PARABOLAS

We begin our study of the conic sections with parabolas, since we have already seen parabolas described as graphs of quadratic functions, $f(x) = ax^2 + bx + c$ $(a \neq 0)$. It turns out that we can also describe parabolas in terms of distances.

Definition 1. Let F be a point in the plane and D be a line not containing F. A **parabola** is the set of all points equidistant from F and D. The point F is called the **focus** of the parabola and the line D is called the **directrix** of the parabola.

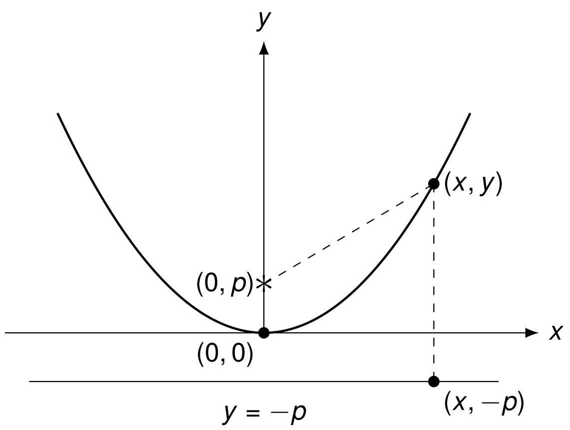
In the GeoGebra interactive below, click and drag the focus, F up and down to generate different parabolas. Click and drag the point A along the parabola. Each dashed line from the point F to a point on the curve has the same length as the dashed line from the point on the curve to the line D.

GeoGebra link: `https://www.geogebra.org/m/swpj4meu`

The point suggestively labeled V is, as you may expect, the **vertex**. The vertex is the point on the parabola closest to the focus.

We want to use only the distance definition of parabola to derive the equation of a parabola and, if all is right with the universe, we should get an expression much like those studied in Section **??**.

For simplicity, assume that the vertex is $(0, 0)$ and that the parabola opens upwards. Let p denote the directed[1] distance from the vertex to the focus, which by definition is the same as the distance from the vertex to the directrix. Hence, the focus is $(0, p)$ and the directrix is the line $y = -p$. Our picture becomes



From the definition of parabola, we know the distance from $(0, p)$ to (x, y) is the same as the distance from $(x, -p)$ to (x, y). Using the Distance Formula, Equation **??**, we get

$$\begin{array}{rcll}
\sqrt{(x-0)^2 + (y-p)^2} & = & \sqrt{(x-x)^2 + (y-(-p))^2} & \\
\sqrt{x^2 + (y-p)^2} & = & \sqrt{(y+p)^2} & \\
x^2 + (y-p)^2 & = & (y+p)^2 & \text{square both sides} \\
x^2 + y^2 - 2py + p^2 & = & y^2 + 2py + p^2 & \text{expand quantities} \\
x^2 & = & 4py & \text{gather like terms}
\end{array}$$

Solving for y yields $y = \frac{x^2}{4p} = \frac{1}{4p}x^2$, which is a quadratic function of the form found in Equation **??** with $a = \frac{1}{4p}$ and vertex $(0, 0)$.

We know from previous experience that if the coefficient of x^2 is negative, the parabola opens downwards. In the equation $y = \frac{1}{4p}x^2$ this happens when $p < 0$. In our formulation, we say that p is a 'directed distance'

[1] We'll talk more about what 'directed' means later.

from the vertex to the focus: if $p > 0$, the focus is above the vertex; if $p < 0$, the focus is below the vertex. The **focal length** of a parabola, that is, the length from the vertex to the focus, is therefore $|p|$.

If we choose to place the vertex at an arbitrary point (h, k), we arrive at the following formula using either transformations from Section **??** or re-deriving the formula from Definition **??**.

Formula 1. The Standard Equation of a Vertical[2] Parabola in the *xy*-plane:

The equation of a (vertical) parabola with vertex (h, k) and focal length $|p|$ is

$$(x - h)^2 = 4p(y - k)$$

If $p > 0$, the parabola opens upwards; if $p < 0$, it opens downwards.

Notice that in the standard equation of the parabola above, only one of the variables, x, is squared. As we'll see in the coming sections, this is a quick way to distinguish the equation of a parabola from equations representing the other conic sections.

Before embarking on an example, we take a moment to better illustrate the affect of the focal length $|p|$ on the graph of a parabola. In the GeoGebra interactive below, adjust the slider for p and observe the resulting parabola with focus F. In general, as the focal length $|p|$ increases, the parabola becomes wider.

GeoGebra link: `https://www.geogebra.org/m/rscgdtbm`

The dashed line segment in the interactive above is called the *latus rectum* of the parabola. More specifically, the **latus rectum** of a parabola is the line segment with endpoints on the parabola which contains the focus and is parallel to the directrix.[3]

We leave it to the reader to show that the length of the latus rectum, called the **focal diameter** of the parabola is $|4p| = 4|p|$, which appears ever so conveniently in the standard form as stated in Equation **??**.

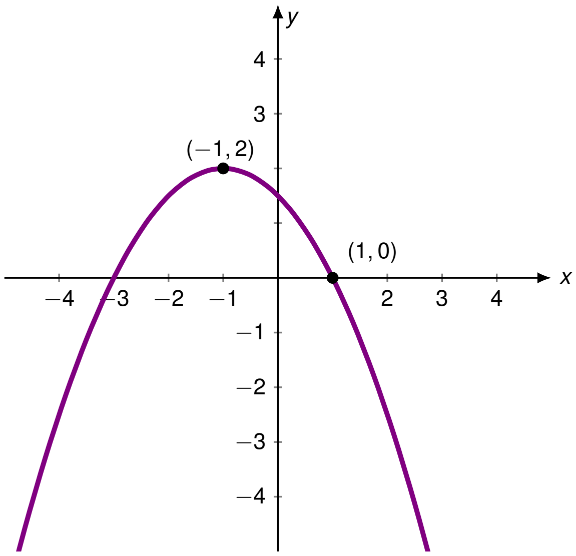
Knowing the focus and focal diameter allows to plot two points on the parabola in addition to the vertex, thus producing a more accurate graph.

Example 1.

1. Graph $(x + 1)^2 = -8(y - 3)$ in the xy-plane. Find the vertex, focus, and directrix. State the focal length, focal diameter, and find the endpoints of the latus rectum.
2. Find the standard form of the equation of the parabola with focus $(2, 1)$ and directrix $y = -4$.
3. Find the standard form of the equation of the parabola sketched below:

[2] That is, a parabola which opens either upwards or downwards.

[3] Hence, the endpoints of the latus rectum are two points on 'opposite' sides of the parabola.



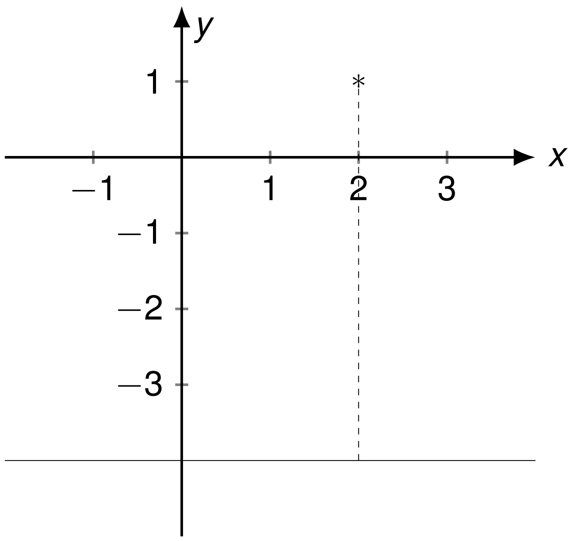
Explanation. 1. Rewriting $(x+1)^2 = -8(y-3)$ as $(x-(-1))^2 = -8(y-3)$, we identify $h=-1$ and $k=3$ in Equation **??**, so the vertex is $(-1,3)$. Additionally, we have $4p=-8$ so $p=-2$. Since $p<0$, the focus is *below* the vertex so the parabola opens *downwards*.

The focal length is $|p|=2$, which means the focus is 2 units below the vertex. From $(-1,3)$, we move down 2 units and find the focus at $(-1,3-2)=(-1,1)$. Likewise the directrix is 2 units above the vertex, or the horizontal line $y=3+2=5$.

The focal diameter is $|4p|=|-8|=8$, which means the parabola is 8 units wide at the focus. Hence, the endpoints of the latus rectum are 4 units to the left and right of the focus. Starting at $(-1,1)$ and moving to the left 4 units, we arrive at $(-1-4,1)=(-5,1)$. Starting at $(-1,1)$ and moving to the right 4 units we arrive at $(-1+4,1)=(3,1)$. The final graph is below.

Desmos link: `https://www.desmos.com/calculator/w1grtpwsfk`

2. We begin by sketching the data given to us below.



Since the focus is $(2,1)$, we know the the vertex lies on the vertical line $x=2$. Moreover, since the vertex is halfway between the focus and directrix, we know the vertex is exactly $\frac{5}{2}$ units *below* the focus at $\left(2, 1 - \frac{5}{2}\right) = \left(2, -\frac{3}{2}\right)$. This gives $h = 2$ and $k = -\frac{3}{2}$. Since the focus of the parabola is $\frac{5}{2}$ units *above* the vertex we know $p = +\frac{5}{2}$. Using Equation **??**, we get our final answer: $(x-2)^2 = 4\left(\frac{5}{2}\right)\left(y - \left(-\frac{3}{2}\right)\right)^2$ or $(x-2)^2 = 10\left(y + \frac{3}{2}\right)^2$.

3. From the graph, we assume the point labeled $(-1, 2)$ is the vertex, which means in the context of Equation **??**, $h = -1$ and $k = 2$. Hence, at this point, we know the equation is $(x - (-1))^2 = 4p(y-2)$, or, more simply $(x+1)^2 = 4p(y-2)$.

   To determine the value of p, we see $(1, 0)$ is on the graph so when $x = 1$, $y = 0$. Substituting these values into our equation gives $(1+1)^2 = 4p(0-2)$ so $4 = -8p$ or $p = -\frac{1}{2}$. (The fact $p < 0$ tracks with the parabola opening downwards.) Hence, $4p = 4\left(-\frac{1}{2}\right) = -2$ so the equation of the parabola is $(x+1)^2 = -2(y-2)$. We leave it to the reader to check our answer analytically and graphically.

We can produce 'horizontal' parabolas in the xy-plane by reflecting our so-called 'vertical' parabolas about the line $y = x$. As you may recall from Section **??**, we accomplish this algebraically by interchanging the variables x and y. Such parabolas necessarily open to the left or to the right, which means that unlike the vertical parabolas, these parabolas do not represent y as a function of x. As we shall see, however, they can *implicitly* describe y as a function of x, provided certain restrictions are in place.

Formula 2. The Standard Equation of a Horizontal Parabola:

The equation of a (horizontal) parabola with vertex (h, k) and focal length $|p|$ is

$$(y-k)^2 = 4p(x-h)$$

If $p > 0$, the parabola opens to the right; if $p < 0$, it opens to the left.

As we saw in Section **??**, when we reflect a horizontal line across the line $y = x$, we obtain a vertical line, and, as a result, the directrix of a *horizontal* parabola is a *vertical* line. Moreover, the focus of a horizontal parabola is either to the *left* or right of the directrix.

In the GeoGebra interactive below adjust the slider for the focal length, p and observe the resuting 'horizontal' parabola.

GeoGebra link: `https://www.geogebra.org/m/vumxajqy`

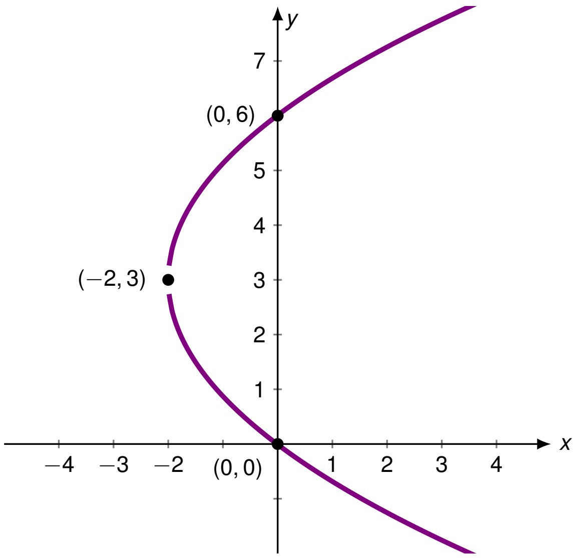
Example 2.

1. For each of the equations below:
   - Graph the equation in the xy-plane.
   - Find the vertex, focus, and directrix. State the focal length, focal diameter, and find the endpoints of the latus rectum.

   (i) $(y-2)^2 = 12(x+1)$.
   (ii) $y^2 + 4y + 8x = 4$

2. Represent each of the parabolas in number **??** as the graphs of two or more explicit functions of x.

3. Find the standard form of the parabola satisfying the following characteristics:
   (i) The focus is $(-4, 2)$ and the directrix is the y-axis.
   (ii) The parabola whose graph is sketched below:



Explanation. 1. (i) Rewriting $(y-2)^2 = 12(x+1)$ as $(y-2)^2 = 12(x-(-1))$, we identify $h=-1$ and $k=2$ so per Equation **??**, the vertex is $(-1,2)$. We also see that $4p=12$ so $p=3$. Since $p>0$, this means the focus is to the *right* of the vertex so the parabola opens to the *right*.

The focal length is $|p|=3$, which means the focus is 3 units to the right of the vertex. From $(-1,2)$, we move 3 units to the right and find the focus at $(-1+3,2)=(2,2)$. Likewise the directrix is 3 units to the left of the vertex, the vertical line $x=-1-3=-4$.

The focal diameter is $|4p|=|12|=12$, which means the parabola is 12 units wide at the focus. Hence, the endpoints of the latus rectum are 6 units above and below the focus. Starting at $(2,2)$ and moving down 6 units, we arrive at $(2,2-6)=(2,-4)$. Starting at $(2,2)$ and moving up 6 units we arrive at $(2,2+6)=(2,8)$. The final graph is below.

Desmos link: `https://www.desmos.com/calculator/0wspllmyi9`

(ii) Unlike the previous example, the equation $y^2+4y+8x=4$ is not in the form described in Equation **??**. In order to get an equivalent equation in the form prescribed by Equation **??**, we need to complete the square in y on the left-hand side of the equation. Once that is done, we factor out the coefficient of x on the other side of the equation below.

$$\begin{array}{rcll} y^2+4y+8x &=& 4 & \\ y^2+4y &=& -8x+4 & \\ y^2+4y+4 &=& -8x+4+4 & \text{complete the square in } y. \\ (y+2)^2 &=& -8x+8 & \text{factor} \\ (y+2)^2 &=& -8(x-1) & \end{array}$$

The equation $(y+2)^2=-8(x-1)$, rewritten as $(y-(-2))^2=-8(x-1)$ is in the form given in Equation **??**. Identifying $h=1$ and $k=-2$, we get the vertex is $(1,-2)$. Moreover, we see $4p=-8$ so that $p=-2$. The fact that $p<0$, means the focus will be the *left* of the vertex so the parabola will open to the *left*.

Since the focal length is $|p|=2$, the focus is 2 units to the left of the vertex. From $(1,-2)$ and move left 2 units and arrive at the focus $(1-2,-2)=(-1,-2)$. Similarly, the directrix is 2 units to the right of the vertex, the vertical line $x=1+2=3$.

Since the focal diameter is $|4p|$ is 8, the parabola is 8 units wide at the focus. Starting at the focus $(-1,-2)$ we move down 4 units and get $(-1,-2-4)=(-1,-6)$. Moving up 4 units from the focus we get $(-1,-2+4)=(-1,2)$. Hence, $(-1,-6)$ and $(-1,2)$ are the endpoints of the latus rectum. The final graph is below.

Desmos link: `https://www.desmos.com/calculator/2fh7y50igq`

2. To describe these parabolas as graphs of functions of x, we solve each equation for y in terms of x.

   (i) Starting with $(y-2)^2 = 12(x+1)$, we extract square roots to get $y-2 = \pm\sqrt{12(x+1)}$. Isolating y, we get $y = 2 \pm \sqrt{12(x+1)}$ which simplifies to $y = 2 \pm 2\sqrt{3x+3}$.

   We let $f(x) = 2 + \sqrt{3x+3}$ and $g(x) = 2 - \sqrt{3x+3}$. Note that since $\sqrt{3x+3} \geq 0$ by definition, $f(x) = 2 + \sqrt{3x+3} \geq 2$ which means the graph of f describes the *upper* half of the parabola. Similarly, the graph of $g(x) = 2 - \sqrt{3x+3}$ describes the *lower* half of the parabola.

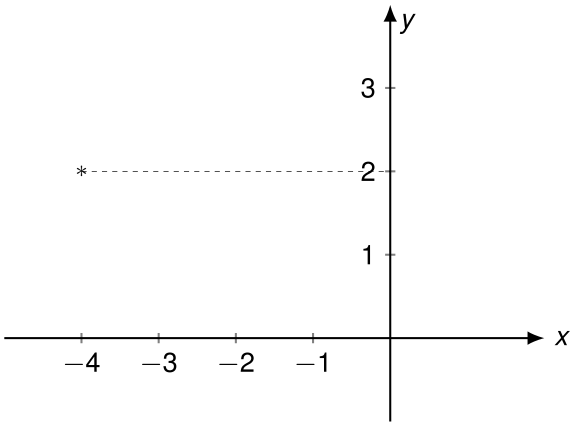
   (ii) We can solve $y^2 + 4y + 8x = 4$ for y by completing the square or using the quadratic formula. We leave the former to the reader,[4] and proceed with the latter for the sake of practice.

   To use the quadratic formula, we need to set the equation to 0: $y^2 + 4y + 8x - 4 = 0$. Since we are solving for y, we identify $a = 1$, $b = 4$ and $c = 8x - 4$. We find the discriminant $b^2 - 4ac = (4)^2 - 4(1)(8x-4) = 32 - 32x$ so

$$y = \frac{-4 \pm \sqrt{32-32x}}{2} = \frac{-4 \pm \sqrt{32(1-x)}}{2} = \frac{-4 \pm 4\sqrt{2(1-x)}}{2} = -2 \pm 2\sqrt{2-2x}.$$

   We identify $f(x) = -2 + 2\sqrt{2-2x}$ and $g(x) = -2 - 2\sqrt{2-2x}$. Since $\sqrt{2-2x} \geq 0$, we see the graph of f traces out the *upper* half of the parabola while the graph of g traces out the *lower* half of the parabola.

3. (i) We sketch the data below.



   Since the focus is $(-4, 2)$ and the directrix is a vertical line, we know the vertex must lie on the horizontal line $y = 2$. Moreover, we know the vertex must lie midway between the focus and directrix which in this case is $(-2, 2)$. This gives $h = -2$ and $k = 2$. Since the focus is 2 units to the *left* of the vertex, we know $p = -2$. Using Equation **??**, we get our answer as $(y-2)^2 = 4(-2)(x-(-2))$ or $(y-2)^2 = -8(x+2)$.

   (ii) From the graph, we may infer the vertex of the parabola is $(-2, 3)$ so $h = -2$ and $k = 3$. Per Equation **??**, we have $(y-3)^2 = 4p(x-(-2))$ or $(y-3)^2 = 4p(x+2)$. Since the graph contains $(0, 6)$, we substitute $x = 0$ and $y = 6$ to get $(6-3)^2 = 4p(0+2)$. We find $p = \frac{9}{8}$ which gives our final answer $(y-3)^2 = 4\left(\frac{9}{8}\right)(x+2)$ or $(y-3)^2 = \frac{9}{2}(x+2)$.

As we have seen, not all equations which describe parabolas will immediately match Equation **??** or Equation **??**. Indeed, completing the square as we did with the equation in number **??** in Example **??** will be a necessary skill not only in this section, but in the rest of this chapter.

For parabolas, we summarize the procedure for putting an equation of a parabola into standard form below. Of key importance is that in the equation for a parabola, one, and only one, of the variables are squared.

To Write the Equation of a Parabola in Standard Form

[4] The standard form of the parabola will make an appearance using this route.

1. Group the variable which is squared on one side of the equation and position the non-squared variable and the constant on the other side.
2. Complete the square if necessary and divide by the coefficient of the perfect square.
3. Factor out the coefficient of the non-squared variable from it and the constant.

In studying quadratic functions, we have seen parabolas used to model physical phenomena such as the trajectories of projectiles. Other applications of the parabola concern its 'reflective property' which necessitates knowing about the focus of a parabola. For example, many satellite dishes are formed in the shape of a **paraboloid of revolution** as depicted below.

GeoGebra link: https://www.geogebra.org/m/xrvhewz7

Every cross section through the vertex of the paraboloid is a parabola with the same focus. To see why this is important, imagine the dashed lines below as electromagnetic waves heading towards a parabolic dish. It turns out that the waves reflect off the parabola and concentrate at the focus which then becomes the optimal place for the receiver.

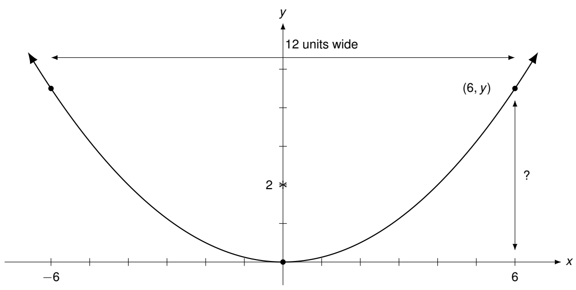
If, on the other hand, we imagine the dashed lines as emanating from the focus, we see that the waves are reflected off the parabola in a coherent fashion as in the case in a flashlight. Here, the bulb is placed at the focus and the light rays are reflected off a parabolic mirror to give directional light.

In the GeoGebra interactive below, click and drag the point on the parabola A to see how a rays from a light source at the focus F reflect off a parabolic mirror. Additionally, clicking and dragging the focus F allows us to observe this phenomenon on a variety of parabolas.

GeoGebra link: https://www.geogebra.org/m/kqyxjura

Example 3. A satellite dish is to be constructed in the shape of a paraboloid of revolution. If the receiver placed at the focus is located 2 ft above the vertex of the dish, and the dish is to be 12 feet wide, how deep will the dish be?

Explanation. One way to approach this problem is to determine the equation of the parabola suggested to us by this data. For simplicity, we'll assume the vertex is $(0, 0)$ and the parabola opens upwards. Our standard form for such a parabola is $x^2 = 4py$. Since the focus is 2 units above the vertex, we know $p = 2$, so we have $x^2 = 8y$. Visually,



Since the parabola is 12 feet wide, we know the edge is 6 feet from the vertex. To find the depth, we are looking for the y value when $x = 6$. Substituting $x = 6$ into the equation of the parabola yields $6^2 = 8y$ or $y = \frac{36}{8} = \frac{9}{2} = 4.5$. Hence, the dish will be 4.5 feet deep.