 4(\underline{4}) & \text{complete the square in } x \\
4(x+2)^2 + (y-1)^2 & = & 4 & \text{factor} \\
\dfrac{4(x+2)^2 + (y-1)^2}{4} & = & \dfrac{4}{4} & \text{divide through by 4} \\
(x+2)^2 + \dfrac{(y-1)^2}{4} & = & 1 & \\
\dfrac{(x-(-2))^2}{(1)^2} + \dfrac{(y-1)^2}{(2)^2} & = & 1 & \text{Rewrite in the desired form.}
\end{array}$$

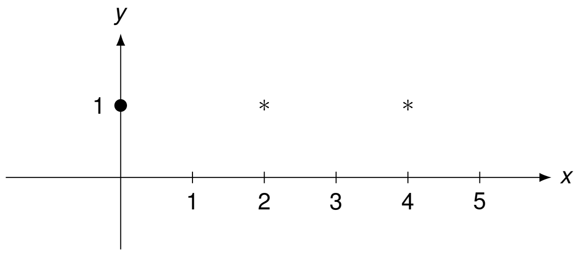
   We identify $h = -2$ and $k = 1$ so the ellipse is centered at $(-2, 1)$. With $a = 1$, we move 1 unit to the left and to the right an obtain the points $(-2-1, 1) = (-3, 1)$ and $(-2+1, 1) = (-1, 1)$. With $b = 2$, we move 2 units up and down from the center to obtain two more points on the graph of the ellipse: $(-2, 1+2) = (-2, 3)$ and $(-2, 1-2) = (-2, -1)$.

   However, the graph of $f(x) = 1 + 2\sqrt{-x^2-4x-3}$ cannot be the *entire* ellipse, else it would violate the vertical line test. This means the graph of f must be a *portion* of the ellipse. Since, by definition $\sqrt{-x^2-4x-3} \geq 0$, we know $f(x) = 1 + 2\sqrt{-x^2-4x-3} \geq 1$. Hence, the graph of f must be the *upper* half of the ellipse, as denoted below.

[3] Indeed, we leave it to the reader to show $-x^2 - 4x - 3 \geq 0$ on $[-3, -1]$.



3. (i) We plot the information given to us below and notice immediately that the major axis is horizontal, which means $a > b$.



Since the center is the midpoint of the foci, we know the center of the ellipse is $(3, 1)$ which means $h = 3$ and $k = 1$. Since one vertex is at $(0, 1)$, which is 3 units from the center, we have that $a = 3$, so $a^2 = 9$. At this point, all that remains is to find b^2.

Since the foci, $(2, 1)$ and $(4, 1)$, are 1 unit away from the center, we have $c = 1$. Putting this together with the fact that $a > b$, we get $c = \sqrt{a^2 - b^2}$, or $1 = \sqrt{9 - b^2}$. Squaring both sides gives $1 = 9 - b^2$ or $b^2 = 8$. Feeding all of this data into Equation **??**, we get our final answer:

$$\frac{(x-3)^2}{9} + \frac{(y-1)^2}{8} = 1.$$

(ii) From the diagram, we infer the ellipse is taller than it is wide. More specifically, the labeled points $(0, 0)$ and $(6, 0)$ are the endpoints of the minor axis. This gives the center is $(3, 0)$, so $h = 3$ and $k = 0$. Moreover, we have $a = 3$.

While it certainly *appears* that the vertices are $(3, \pm 4)$, in which case we'd have $b = 4$, these points aren't labeled. Instead, we use the labeled point $(1, 3)$ to calculate b^2. At this stage, we know the equation of the ellipse is

$$\frac{(x-3)^2}{9} + \frac{y^2}{b^2} = 1,$$

so upon substituting $x = 1$ and $y = 3$, we obtain $\frac{4}{9} + \frac{9}{b^2} = 1$. Solving this equation, we get $b^2 = \frac{81}{5}$. Hence, our final answer is:

$$\frac{(x-3)^2}{9} + \frac{5(y-1)^2}{81} = 1.$$

Note the vertices of the ellipse are $\left(3, \pm\sqrt{\frac{81}{5}}\right) \approx (3, \pm 4)$ but they are not *exactly* $(3, \pm 4)$.

As seen in Example **??** above, it is often necessary to algebraically manipulate a given equation into the standard form of Equation **??** in order to graph. We summarize one approach below.

To Write the Equation of an Ellipse in Standard Form

1. Group common variables together on one side of the equation and put the constant on the other.
2. Complete the square on both variables as needed.
3. Divide both sides, if needed, to obtain 1 on one side of the equation.

If we think of a circle as being 'perfectly round,' then ellipses, being deformed circles, have varying degrees of 'roundness.' We quantify this idea with the notion of *eccentricity* defined formally below.

Definition 2. The **eccentricity** of an ellipse, denoted e, is the following ratio:

$$e = \frac{\text{distance from the center to a focus}}{\text{distance from the center to a vertex}}$$

In an ellipse, the foci are closer to the center than the vertices, so $0 < e < 1$. Using the GeoGebra interactive below, we can adjust the distances from the foci to the center to produce a wide range of ellipses and their corresponding eccentricities. In general, the closer the eccentricity is to 0, the less 'eccentric' or more 'circular' the ellipse appears. On the other hand, the closer the eccentricity is to 1, the more 'eccentric' the ellipse is and it appears less 'circular.'

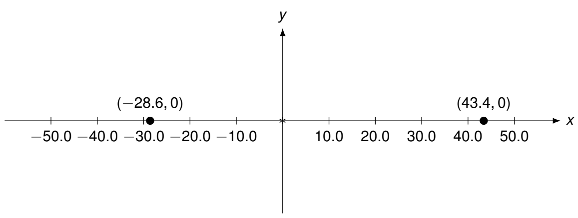
GeoGebra link: `https://www.geogebra.org/m/d6hg4zbs`

According to Kepler's Laws of Planetary Motion, each planet orbits the Sun in an elliptical path with the Sun at one focus. The eccentricity is therefore an important orbital parameter. We investigate the orbit of Mercury in the following example.

Example 2. According to NASA, Mercury orbits the Sun in an elliptical orbit. If at perihelion,[4] Mercury is 28.6 megamiles from the Sun and at aphelion,[5] Mercury is 43.4 megamiles from the Sun, find the eccentricity of Mercury's orbit, rounded to three decimal places.

Explanation. Per Kepler's Laws, the orbit of Mercury is an ellipse with the Sun at one focus. Since we are told to assume the Sun is positioned at $(0, 0)$, we are free to choose if the major axis of the ellipse lies on the x- or y-axis. We choose the former.

With the information given, we know one vertex is 28.6 units away from the focus and the other is 43.4 units. Again, we are free to chose which direction is which, so we decide to put one vertex at $(-28.6, 0)$ and the other is $(43.4, 0)$. Schematically, we have:



Since the center is the midpoint of the vertices, we find the center to be $(7.4, 0)$. This means the focus $(0, 0)$ is 7.4 units from the center and each vertex is 36 units from the center. This is precisely what we need to determine the eccentricity of Mercury's orbit:

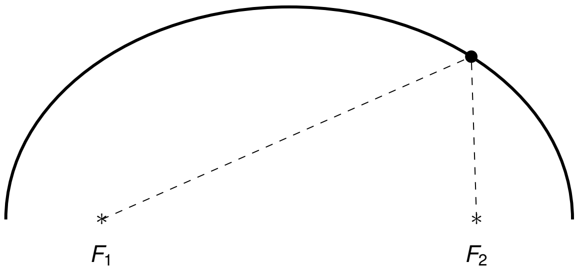
[4] the closest Mercury is to the Sun

[5] the farthest Mercury is from the Sun

$$e = \frac{\text{distance from the center to a focus}}{\text{distance from the center to a vertex}} = \frac{7.4}{36} \approx 0.205.$$

To our delight, we find answer agrees with NASA. With an eccentricity of 0.205, we expect Mercury's orbit to be fairly 'round.' In Exercise **??**, we invite the reader to find the equation of Mercury's orbit to confirm this conclusion graphically.

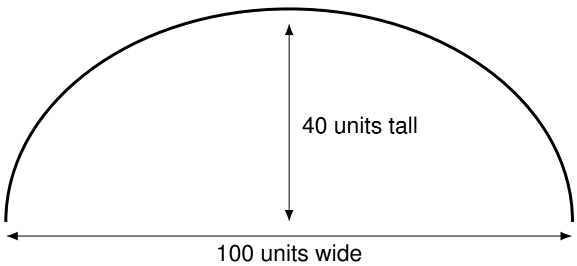
As with parabolas, ellipses have a reflective property. If we imagine the dashed lines below representing sound waves, then it can be shown that the waves emanating from one focus reflect off the top of the ellipse and head towards the other focus.



Such geometry is exploited in the construction of so-called 'Whispering Galleries'. If a person whispers at one focus, a person standing at the other focus will hear the first person as if they were standing right next to them. We explore the Whispering Galleries in our last example.

Example 3. Jamie and Jason want to exchange secrets (terrible secrets) from across a crowded whispering gallery. Recall that a whispering gallery is a room which, in cross section, is half of an ellipse. If the room is 40 feet high at the center and 100 feet wide at the floor, how far from the outer wall should each of them stand so that they will be positioned at the foci of the ellipse?

Explanation. Ultimately, we are looking for information about the relative position of foci and the vertices, so we assume the whispering gallery can be represented by the upper half of an ellipse centered at the origin with its major axis along the x-axis. Graphing the data yields the schematic below.



Since the ellipse is 100 units wide and 40 units tall, we get $a = 50$ and $b = 40$, respectively. From this, we get $c = \sqrt{50^2 - 40^2} = \sqrt{900} = 30$, which means the foci are 30 units from the center. Hence, the foci are $50 - 30 = 20$ units from the vertices so Jason and Jamie should stand 20 feet from opposite ends of the gallery to exchange their secrets in what amounts to very public privacy.