Small test-subset (wim)

2026/08/04

Contents

0.1 ABOUT THIS PROJECT

PreCalculus

Precalculus by Carl Stitz and Jeff Zeager began with a sabbatical in 2008 when Paul Zachlin was sharing an office with Carl. Later, when Paul began publishing textbooks on Ximera with Anna Davis, he knew that it would be great to publish this book on the Ximera server in order to incorporate modern tools that can only be accessed online which increase the learning experience.

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Many ancillaries that can be used with this text can be found at stitz-zeager.com We are grateful for the opportunity to share this Precalculus book with you, and we welcome your feedback. You may also be interested in other publications, Linear Algebra, An Interactive Introduction, and Differential Equations, An Interactive Introduction.

1.1 CONSTANT AND LINEAR FUNCTIONS

Now that we have defined the concept of a function, we'll spend the rest of Chapter **??** revisiting families of curves from prior courses in Algebra by viewing them through a 'function lens'. We start with lines and refer the reader to Section **??** for a review of the basic properties of lines. The simplest lines are vertical and horizontal lines. We leave it to the reader (see Exercise **??**) to think about why we eschew vertical lines in our discussion here, and begin with a functional description of horizontal lines.

Consider the horizontal lines graphed in the xy-plane as shown below. The Vertical Line Test, Theorem **??**, tells us that each describes y as a function of x so the question becomes how to represent these functions algebraically. The key here is to remember that the equation relating the independent variable x, the dependent variable y, and the function f is given by $y = f(x)$.

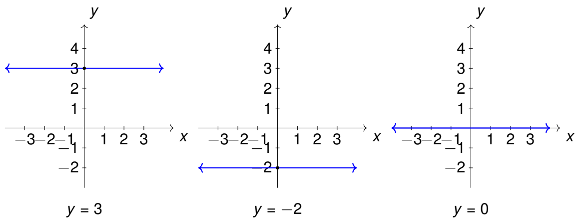


Figure 1: Graphs of constant functions

Alt text: Three examples of horizontal lines

In the graph on the left, y always equals 3 so we have $f(x) = 3$. Procedurally, '$f(x) = 3$' says that the rule f takes the input x, and, regardless of that input, gives the output 3. This is an example of what is called a **constant** function - a function which returns the **same** value regardless of the input. Likewise, the function represented by the graph in the middle is $f(x) = -2$, and the graph on the right (the x-axis) is the graph of $f(x) = 0$. In general, we have the following definition:

Definition 1. A **constant function** is a function of the form

$$f(x) = b$$

where b is real number. The domain of a constant function is $(-\infty, \infty)$.

Some remarks about Definition 1 are in order. First, note that we are using 'x' as the independent variable, 'f' as the function name, and the letter 'b' as a **parameter**. In this context, a parameter is a fixed, but arbitrary, constant used to describe a **family** of functions. Different values of b determine different constant functions. For example, $b = 3$ gives $f(x) = 3$, $b = -2$ gives $f(x) = -2$, and so on. Once b is chosen, however, it does not change as the independent variable, x, changes.

Also note that we are using the generic defaults for function names and independent variables, namely f and x, respectively. The functions $G(t) = \sqrt{\pi}$ and $Z(\rho) = 0$ are also fine examples of constant functions. Recall that inherent in the definition of a function is the notion of domain, so we record (as part of the definition) that a constant function has domain

$(-\infty, \infty)$. The range of a constant function is the set $\{b\}$. The value b in this case is both the maximum and minimum of f, attained at each value in its domain.[1]

The next example showcases an application of constant functions and introduces the notion of a **piecewise-defined** function.

Example 1. The price of admission to see a matinee showing at a local movie theater is a function of the age of the ticket holder. If a person is aged A years, the price per ticket is $p(A)$ dollars and is given by:

$$p(A) = \begin{cases} 5.75 & \text{if } 0 \leq A < 6 \text{ or } A \geq 50, \\ 7.25 & \text{if } 6 \leq A < 50 \end{cases}$$

1. Find and interpret $p(3)$, $p(6)$ and $p(62)$.
2. Explain the pricing structure verbally.
3. Graph p.

Explanation. The function p described above is an example of a **piecewise-defined** function because the **rule** to determine outputs, not just the value of the output, changes depending on the inputs.

1. To find $p(3)$, we note that the value $A = 3$ satisfies the inequality $0 \leq A < 6$ so we use the rule $p(A) = 5.75$. Hence, $p(3) = 5.75$ which means a ticket for a 3 year old is \$5.75. The next age, $A = 6$, just barely satisfies the inequality $6 \leq A < 50$ so we use the rule $p(A) = 7.25$, This yields $p(6) = 7.25$ which means a ticket for a 6 year old is \$7.25. Lastly, $A = 62$ satisfies the inequality $A \geq 50$, so we are back to the rule $p(A) = 5.75$. Thus $p(62) = 5.75$ which means someone 62 years young gets in for \$5.75.

2. Now that we've had some practice interpreting function values, we can begin to verbalize what the function is really saying. In the first 'piece' of the function, the inequality $0 \leq A < 6$ describes ticket holders under the age of 6 years and the inequality $A \geq 50$ describes ticket holders fifty years old or or older. For folks in these two age demographics, $p(A) = 5.75$ so the price per ticket is \$5.75. For everyone else, that is for folks at least 6 but younger than 50, the price is \$7.25 per ticket.

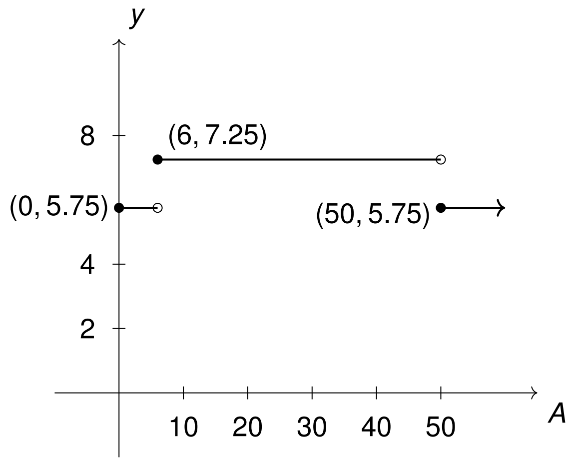
3. The independent variable here is specified as A, so we'll label our horizontal axis that way. The dependent variable remains unspecified so we can use the default y. The graph of $y = p(A)$ consists of three horizontal line pieces: the first is $y = 5.75$ for $0 \leq A < 6$, the second piece is $y = 7.25$ for $6 \leq A < 50$, and the last piece is $y = 5.75$ for $A \geq 50$.

   For the first piece, note that $A = 0$ is included in the inequality $0 \leq A < 6$ but $A = 6$ is not. For this reason, we have a point indicated at $(0, 5.75)$ but leave a hole[2] at

[1] It gets much weirder than that as we explore other more complicated functions. The key is to pay attention to the precision in the definitions of the terms involved in the discussion. Stay tuned!

[2] See our discussion about holes in graphs in Example **??** in Section **??**.

$(6, 5.75)$. Similarly, to graph the second piece, we begin with a point at $(6, 7.25)$ and continue the horizontal line to a hole at $(50, 7.25)$. Lastly, we finish the graph with a point at $(50, 5.75)$ and continue to the right indefinitely.[3] Note the scaling on the horizontal axis compared to the vertical axis.



$$y = p(A) = \begin{cases} 5.75 & \text{if } 0 \leq A < 6 \text{ or } A \geq 50, \\ 7.25 & \text{if } 6 \leq A < 50 \end{cases}$$

Figure 2: Graphs of constant functions

One of the favorite piecewise-defined functions in mathematical circles is the **greatest integer of x**, denoted by $\lfloor x \rfloor$. In Section **??** we defined the set of **integers** as $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.[4] The value $\lfloor x \rfloor$ is defined to be the largest integer k with $k \leq x$. That is, $\lfloor x \rfloor$ is the unique integer k such that $k \leq x < k + 1$. Said differently, given any real number x, if x is an integer, then $\lfloor x \rfloor = x$. If not, then x lies in an interval between two integers, k and $k + 1$ and we choose $\lfloor x \rfloor = k$, the left endpoint.

Example 2. Let $\lfloor x \rfloor$ denote the greatest integer function.

1. Find $\lfloor 0.785 \rfloor$, $\lfloor 117 \rfloor$, $\lfloor -2.001 \rfloor$ and $\lfloor \pi + 6 \rfloor$
2. Explain how we can view $\lfloor x \rfloor$ as a piecewise-defined function and use this to graph $y = \lfloor x \rfloor$.

Explanation. 1. To find $\lfloor 0.785 \rfloor$, we note that $0 \leq 0.785 < 1$ so $\lfloor 0.785 \rfloor = 0$. Given that 117 is an integer, we have $\lfloor 117 \rfloor = 117$. To find $\lfloor -2.001 \rfloor$, we note that $-3 \leq -2.001 < -2$, so $\lfloor -2.001 \rfloor = -3$. Finally, with $\pi \approx 3.14$, we get $\pi + 6 \approx 9.14$ and $9 \leq \pi + 6 < 10$ so $\lfloor \pi + 6 \rfloor = 9$.

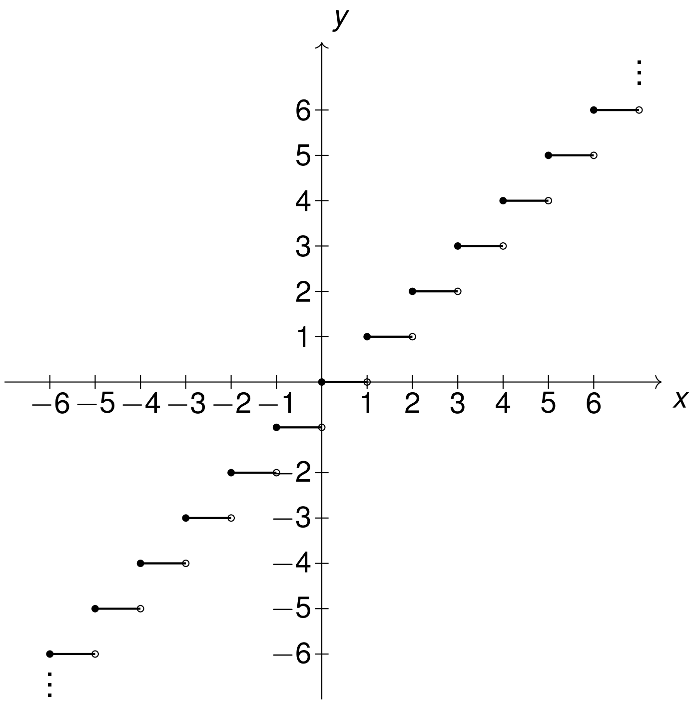
[3]The domain of p is $[0, \infty)$ by definition, even though few 327 year olds are out and about these days.
[4]The use of the letter $\mathbb{Z}$ for the integers is ostensibly because the German word **zahlen** means 'to count'.

2. The first step in evaluating $\lfloor x \rfloor$ is to determine the interval $[k, k+1)$ containing x so it seems reasonable that these are the intervals which produce the 'pieces'. In this case, there happen to be infinitely many pieces. The inequality '$k \leq x < k+1$' includes the left endpoint but excludes the right endpoint, so we have points at the left endpoints of our horizontal line segments while we have holes at the right endpoints.

A partial description of $\lfloor x \rfloor$ is given alongside a partial graph at the top of the next page. (A full description or a complete graph would require infinitely large paper!) We use the vertical dots $\vdots$ to indicate that both the rule and the graph continue indefinitely following the established pattern.[5]

$$\lfloor x \rfloor = \begin{cases} \vdots & \\ -5 & \text{if } -5 \leq x < -4 \\ -4 & \text{if } -4 \leq x < -3 \\ -3 & \text{if } -3 \leq x < -2 \\ -2 & \text{if } -2 \leq x < -1 \\ -1 & \text{if } -1 \leq x < 0 \\ 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } 1 \leq x < 2 \\ 2 & \text{if } 2 \leq x < 3 \\ 3 & \text{if } 3 \leq x < 4 \\ 4 & \text{if } 4 \leq x < 5 \\ 5 & \text{if } 5 \leq x < 6 \\ \vdots & \end{cases}$$

[5] It is always dangerous to leave the rest of the pattern to the reader. See, for instance, this paper.



The graph of $y = \lfloor x \rfloor$.

1.1.A LINEAR FUNCTIONS

Now that we've discussed the functions which correspond to horizontal lines, $y = b$, we move to discussing the functions which can be represented by lines of the form $y = mx + b$ where $m \neq 0$. These functions are called **linear** functions and are described below.

Definition 2. A **linear function** is a function of the form

$$f(x) = mx + b,$$

where m and b are real numbers with $m \neq 0$. The domain of a linear function is $(-\infty, \infty)$.

As with Definition 1, in Definition 2, x is the independent variable, f is the function name, and both m and b are parameters. Notice that m is restricted by $m \neq 0$ for if $m = 0$ then the function $f(x) = mx + b$ would reduce to the constant function $f(x) = b$. The domain of linear functions, like that of constant functions, is specified as $(-\infty, \infty)$

Recall[6] that the form of the line $y = mx + b$ is called the slope-intercept form of the line and the slope, m, and the y-intercept $(0, b)$, are easily determined when the line is written this way. Likewise, the form of the function in Definition 2, $f(x) = mx + b$, is often called the **slope-intercept form** of a linear function.

The graph of a linear function is the graph of the line $y = mx + b$. Lines are uniquely determined by two points, and two points of geometric interest are the axis intercepts. We've already reminded you of the y-intercept, $(0, b)$, which is obtained by setting $x = 0$. Similarly, to find the x-intercept, we set $y = 0$ and solve $mx+b = 0$ for x. We leave this to the reader in Exercise **??**. In addition to having special graphical significance, axis intercepts quite often play important roles in applications involving both linear and non-linear functions. For that reason, we take the time to define them here using function notation.

Definition 3. Suppose f is a function represented by the graph of $y = f(x)$.

- If 0 is in the domain of f then the point $(0, f(0))$ is the ***y*-intercept** of the graph of $y = f(x)$.

  That is, $(0, f(0))$ is where the graph meets the y-axis.

- If 0 is in the range of f then the solutions to $f(x) = 0$ are called the **zeros** of f. If c is a zero of f then the point $(c, 0)$ is an ***x*-intercept** of the graph of $y = f(x)$.

  That is, $(c, 0)$ is where the graph meets the x-axis.

As is customary in this text, Definition 3 uses the default independent variable x, function name f, and dependent variable y, so these letters will change depending on the context. Also note that the 'zeros' of a function are the solutions to $f(x) = 0$ - so they are **real numbers**. The x-intercepts are, on the other hand, **points** on the graph. As a quick example, consider $f(x) = x - 3$. The zeros of f are found by solving $f(x) = 0$, or $x - 3 = 0$. We get one solution, $x = 3$. Therefore, $x = 3$ is the **zero** of f that corresponds graphically to the ***x*-intercept** $(3, 0)$.

[6] or see Section **??**

1.1.A LINEAR FUNCTIONS

We now turn our attention to slope. The role of slope, or more generally a 'rate of change', in Science and Mathematics cannot be overstated.[7] As you may recall, or quickly read about on page **??**, the slope of a line that has been graphed in the xy-plane is defined geometrically as follows:

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x},$$

where the capital Greek letter 'Δ' denotes 'change in.'[8] In this course, it is vital that we regard the slope of a linear function as a rate of change of **function outputs** to **function inputs**. That is, given the graph of a linear function $y = f(x) = mx + b$:

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{\Delta[f(x)]}{\Delta x} = \frac{\Delta\text{outputs}}{\Delta\text{inputs}}.$$

What is important to note here is that for linear functions, the rate of change m is constant for all values in the domain.[9] We'll see the importance of this statement in the upcoming examples.

Geometrically, the sign of the slope has a profound impact on the graph of the line. Recall that if the slope $m > 0$, the line rises as we read from left to right; if $m < 0$, the line falls as we read from left to right; if $m = 0$, we have a horizontal line and the graph plateaus. We define these notions more precisely for general functions in the following definition.

Definition 4. Let f be a function defined on an interval I. Then f is said to be:

- **increasing** on I if, whenever $a < b$, then $f(a) < f(b)$. (i.e., as inputs increase, outputs **increase**.)

  NOTE: The graph of an increasing function **rises** as one moves from left to right.

- **decreasing** on I if, whenever $a < b$, then $f(a) > f(b)$. (i.e., as inputs increase, outputs **decrease**.)

  NOTE: The graph of a decreasing function **falls** as one moves from left to right.

- **constant** on I if $f(a) = f(b)$ for all a, b in I. (i.e., outputs don't change with inputs.)

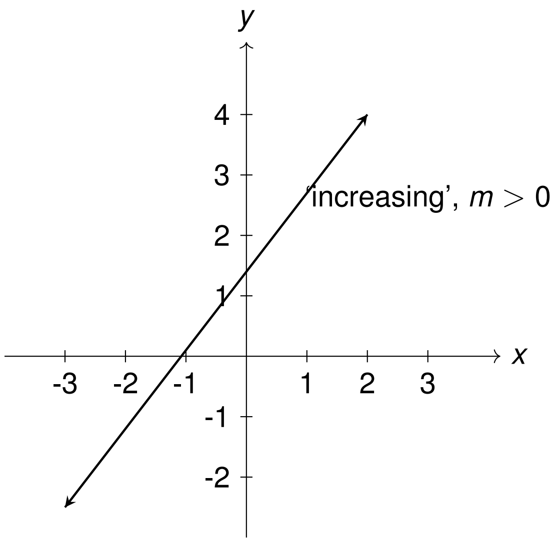
  NOTE: The graph of a function that is constant over an interval is a horizontal line.

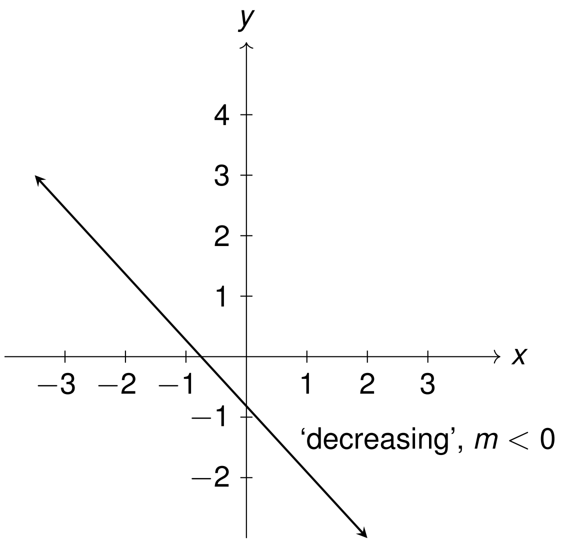
Again, as with Definition 3, Definition 4 applies to any function, not just linear and constant functions. Also, note that, like Definition **??**, Definition 4 blurs the line between the function, f, and its outputs, $f(x)$, because the verbiage 'f is increasing' is really a statement about the outputs, $f(x)$. Finally, when we ask 'where' a function is increasing, decreasing or constant, we are looking for an interval of **inputs**. We'll have more to say about this in later sections, but for now, we summarize these ideas graphically below.

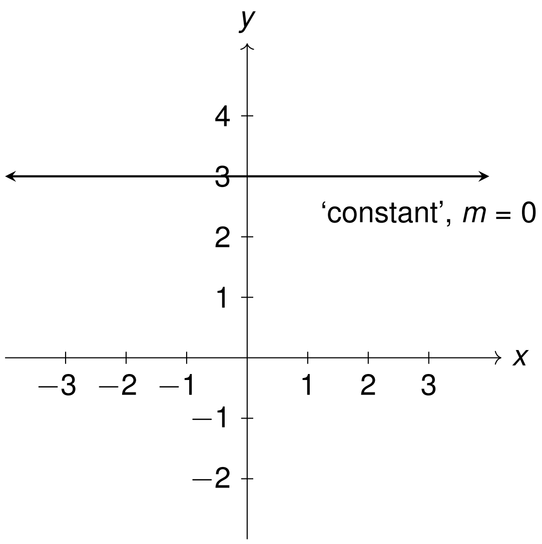
[7] The first half of any introductory Calculus course is about slope.

[8] More specifically, if (x_0, y_0) and (x_1, y_1) are two distinct points in the plane, then $\Delta x = x_1 - x_0$ and $\Delta y = y_1 - y_0$.

[9] See Exercise **??** for more details.







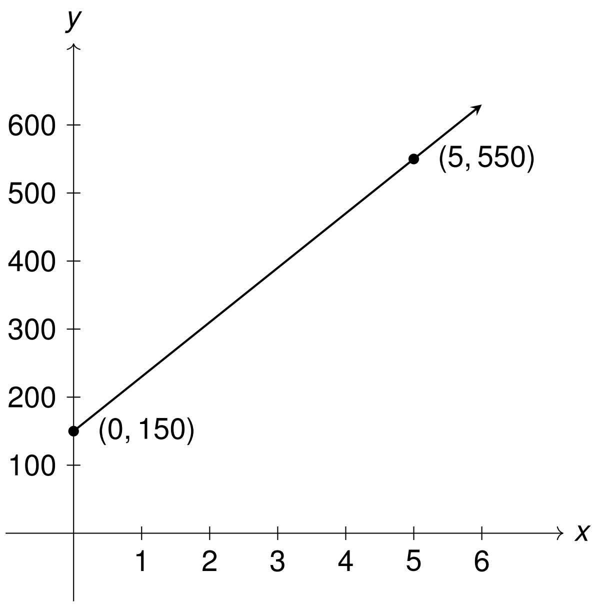
From the graphs above, we see that regardless if $m > 0$ or $m < 0$, the range of linear functions is $(-\infty, \infty)$. Therefore, linear functions have no maximum or minimum.[10]

Example 3. The cost, in dollars, to produce x PortaBoy[11] game systems for a local retailer is given by $C(x) = 80x + 150$ for $x \geq 0$.

1. Find and interpret $C(0)$ and $C(5)$ and use these to graph $y = C(x)$.
2. Explain the significance of the restriction on the domain, $x \geq 0$.
3. Interpret the slope of $y = C(x)$ geometrically and as a rate of change.
4. How many PortaBoys can be produced for \$15,000?

Solution.

1. To find $C(0)$, we substitute 0 for x in the formula $C(x)$ and obtain: $C(0) = 80(0) + 150 = 150$. Given that x represents the number of PortaBoys produced and $C(x)$ represents the cost to produce said PortaBoys, $C(0) = 150$ means it costs \$150 even if we don't produce any PortaBoys at all. At first, this may not seem realistic, but that \$150 is often called the **fixed** or **start-up** cost of the venture. Things like re-tooling equipment, leasing space, or any other 'up front' costs get lumped into the fixed cost. To find $C(5)$, we substitute 5 for x in the formula $C(x)$: $C(5) = 80(5) + 150 = 550$. This means it costs \$550 to produce 5 PortaBoys for the local retailer. These two computations give us two points on the graph: $(0, C(0))$ and $(5, C(5))$. Along with the domain restriction $x \geq 0$, we get:



$y = C(x)$

[10]This is one of the more pedantic reasons why we distinguish between constant and linear functions. See the discussion concerning the range of a constant function on page 6.

[11]The similarity of this name to PortaJohn is deliberate.

2. In this context, x represents the number of PortaBoys produced. It makes no sense to produce a negative quantity of game systems,[12] so $x \geq 0$.

3. The cost function $C(x) = 80x + 150$ is in slope-intercept form so we recognize the slope as the coefficient of x, $m = 80$. With $m > 0$, the function C is always increasing. This means that it costs more money to make more game systems. To interpret the slope as a rate of change, we note that the output, $C(x)$, is the cost in dollars, while the input, x, is the number of PortaBoys produced:
$$m = 80 = \frac{80}{1} = \frac{\Delta[C(x)]}{\Delta x} = \frac{\$80}{\text{1 PortaBoy produced}}.$$
Hence, the cost to produce PortaBoys is increasing at a rate of \$80 per PortaBoy produced. This is often called the **variable cost** for the venture.

4. To find how many PortaBoys can be produced for \$15,000, we solve $C(x) = 15000$, which means $80x + 150 = 15000$. This yields $x = 185.625$. We can produce only a whole number amount of PortaBoys so we are left with two options: produce 185 or 186 PortaBoys. Given that $C(185) = 14950$ and $C(186) = 15030$, we would be over budget if we produced 186 PortaBoys. Hence, we can produce 185 PortaBoys for \$15,000 (with \$50 to spare).

A couple of remarks about Example 3 are in order. First, if x represents the number of PortaBoy game systems being produced, then x can really only take on whole number values. We will revisit this scenario in Section **??** where we will see how the approach presented here allows us to use more elegant techniques when analyzing the situation than a discrete data set would allow.[13]

Second, once we know that the variable cost is \$80 per PortaBoy, we can revisit a computation we did earlier in the example. We computed $C(185) = 14950$ and needed to compute $C(186)$. With 186 being just one more PortaBoy than 185, we can use the variable cost to get
$$C(186) = C(185) + 80(1) = 14950 + 80 = 15030,$$
which agrees with our earlier computation.[14] If we wanted to find $C(300)$, we could do something similar. Using $300 - 185 = 115$, we can find $C(300)$ as follows:
$$C(300) = C(185) + 80(115) = 14950 + 9200 = 24150.$$
In general, we could rewrite $C(x) = C(185) + 80(x - 115)$. This same reasoning shows that for any x_0 in the domain of C, we have $C(x) = C(x_0) + 80(x - x_0)$ - a fact we invite the reader to verify.[15]

[12] Actually, it makes no sense to produce a fractional part of a game system, either, which we'll discuss later in this example.

[13] This is an example of using a 'continuous' variable to model a 'discrete' scenario. Contrast this with the discussion following Example **??** in Section **??**.

[14] The cost to produce 'just one more item' is called the **marginal cost**. The difference between variable and marginal costs in this case are the units used: the variable cost is \$80 per Portaboy whereas the marginal cost is simply \$80.

[15] In the case $x_0 = 0$, this formula reduces to $C(x) = C(0) + 80(x - 0) = 150 + 80x = 80x + 150$. To show the formula in general, consider $C(x_0) = 80x_0 + 150 \ldots$

Indeed, the computations above are at the heart of what it means to be a linear function: linear functions change at a constant rate known as the slope. To better see this algebraically, recall that given a point (x_0, y_0) on a line along with the slope, m, the **point-slope form of the line** is: $y - y_0 = m(x - x_0)$.[16] Rewriting, we get $y = y_0 + m(x - x_0)$ and setting $y = f(x)$ and $y_0 = f(x_0)$ yields:

Formula 1. The **point-slope form** of a linear function is

$$f(x) = f(x_0) + m(x - x_0)$$

A few remarks are in order. First note that if the point $(x_0, f(x_0))$ is the y-intercept $(0, b)$, Equation 1 immediately reduces to the slope-intercept form of the line: $f(x) = f(x_0) + m(x - x_0) = b + m(x - 0) = mx + b$, so you can use Equation 1 exclusively from this point forward.[17]

Second, if we write $\Delta x = x - x_0$, then $x = x_0 + \Delta x$ so we can rewrite Equation 1 as follows:

$$\begin{array}{ccccc} f(x_0 + \Delta x) & = & f(x_0) & + & m\Delta x \\ \text{(new output)} & = & \text{(known output)} & + & \text{(change in outputs)} \end{array}$$

In other words, changing the **input** by Δx results in changing the **output** by $m\Delta x$. This tracks since

$$m\Delta x = \frac{\Delta[f(x)]}{\Delta x}\Delta x = \Delta[f(x)] = \Delta\text{outputs}.$$

The fact that we can write $\Delta\text{outputs} = m\Delta x$ for any choice of x_0 is another way to see that for linear functions, the rate of change is constant. That is, the rate of change, m, is the same for all values x_0 in the domain. We'll put Equation 1 to good use in the next example.

Example 4. The local retailer in Example 3 is trying to mathematically model the relationship between the number of PortaBoy systems sold and the price per system. Suppose 20 systems were sold when the price was \$220 per system but when the systems went on sale for \$190 each, sales doubled.

1. Find a formula for a linear function p which represents the price $p(x)$ as a function of the number of systems sold, x. Graph $y = p(x)$, find and interpret the intercepts, and determine a reasonable domain for p.

2. Interpret the slope of $p(x)$ in terms of price and game system sales.

3. If the retailer wants to sell 150 PortaBoys next week, what should the price be?

4. How many systems would sell if the price per system were set at \$150?

Explanation. 1. We are asked to find a linear function $p(x)$ ostensibly because the retailer has only two data points and two points are all that is needed to determine a unique line. We know that 20 PortaBoys were sold when the price was 220 dollars and double that, so 40 units, were sold when the price was 190 dollars. Using

[16] See Section **??** for a review of this form.

[17] In other words, the slope intercept form of a line is just a special case of the point-slope form.

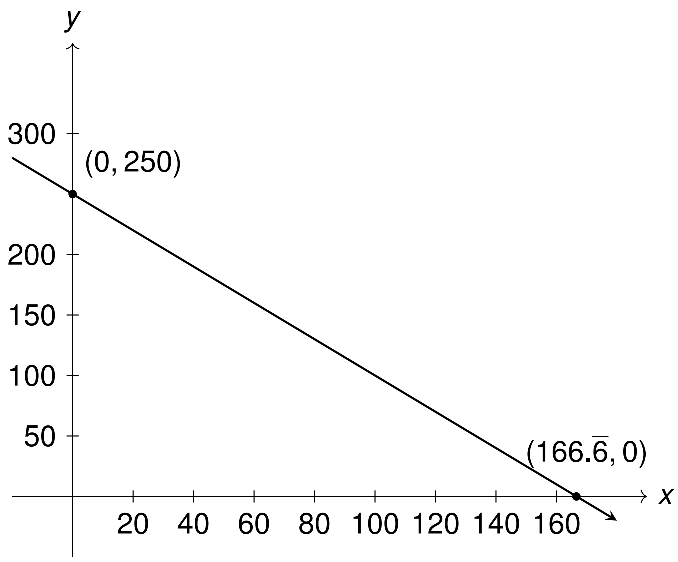
the language of function notation, these statements translate to $p(20) = 220$ and $p(40) = 190$, respectively. We first find the slope

$$m = \frac{\Delta[p(x)]}{\Delta x} = \frac{190 - 220}{40 - 20} = \frac{-30}{20} = -1.5$$

and then substitute it and a pair $(x_0, p(x_0))$ into the point-slope formula. We have two choices: $x_0 = 20$ and $p(x_0) = 220$ or $x_0 = 40$ and $p(x_0) = 190$. We'll choose the former and invite the reader to use the latter - both will result in the same simplified expression. The point-slope formula yields

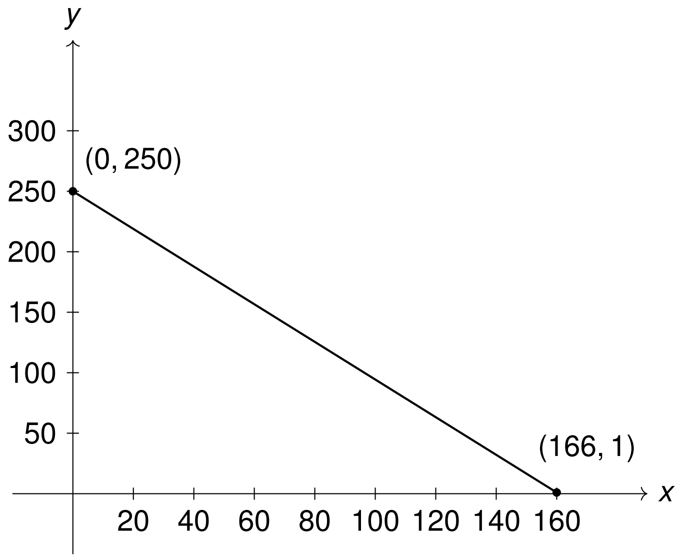
$$p(x) = p(x_0) + m(x - x_0) = 220 + (-1.5)(x - 40)$$

which simplifies to $p(x) = -1.5x + 250$. (To check this algebraically, we can verify that $p(20) = 220$ and $p(40) = 190$.) To find the y-intercept of the graph, we substitute $x = 0$ and find $p(0) = 250$. Hence our y-intercept is $(0, 250)$. To find the x-intercept, we set $p(x) = 0$. Solving $-1.5x + 250 = 0$ gives $x = 166.\overline{6}$, so our x-intercept is $(166.\overline{6}, 0)$.[18] The graph on the left is that of the line $y = -1.5x + 250$.



$y = -1.5x + 250$

[18] The exact value is $x = \frac{500}{3}$. Recall that the bar over the 6 indicates that the decimal repeats. See page **??** for details.



$y = p(x)$

To determine a reasonable domain for p, we certainly require $x \geq 0$, because we can't sell a negative number of game systems.[19] Next, we require $p(x) \geq 0$, otherwise we'd be **paying** customers to 'buy' PortaBoys. Solving $-1.5x + 250 \geq 0$ results in $x \leq 166.\overline{6}$. This shouldn't be too surprising since our graph passes through the x-axis at $(166.\overline{6}, 0)$, going from positive y-values (hence, positive $p(x)$ values) to negative y (hence negative $p(x)$ values).[20]

Given that x represents the number of PortaBoys sold, we need to choose to end the domain at either $x = 166$ or $x = 167$. We have that $p(166) = 1 > 0$ but $p(167) = -0.5 < 0$ so we settle on the domain $[0, 166]$. Our final answer is $p(x) = -1.5x + 250$ restricted to $0 \leq x \leq 166$ which is graphed above on the right.

2. The slope $m = -1.5$ represents the rate of change of the price of a system with respect to sales of PortaBoys. The slope is negative so we have that the price is **decreasing** at a rate of \$1.50 per PortaBoy sold. (Said differently, you can sell one more PortaBoy for every \$1.50 drop in price.)

3. To determine the price which will move 150 PortaBoys, we find $p(150) = -1.5(150) + 250 = 25$. That is, the price would have to be \$25 per system.

4. If the price of a PortaBoy were set at \$150, we'd have $p(x) = 150$, or $-1.5x + 250 = 150$. This yields $-1.5x = -100$ or $x = 66.\overline{6}$. Again our algebraic solution lies between two whole numbers, so we find $p(66) = 151$ and $p(67) = 149.5$. If the price were set at \$150, we'd sell 66 systems, since to sell 67 systems, we'd have to drop the price just under \$150.

The function p in Example 4 is called the **price-demand** function (or, sometimes called more simply a 'demand function') because it returns the price $p(x)$ associated with a certain

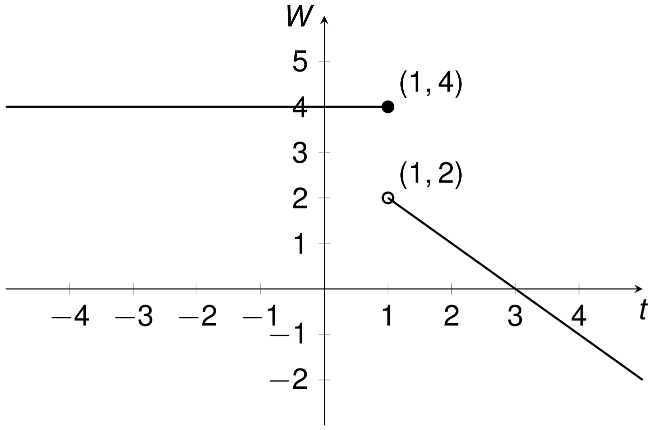
[19]ignoring returns, that is.

[20]We'll discuss these sorts of connections in greater depth in Section 5.

demand x - that is, how many products will sell.[21] These functions, along with cost functions like the one in Example 3, will be revisited in Example **??**.

Our next two examples focus on writing formulas for piecewise-defined functions, the second of which models a real-world situation.

Example 5. Find a formula for the function L graphed below.



Explanation. From the graph of $W = L(t)$ we see that there are two distinct pieces. Taking note of the point at $(1, 4)$, we get $L(t) = 4$ for $t \leq 1$. To represent L for $t > 1$, we use the point-slope form of a linear function: $L(t) = L(t_0) + m(t - t_0)$. The only 'point' labeled with this part of the graph is the hole at $(1, 2)$ and it isn't technically part of the graph, so we will avoid using it.[22] Instead, we infer from the graph two other points: $(2, 1)$ and $(3, 0)$. We get the slope to be

$$m = \frac{\Delta W}{\Delta t} = \frac{\Delta[L(t)]}{\Delta t} = \frac{3-2}{0-1} = -1.$$

Next, we choose a point to plug into $L(t) = L(t_0) + m(t - t_0)$. We have two options: $t_0 = 2$ and $L(t_0) = 1$ or $t_0 = 3$ and $L(t_0) = 0$. Using the latter, we get $L(t) = 0 + (-1)(t - 3)$, or $L(t) = -t + 3$. Putting this together with the first part, we get:

$$L(t) = \begin{cases} 4 & \text{if} \quad t \leq 1 \\ -t+3 & \text{if} \quad t > 1 \end{cases}$$

Note that when $t = 1$ is substituted into the expression $-t + 3$, we get 2, so the hole at $(1, 2)$ checks.[23]

Example 6. A popular Fōn-i smartphone carrier offers the following smartphone data plan: use any amount of data up to and including 4 gigabytes for \$60 per month with an 'overage' charge of \$5 per gigabyte. Determine a formula that computes the cost in dollars as a function of using g gigabytes of data per month. Graph your answer.

[21] It may seem counter-intuitive to express price as a function of demand. Shouldn't the price determine how many systems people will buy? We will address this issue later.

[22] We actually could use the point $(1, 2)$ to find the equation of the **line** containing $(1, 2)$ and, say $(3, 0)$, which is $y = -t + 3$. It's just that the graph of $L(t)$ and the line $y = -t + 3$ only agree for $t > 1$, so it would be incorrect to write $L(1) = 2$.

[23] Alternatively, for t values larger than 1 but getting closer and closer to 1, $L(t) \approx 2$.

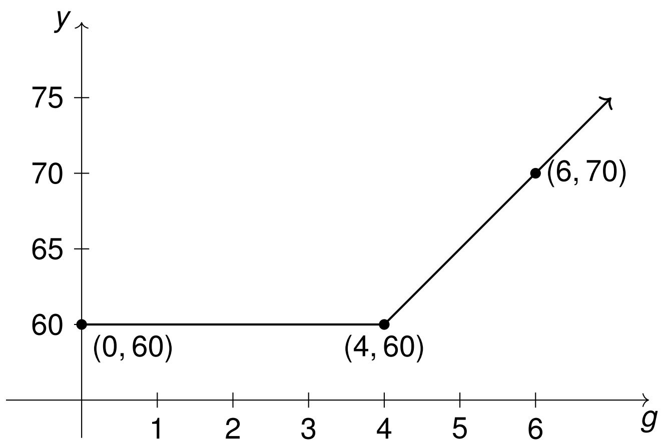
Explanation. It is clear from context that we are to use the variable g (for 'g'igabytes) as the independent variable. We are asked to compute the cost so it seems natural to name the function C. Hence, we are after a formula for $C(g)$. Knowing that g represents the amount of data used each month, we must have $g \geq 0$. In order to get a feel for the formula for $C(g)$, we can choose some specific values for g and determine the cost, $C(g)$. For example, if we use no data at all, 1 gigabyte of data, or 3.796 gigabytes of data, the cost is the same: \$60. Indeed, per the plan, for any amount of data up to and including 4 gigabytes, the cost is \$60.

Translating this to function notation means $C(0) = 60$, $C(1) = 60$, $C(3.796) = 60$, and, in general, $C(g) = 60$ for $0 \leq g \leq 4$. What happens if we use more than 4 gigabytes? Let's say we use 6 gigabytes. Per the plan, we are charged \$60 for the first 4 and then \$5 for each gigabyte over 4. Using 6 gigabytes means that we are 2 gigabytes over and our overage charge is (\$5)(2) = \$10. The total cost is the base plus the overages or \$60 + \$10 = \$70. In general, if $g > 4$, the expression $(g - 4)$ computes the amount of data used over 4 gigabytes. Our base plus overage then comes to: $60 + 5(g - 4) = 5g + 40$. Putting this together with our previous work, we get

$$C(g) = \begin{cases} 60 & \text{if} \quad 0 \leq g \leq 4 \\ 5g + 40 & \text{if} \quad g > 4 \end{cases}$$

To graph C, we graph $y = C(g)$. For $0 \leq g \leq 4$, we have the horizontal line $y = 60$ from $(0, 60)$ to $(4, 60)$. For $g > 4$, we have the line $y = 5g + 40$. Even though the inequality $g > 4$ is strict, we nevertheless substitute $g = 4$ into the formula $y = 5g + 40$ and get $y = 60$. Normally, this would produce a hole at $(4, 60)$, but in this case, the point $(4, 60)$ is already on the graph from the first piece of the function. Essentially, the point $(4, 60)$ from $C(g) = 60$ for $0 \leq g \leq 4$ 'plugs' the hole from $C(g) = 5g + 40$ when $g > 4$.

We are graphing a line so we need to plot just one more point to determine the graph. From our work above, we know $C(6) = 70$, so we use $(6, 70)$ as our second point. Our graph is below. As with the graphs shown on page **??** from Example **??**, we use '$\asymp$' to denote a break in the vertical axis in order to better display the graph.



$y = C(g)$

1.1.B LINEAR REGRESSION

We have demonstrated in this section that constant, linear, and piecewise combinations of these two function types can be used to model a variety of phenomena inspired by real-world situations. What happens, as if often the case in real-world situations, when we are given data sets that are not precisely linear, but still have a definite linear trend? An example of this is Skippy's time and temperature data from Example **??** in Section **??**.

In that example, t represented the time (number of hours after 6 a.m.) and T represented the outdoor temperature in degrees Fahrenheit. The data Skippy collected along with a plot of the function $T = f(t)$ are given below. Even though the data points as t varies from $t = 0$ to $t = 8$ do not all lie on the same line - a fact we could prove analytically by checking slopes - there does appear to be a linear **trend** evident. The same can be said for the data as t varies from $t = 8$ to $t = 12$. As we'll see, there are statistical methods which can produce linear functions that are in some sense 'closest' to all of the data, and they are represented below by the dashed lines below on the right.

t: hours after 6 a.m.	T: temperature °F
0	64
2	67
4	75
6	80
8	83
10	83
12	82

Desmos link: https://www.desmos.com/calculator/aqedfuqmuw

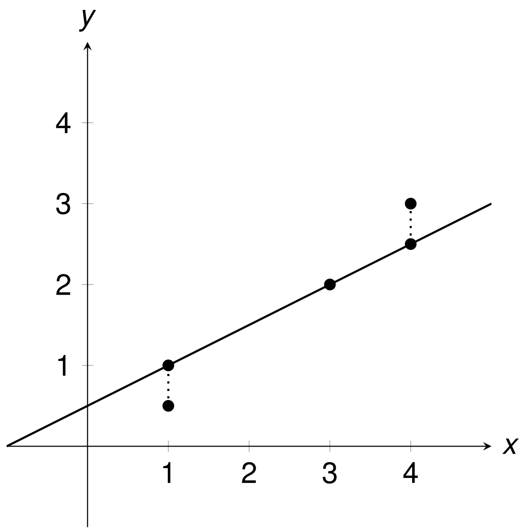
How do we measure how 'close' a set of points is to a given line? Let's leave Skippy's data for the moment and focus on a smaller data set. Suppose we collected three data points: $\{(1, 0.5), (3, 2), (4, 3)\}$. At the top of the next page (on the left) we plot these points along with the line $y = 0.5x + 0.5$. The way we measure how close the line is to these points is by computing the **total squared (vertical) error** between the data points and the line as follows. For each of our data points, we find the vertical distance between the point and the line. To accomplish this, we need to find a point on the line directly above or below each data point. In other words, we need a point on the line with the same x-coordinate as our data point.

For example, to find the point on the line directly above $(1, 0.5)$, we plug $x = 1$ into $y = 0.5x + 0.5$ and we get the point $(1, 1)$. Similarly, we find $(3, 2)$ is on the line already and $(4, 2.5)$ is the point on the line directly beneath $(4, 3)$. We find the total squared error E by taking the sum of the squares of the differences of the y-coordinates of each data point and its corresponding point on the line. For the data and line in this discussion $E = (0.5 - 1)^2 + (2 - 2)^2 + (3 - 2.5)^2 = 0.5$.

Using advanced mathematical machinery,[24] it is possible to find the line which results in the lowest value of E. This line is called the **least squares regression line**, or sometimes

[24] like Calculus or Linear Algebra . . .

the 'line of best fit'. The formula for the line of best fit requires notation we won't present until Chapter **??**, so we will revisit it then. Most graphing utilities have a built-in regression feature, so at this point we turn the computations over to the technology. A screenshot from desmos is given on the right at the top of the next page.



Desmos link: `https://www.desmos.com/calculator/ofrcaiemar`

Our graphing utility produces the model[25] $y = mx + b$ where the slope is $m \approx 0.821$ and the y-coordinate of the y-intercept is $b \approx -0.357$. The value r is the **correlation coefficient** and is a measure of how close the data is to being on the same line. The closer $|r|$ is to 1, the better the linear fit.[26] Having $r \approx 0.997$ tells us that the points have a strong, positive correlation - that is, they are very close to being on a line with a positive slope, namely $y = 0.821x - 0.357$. Indeed, the total squared error between our data set and this line is $E \approx 0.018$. The mathematics tells us that this is the smallest we can get E by modifying the parameters m and b, even though none of the data points actually lie on the line.

Now that we have this new mathematical machinery, let's revisit Skippy's time and temperature data.

Example 7.

1. Use a graphing utility to find best fit linear models for each of the data sets below. Comment on the fit and interpret the slope of each.

[25]We chose to use three decimal places for the approximations in this demonstration. How many you get to use in reality varies from one application to another.

[26]The value r^2 is called the **coefficient of determination** and is also a measure of the goodness of fit. We refer the interested reader to a course in Statistics to explore the significance of r and r^2.

t: hours after 6 a.m.	T: temperature °F
0	64
2	67
4	75
6	80
8	83

t: hours after 6 a.m.	T: temperature °F
8	83
10	83
12	82

2. Use your models to predict the temperature at 7 a.m. and 3 p.m., rounded to one decimal place.

Explanation. 1. For our first set of data, we get the line $T = F(t) = 2.55t + 63.6$. The value $r = 0.987$ tells us that it is a fairly good fit and we see this graphically, too.[27] Thus we can be confident in using this model to predict the temperature during between the hours of 6 a.m. and 2 p.m. with reasonable accuracy.

To interpret the slope, we recognize t as the independent variable (input) and T as the dependent variable (output), so the slope $m = \dfrac{\Delta T}{\Delta t}$ is the rate of change of temperature with respect to time. In this case, $m = 2.55$ means that the temperature is increasing (getting warmer) at a rate of 2.55°F per hour. A screenshot from Desmos is given below.

Desmos link: `https://www.desmos.com/calculator/jhkc4szlj7`

For the second set of data, we get $T = G(t) = -0.25t + 85.167$ and we have $r = -0.866$. Here, the negative sign on r indicates a negative correlation which means our line has a negative slope.[28] While the fit looks OK, it certainly isn't as strong as with the first data set, so using this model to predict the temperature between 2 p.m. and 6 p.m. (let alone beyond) is a bit risky.

The slope in this case is $m = -0.25$ which corresponds to the temperature decreasing (getting cooler) at a rate of 0.25°F per hour. That's what a negative correlation means - an increase in input (more time passes) yields a decrease in output (cooler temperatures). As screenshot from Desmos is given at the top of the next page.

Desmos link: `https://www.desmos.com/calculator/rqhhlc0enf`

[27] We use F as the name of the function here to distinguish it from f - the function determined solely by the given set of data.

[28] We use G as the function name here to distinguish it from the given function f and the regression for the first data set, F.

2. The time 7 a.m. corresponds to $t = 1$. This falls between $t = 0$ and $t = 8$ so we use our first model. Substituting $t = 1$ gives $T = F(1) = 2.55(1) + 63.6 \approx 66.2$. Therefore, the model predicts the temperature to be 66.2°F at 7 a.m.. Likewise, 3 p.m. corresponds to $t = 9$. This is greater than 8, so we use the second model: $T = G(9) = -0.25(9) + 85.167 \approx 82.9$. The model predicts the temperature at 3 p.m. to be 82.9°F. Based on the goodness of fit of each model, we have more confidence in the former prediction than in the latter.

Examples 3, 4 and 7 (among others) represent three different levels of mathematical modeling. In Example 3, the mathematical model (the cost function) was provided and our task was to use the model to **interpret** the mathematics in that context. In Example 4, we were given a minimal amount of information, namely, two data points, and then asked to **construct** a model which fit those data exactly. Lastly, in Example 7, we were given several data points and we used statistical methods to construct a **best fit** model to the data.

The validity of the models rests on the validity of the underlying assumptions used to create the models. For instance, is there any reason to assume a price-demand function would be linear? Is it reasonable to assume that the temperature changes at a constant rate? These are questions for economists and scientists. Mathematicians often take on a role of equal parts translator and prophet: they codify ideas into formulas and then use them to make predictions about yet-to-be observed phenomena.

1.1.C THE AVERAGE RATE OF CHANGE OF A FUNCTION

As mentioned earlier in the section, the concepts of slope and the more general rates of change are important concepts not just in Mathematics, but also in other fields. Many important phenomena are modeled using non-linear functions, and while the rates of change of these functions are not constant, we can sample the function at two points and compute what is known as an **average rate of change** between them to give some sense as to the function's behavior over that interval.[29]

Definition 5. Let f be a function defined on the interval $[a, b]$.

The **average rate of change** of f over $[a, b]$ is defined as:

$$\frac{\Delta[f(x)]}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

Geometrically, the average rate of change is the slope of the line[30] containing $(a, f(a))$ and $(b, f(b))$.

Desmos link: https://www.desmos.com/calculator/2ctplusi5g

As with Definitions **??** and 4, the wording in Definition 5, while referring to the function f, is really making a statement about its outputs $f(x)$.

If f is increasing over $[a, b]$, then the average rate of change will be positive. Likewise, if f is decreasing or constant, the average rate of change will be negative or 0, respectively. (Think about this for a moment.) However, as the next example demonstrates, the converses of these statements aren't always true.[31]

Example 8. The formula $s(t) = -5t^2 + 100t$ for $0 \leq t \leq 20$ gives the height, $s(t)$, measured in feet, of a model rocket above the Moon's surface as a function of the time t, in seconds after lift-off.

1. Find $s(0)$, $s(5)$, $s(10)$, $s(15)$ and $s(20)$ and use these along with a graphing utility to graph $y = s(t)$.
2. State the range of s and interpret the extrema, if any exist.
3. Find and interpret the t- and y-intercepts.
4. Find and interpret the interval(s) over which s is increasing, decreasing or constant.
5. Find and interpret the average rate of change of s over the intervals $[0, 5]$, $[5, 10]$, $[10, 20]$ and $[5, 15]$.

[29] We are basically pretending that the function is linear on a short interval to see what we can say about its behavior.

[30] This line is called a **secant line**.

[31] For example, the average rate of change over an interval could be positive yet the function could decrease over part of that interval and then increase on a different part.

1.1.C THE AVERAGE RATE OF CHANGE OF A FUNCTION

Explanation. 1. To find $s(0)$, we substitute $t = 0$ into the formula for $s(t)$: $s(0) = -5(0)^2 + 100(0) = 0$. Similarly, $s(5) = -5(5)^2 + 100(5) = -5(25) + 500 = -125 + 500 = 375$. Continuing, we obtain: $s(10) = 500$, $s(15) = 375$ and $s(20) = 0$. We construct a table of values and with a graphing utility we obtain:

Desmos link: `https://www.desmos.com/calculator/gh2t8utkoz`

2. Projecting the graph to the y-axis, we see that the range of s is $[0, 500]$ so the minimum of s is 0 and the maximum is 500. This means that the rocket at some point is on the surface of the Moon and reaches its highest altitude of 500 feet above the lunar surface.

3. The first intercept we see is $(0, 0)$ which is both a t- and a y-intercept. Since t is the time after lift-off and $y = s(t)$ is the height above the Moon's surface, the point $(0, 0)$ means that the model rocket was launched ($t = 0$) from the Moon's surface ($s(t) = 0$). The remaining intercept, $(20, 0)$, is another t-intercept. This means that 20 seconds after lift-off ($t = 20$), the model rocket returns to the Moon's surface ($s(t) = 0$). That is, the 'time of flight' of the model rocket is 20 seconds.

4. Referring to Definition 4, s increases over the interval $[0, 10]$, since for those values of t, as we read from left to right, the graph of the function is rising meaning the y values (hence $s(t)$ values) are getting larger. Thus the model rocket is heading upwards for the first 10 seconds of its flight. We find that s decreases over the interval $[10, 20]$, indicating once it has reached its highest altitude of 500 feet 10 seconds into the flight, the rocket begins to fall back to the surface of the Moon, landing 20 seconds after lift-off.

5. To find the average rate of change of s over the interval $[0, 5]$ we compute

$$\frac{\Delta[s(t)]}{\Delta t} = \frac{s(5) - s(0)}{5 - 0} = \frac{375 \text{ feet}}{5 \text{ seconds}} = 75 \text{ feet per second.}$$

In other words, the height is **increasing** at an **average rate** of 75 feet per second during the first 5 seconds of flight. The rate here is called the **average velocity** of the rocket over this interval. Velocity differs from speed in that velocity comes with a direction. In this case, a positive velocity indicates that the rocket is traveling **upwards**, since when s is increasing, the model rocket is climbing higher.

Similarly, the average rate of change of s over the interval $[5, 10]$ works out to be 25. This means that the average velocity over the next 5 seconds of the flight has slowed to 25 feet per second. The model rocket is still, on average, traveling upwards, albeit more slowly than before.

Over the interval $[10, 20]$, the average rate of change of s works out to be -50. This means that, on average, the rocket is **falling** at a rate of 50 feet per second. The rocket has managed to fall from its highest point 500 feet above the surface of the Moon back to the Moon's surface in 10 seconds so this makes sense. Finally, the average rate of change of s over $[5, 15]$ is 0. This means that the model is the same height above the ground after 5 seconds (375 feet) as it is after 15 seconds.

Geometrically, the average rate of change of a function over an interval can be interpreted as the slope of a secant line. Below on the left is a dotted line containing $(0, 0)$ and $(5, 375)$ (which has slope 75) along with a dotted line containing the points $(5, 375)$ and $(10, 500)$ (which has slope 25). Visually, the lines help demonstrate that, while s is increasing over $[0, 10]$, the rate of increase is slowing down as t nears 10.

Desmos link: https://www.desmos.com/calculator/qj3xoaviev

The graph above on the right depicts a dotted line through $(10, 500)$ and $(20, 0)$ indicating a net decrease over that interval. We also have a horizontal line (0 slope) containing the points $(5, 375)$ and $(15, 375)$, which shows no net change between those two points, despite the fact that the rocket rose to its maximum height then began its descent during the interval $[5, 15]$.

An important lesson from the last example is that average rates of change give us a snapshot of what is happening **at the endpoints** of an interval, but not necessarily what happens **over the course** of the interval. Calculus gives us tools to compute slopes **at** points which correspond to **instantaneous** rates of changes. While we don't quite have the machinery to properly express these ideas, we can hint at them in the Exercises. Speaking of exercises . . .

1.2 ABSOLUTE VALUE FUNCTIONS

1.2.0.1 Graphs of Absolute Value Functions

In Section **??**, we revisited lines in a function context. In this section, we revisit the absolute value in a similar manner, so it may be useful to refresh yourself with the basics in Section **??**. Recall that the absolute value of a real number x, denoted $|x|$, can be defined as the distance from x to 0 on the real number line.[1] This definition is very useful for several applications, and lends itself well to solving equations and inequalities such as $|x-2|+1 = 5$ or $2|t+1| > 4$.

We now wish to explore solving more complicated equations and inequalities, such as $|x-2|+1 = x$ and $2|t+1| \geq t+4$. We'll approach these types of problems from a function standpoint and use the interplay between the graphical and analytical representations of these functions to obtain solutions. The key to this section is understanding the absolute value from that function (or procedural) standpoint.

Consider a real number $x \geq 0$ such as $x = 0$, $x = \pi$ or $x = 117.42$. When computing absolute values, we find $|0| = 0$, $|\pi| = \pi$ and $|117.42| = 117.42$. In general, if $x \geq 0$, the absolute value function does nothing to change the input, so $|x| = x$. On the other hand, if $x < 0$, say $x = -1$, $x = -\sqrt{42}$ or $x = -117.42$, we get $|-1| = 1$, $|-\sqrt{42}| = \sqrt{42}$ and $|-117.42| = 117.42$. That is, if $x < 0$, $|x|$ returns the exact **opposite** of the input x, so $|x| = -x$.

Putting these two observations together, we have the following.

Definition 1. The **absolute value** of a real number x, denoted $|x|$, is given by

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

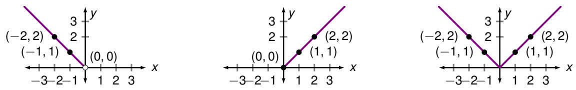
In Definition 1, it is **absolutely** essential to read '$-x$' as 'the **opposite** of x' as **opposed** to 'negative x' in order to avoid serious errors later. To see that this description agrees with our previous experience, consider $|117.42|$. Given that $117.42 \geq 0$, we use the rule $|x| = x$. Hence, $|117.42| = 117.42$. Likewise, $|0| = 0$. To compute $|-\sqrt{42}|$, we note that $-\sqrt{42} < 0$ we use the rule $|x| = -x$ in this case. We get $|-\sqrt{42}| = -(-\sqrt{42})$ (the opposite of $-\sqrt{42}$), so $|-\sqrt{42}| = -(-\sqrt{42}) = \sqrt{42}$.

Another way to view Definition 1 is to think of $-x = (-1)x$ and $x = (1)x$. That is, $|x|$ multiplies negative inputs by -1 and non-negative inputs by 1. This viewpoint is especially useful in graphing $f(x) = |x|$. For $x < 0$, $|x| = (-1)x$, so the graph of $y = |x|$ is the graph of $y = -x = (-1)x$: a line with slope -1 and y-intercept $(0,0)$. Likewise, for $x \geq 0$, $|x| = x$, so the graph of $y = |x|$ is the graph of $y = x = (1)x$: a line with slope 1 and y-intercept $(0,0)$.

Below we graph each piece and then put them together. Note that when graphing $f(x) = |x|$ for $x < 0$, we have a hole at $(0,0)$ because the inequality $x < 0$ is strict. However, the point $(0,0)$ is included in the graph of $f(x) = |x|$ for $x \geq 0$, so there is no hole in our final graph.

[1] More generally, $|x-c|$ is the distance from x to c on the number line.

1.2 ABSOLUTE VALUE FUNCTIONS



The graph of $f(x) = |x|$ is a very distinctive '$\vee$' shape and is worth remembering. The point $(0, 0)$ on the graph is called the **vertex**. This terminology makes sense from a geometric viewpoint because $(0, 0)$ is the point where two lines meet to form an angle. We will also see this term used in Section **??** where, more generally, it corresponds to the graphical location of the sole maximum or minimum of a quadratic function.

We put Definition 1 to good use in the next example and review the basics of graphing along the way.

Example 1. For each of the functions below, analytically find the zeros of the function and the axis intercepts of the graph, if any exist. Rewrite the function using Definition 1 as a piecewise-defined function and sketch its graph. From the graph, determine the vertex, find the range of the function and any extrema, and then list the intervals over which the function is increasing, decreasing or constant.

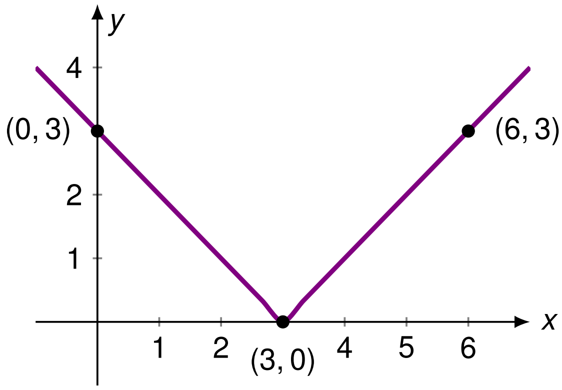
1. $f(x) = |x - 3|$
2. $g(t) = |t| - 3$
3. $h(u) = |2u - 1| - 3$
4. $i(w) = 4 - 2|3w - 1|$

Solution. In what follows below, we will be doing quite a bit of substitution. As we have mentioned before, when substituting one expression in for another, the use of parentheses or other grouping symbols is highly recommended. Also, the dependent variable wasn't specified so we use the default y in each case.

1. To find the zeros of f, we solve $f(x) = 0$ or $|x - 3| = 0$. We get $x = 3$ so the sole x-intercept of the graph of f is $(3, 0)$. To find the y-intercept, we compute $f(0) = |0 - 3| = 3$ and obtain $(0, 3)$. Using Definition 1 to rewrite the expression for $f(x)$ means that we substitute the expression $x - 3$ in for x and simplify. Note that when substituting the $x - 3$ in for x, we do so for **every** instance of x – both in the formula (output) as well as the inequality (input).

$$f(x) = |x-3| = \begin{cases} -(x-3) & \text{if } (x-3) < 0 \\ (x-3) & \text{if } (x-3) \geq 0 \end{cases} \longrightarrow \quad f(x) = \begin{cases} -x+3 & \text{if } x < 3 \\ x-3 & \text{if } x \geq 3 \end{cases}$$

   As both pieces of the graph of f are lines, we need just two points for each piece. We already have two points for the graph: $(0, 3)$ and $(3, 0)$. These two points both lie on the line $y = -x + 3$ but the strictness of the inequality means $f(x) = -x + 3$ only for $x < 3$, <u>not</u> $x = 3$, so we would have a hole at $(3, 0)$ instead of a point there. For $x \geq 3$, $f(x) = x - 3$, so the hole we thought we had at $(3, 0)$ gets plugged because $f(3) = 3 - 3 = 0$. We need just one more point for $f(x)$ where $x \geq 3$ and choose somewhat arbitrarily $x = 6$. We find $f(6) = |6 - 3| = 3$ so our final point on the graph is $(6, 3)$.

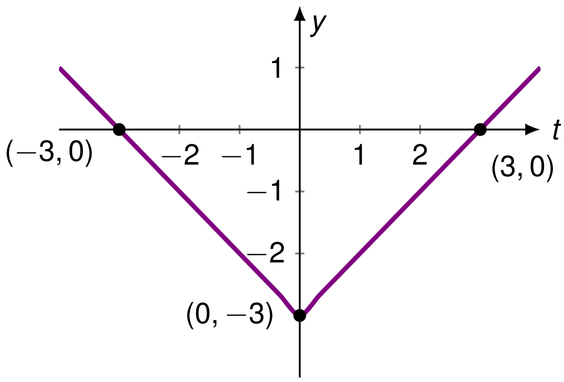


Now that we have a complete graph,[2] we see that the vertex is $(3, 0)$ and the range is $[0, \infty)$. The minimum of f is 0 when $x = 3$ and f has no maximum. Also, f is decreasing over $(-\infty, 3]$ and increasing on $[3, \infty)$.

2. To find the zeros of g, we solve $g(t) = |t| - 3 = 0$ and get $|t| = 3$ or $t = \pm 3$. Hence, the t-intercepts of the graph of g are $(-3, 0)$ and $(3, 0)$. To find the y-intercept, we compute $g(0) = |0| - 3 = -3$ and get $(0, -3)$. To rewrite $g(t)$ has a piecewise defined function, we first substitute t in for x in Definition 1 to get a piecewise definition of $|t|$. This breaks the domain into two pieces: $t < 0$ and $t \geq 0$. For $t < 0$, $|t| = -t$, so $g(t) = |t| - 3 = (-t) - 3 = -t - 3$. Likewise, for $t \geq 0$, $|t| = t$ so $g(t) = |t| - 3 = t - 3$.

$$|t| = \begin{cases} -t & \text{if } t < 0 \\ t & \text{if } t \geq 0 \end{cases} \quad \longrightarrow \quad g(t) = |t| - 3 = \begin{cases} -t - 3 & \text{if } t < 0 \\ t - 3 & \text{if } t \geq 0 \end{cases}$$

Once again, we have two lines to graph, but in this case we have three points: $(-3, 0)$, $(0, -3)$ and $(3, 0)$. Both $(-3, 0)$ and $(0, -3)$ lie on $y = -t - 3$, but $g(t) = -t - 3$ only for $t < 0$. This would yield a hole at $(0, -3)$, but, just like in the previous example, the hole is plugged thanks to the second piece of the function because $g(0) = 0 - 3 = -3$. We also pick up the second t-intercept, $(3, 0)$ and this helps us complete our graph below.



[2] We know it's complete because we did the Math - no trusting technology on this example!

We see that the vertex is $(0, -3)$ and the range is $[-3, \infty)$. The minimum of g is -3 at $t = 0$ and there is no maximum. Also, g is decreasing on $(-\infty, 0]$ and increasing on $[0, \infty)$.

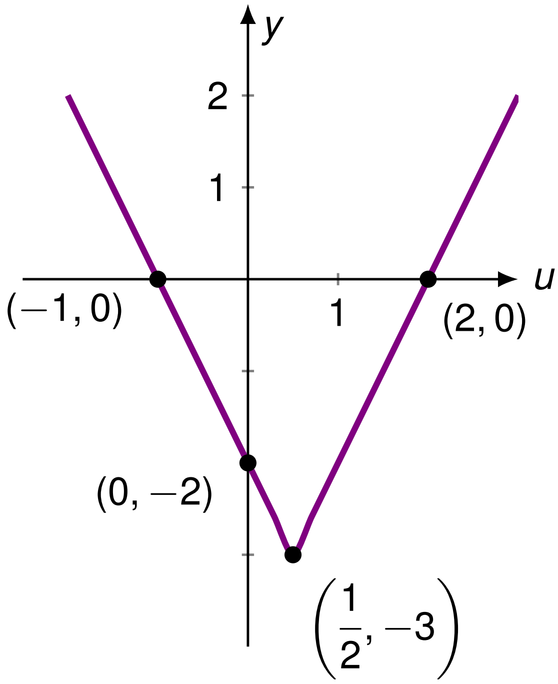
3. Solving $h(u) = |2u - 1| - 3 = 0$ gives $|2u - 1| = 3$ or $2u - 1 = \pm 3$. We get two zeros: $u = -1$ and $u = 2$ which correspond to two u-intercepts: $(-1, 0)$ and $(2, 0)$. We find $h(0) = |2(0) - 1| - 3 = -2$ so our y-intercept is $(0, -2)$. To rewrite $h(u)$ as a piecewise defined function, we first rewrite $|2u - 1|$ as a piecewise function. Substituting the expression $2u - 1$ in for x in Definition 1 gives:

$$|2u-1| = \begin{cases} -(2u-1) & \text{if } 2u-1 < 0 \\ 2u-1 & \text{if } 2u-1 \geq 0 \end{cases} \longrightarrow \quad |2u-1| = \begin{cases} -2u+1 & \text{if } u < \frac{1}{2} \\ 2u-1 & \text{if } u \geq \frac{1}{2} \end{cases}$$

Hence, for $u < \dfrac{1}{2}$, $|2u-1| = -2u+1$ so $h(u) = |2u-1|-3 = (-2u+1)-3 = -2u-2$. Likewise, for $u \geq \dfrac{1}{2}$, $|2u-1| = 2u-1$ so $h(u) = |2u-1|-3 = (2u-1)-3 = 2u-4$.

$$h(u) = |2u-1| - 3 = \begin{cases} (-2u+1)-3 & \text{if } u < \frac{1}{2} \\ (2u-1)-3 & \text{if } u \geq \frac{1}{2} \end{cases} \longrightarrow \quad h(u) = \begin{cases} -2u-2 & \text{if } u < \frac{1}{2} \\ 2u-4 & \text{if } u \geq \frac{1}{2} \end{cases}$$

We have three points to help us graph $y = h(u)$: $(-1, 0)$, $(0, -2)$ and $(2, 0)$. Unlike in the last two examples, these points do not give us information at the value $u = \dfrac{1}{2}$ where the rule for $h(u)$ changes. Substituting $u = \dfrac{1}{2}$ into the expression $-2u-2$ gives -3, so from $h(u) = -2u-2$, $u < \dfrac{1}{2}$, we get a hole at $\left(\dfrac{1}{2}, -3\right)$. However, this hole is filled because $h\left(\dfrac{1}{2}\right) = 2\left(\dfrac{1}{2}\right) - 4 = -3$ and this produces the vertex at $\left(\dfrac{1}{2}, -3\right)$. Our graph is below.



The range of h is $[-3, \infty)$, with the minimum of h being -3 at $t = \frac{1}{2}$. Moreover, h is decreasing on $\left(-\infty, \frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2}, \infty\right)$.

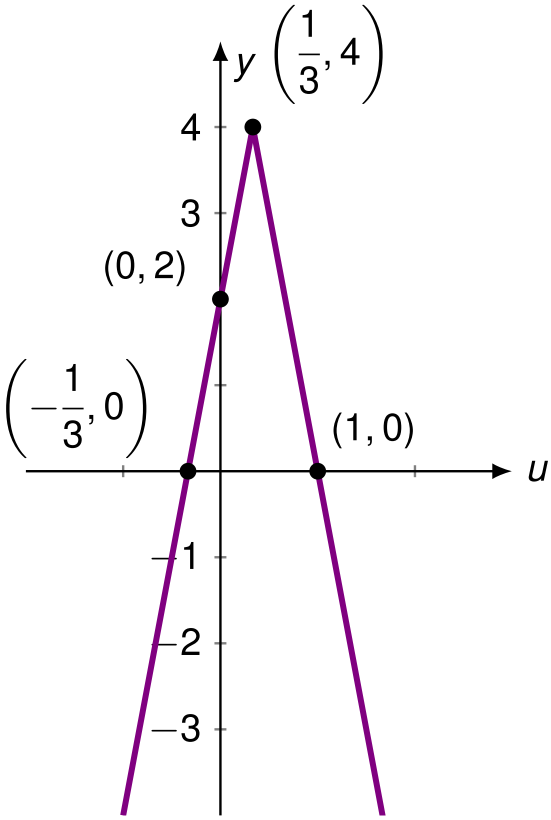
4. Solving $i(w) = 4 - 2|3w - 1| = 0$ yields $|3w - 1| = 2$ or $3w - 1 = \pm 2$. This gives two zeros, $w = -\frac{1}{3}$ and $w = 1$, which correspond to two w-intercepts, $\left(-\frac{1}{3}, 0\right)$ and $(1, 0)$. Also, $i(0) = 4 - 2|3(0) - 1| = 2$, so the y-intercept of the graph is $(0, 2)$. As in the previous example, the first step in rewriting $i(w)$ as a piecewise defined function is to rewrite $|3w - 1|$ as a piecewise function. Once again, we substitute the expression $3w - 1$ in for every occurrence of x in Definition 1:

$$|3w-1| = \begin{cases} -(3w-1) & \text{if } 3w-1 < 0 \\ 3w-1 & \text{if } 3w-1 \geq 0 \end{cases} \longrightarrow |3w-1| = \begin{cases} -3w+1 & \text{if } w < \frac{1}{3} \\ 3w-1 & \text{if } w \geq \frac{1}{3} \end{cases}$$

Thus for $w < \frac{1}{3}$, $|3w-1| = -3w+1$, so $i(w) = 4-2|3w-1| = 4-2(-3w+1) = 6w+2$. Likewise, for $w \geq \frac{1}{3}$, $|3w - 1| = 3w - 1$ so $i(w) = 4 - 2|3w - 1| = 4 - 2(3w - 1) = -6w + 6$.

$$i(w) = 4 - 2|3w-1| = \begin{cases} 4-2(-3w+1) & \text{if } w < \frac{1}{3} \\ 4-2(3w-1) & \text{if } w \geq \frac{1}{3} \end{cases} \longrightarrow i(w) = \begin{cases} 6w+2 & \text{if } w < \frac{1}{3} \\ -6w+6 & \text{if } w \geq \frac{1}{3} \end{cases}$$

As with the previous example, we have three points on the graph of i: $\left(-\frac{1}{3}, 0\right)$, $(0, 2)$ and $(1, 0)$, but no information about happens at $w = \frac{1}{3}$. Substituting this value of w into the formula $6w + 2$ would produce a hole at $\left(\frac{1}{3}, 4\right)$. As we've seen several times already, however, $i\left(\frac{1}{3}\right) = 4$ so we don't have a hole at $\left(\frac{1}{3}, 4\right)$ but, rather, the vertex. Our graph is below.



From the graph we see that the range of i is $(-\infty, 4]$ with the maximum of i being 4 when $w = \frac{1}{3}$. Also, i is increasing over $\left(-\infty, \frac{1}{3}\right]$ and decreasing on $\left[\frac{1}{3}, \infty\right)$. ■

As we take a step back and look at the graphs produced in Example 1, some patterns begin to emerge. Indeed, each of the graphs has the common '$\vee$' shape (in the case of the function i it's a '$\wedge$') with the vertex located at the x-value where the rule for each function changes from one formula to the other. It turns out that, independent variable labels aside, each and every function in Example 1 can be rewritten in the form $F(x) = a|x - h| + k$ for real number parameters a, h and k.

Each of the functions from Example 1 is rewritten in this form below and we record the vertex along with the slopes of the lines in the graph.

- $f(x) = |x - 3| = (1)|x - 3| + 0$: $a = 1$, $h = 3$, $k = 0$; vertex $(3, 0)$; slopes ± 1
- $g(t) = |t| - 3 = (1)|t - 0| + (-3)$: $a = 1$, $h = 0$, $k = -3$; vertex $(0, -3)$; slopes ± 1

- $h(u) = |2u - 1| - 3 = 2\left|u - \dfrac{1}{2}\right| + (-3)$: $a = 2$, $h = \dfrac{1}{2}$, $k = -3$; vertex $\left(\dfrac{1}{2}, -3\right)$; slopes ± 2
- $i(w) = 4 - 2|3w - 1| = -6\left|w - \dfrac{1}{3}\right| + 4$: $a = -6$, $h = \dfrac{1}{3}$, $k = 4$; vertex $\left(\dfrac{1}{3}, 4\right)$; slopes ± 6

These specific examples suggest the following theorem.

Theorem 1. For real numbers a, h and k with $a \neq 0$, the graph of $F(x) = a|x - h| + k$ consists of parts of two lines with slopes $\pm a$ which meet at a vertex (h, k). If $a > 0$, the shape resembles '$\vee$'. If $a < 0$, the shape resembles '$\wedge$'. Moreover, the graph is symmetric about the line $x = h$.

Proof. What separates Mathematics from the other sciences is its ability to actually **prove** patterns like the one stated in the theorem above as opposed to just **verifying** it by working more examples. The proof of Theorem 1 uses the exact same concepts as were used in Example 1, just in a more general context by which we mean using letters as parameters instead of numbers.

The first step is to rewrite $|x - h|$ as a piecewise function.

$$|x - h| = \begin{cases} -(x - h) & \text{if } x - h < 0 \\ x - h & \text{if } x - h \geq 0 \end{cases} \quad \longrightarrow \quad |x - h| = \begin{cases} -x + h & \text{if } x < h \\ x - h & \text{if } x \geq h \end{cases}$$

We plug that work into $F(x)$ to rewrite it as a piecewise function. For $x < h$, we have $|x - h| = -x + h$, so

$$F(x) = a|x - h| + k = a(-x + h) + k = -ax + ah + k = -ax + (ah + k)$$

Similarly, for $x \geq h$, we have $|x - h| = x - h$, so

$$F(x) = a|x - h| + k = a(x - h) + k = ax - ah + k = ax + (-ah + k)$$

Hence,

$$F(x) = a|x - h| + k = \begin{cases} a(-x + h) + k & \text{if } x < h \\ a(x - h) + k & \text{if } x \geq h \end{cases} \quad \longrightarrow \quad F(x) = \begin{cases} -ax + (ah + k) & \text{if } x < h \\ ax + (-ah + k) & \text{if } x \geq h \end{cases}$$

All three parameters, a, h and k, are fixed (but arbitrary) real numbers. Thus, for any given choice of a, h and k the numbers $ah + k$ and $-ah + k$ are also just numbers as opposed to variables. This shows that the graph of F is comprised of pieces of two lines, $y = -ax + (ah + k)$ and $y = ax + (-ah + k)$, the former with slope $-a$ and the latter with slope a. Note that substituting $x = h$ into $y = -ax + (ah + k)$ produces $y = -ah + (ah + k) = k$ and substituting $x = h$ into $y = ax + (-ah + k)$ also produces $y = ah + (-ah + k) = k$. This tells us that the two linear pieces meet at the point (h, k).

If $a > 0$ then $-a < 0$ so the line $y = -ax + (ah + k)$, hence F, is decreasing on $(-\infty, h]$. Similarly, the line $y = ax + (-ah + k)$, hence F, is increasing on $[h, \infty)$. This produces a '$\vee$' shape. On the other hand, if $a < 0$ then $-a > 0$ which produces a '$\wedge$' shape because F is increasing on $(-\infty, h]$ followed by decreasing on $[h, \infty)$. (Said another way, $-a > 0$ means that the first linear piece has a positive slope and $a < 0$ means that the second piece has a negative slope.)

To show that the graph is symmetric about the line $x = h$, we need to show that if we move left or right the same distance away from $x = h$, then we get the same y-value on the graph. Suppose we move Δx to the right or left of h. The y-values are the function values so we need to show that $F(a + \Delta x) = F(a - \Delta x)$. Given that

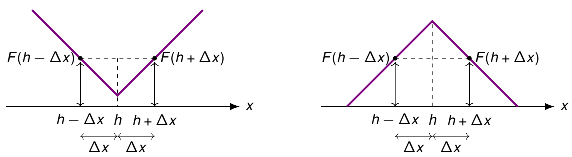
$$F(a + \Delta x) = a|a + \Delta x - a| + k = a|\Delta x| + k$$

and

$$F(a - \Delta x) = a|a - \Delta x - a| + k = a| - \Delta x| + k = a|\Delta x| + k$$

we see that $F(a + \Delta x) = F(a - \Delta x)$. Thus we have shown that the y-values on the graph on either side of $x = h$ are equal provided we move the same distance away from $x = a$. This completes the proof. ■

The line $x = a$ in Theorem 1 is called the **axis of symmetry** of the graph of $y = F(x)$. This language is consistent with the basics of symmetry discussed in Section **??** and we will build upon our work here in several upcoming sections. For now, we simply present two graphs illustrating the concept of the axis of symmetry below.



While Theorem 1 and its proof are specific to the particular family of absolute value functions, there are ideas here that apply to all functions. Thus we wish to take a slight detour away from the main narrative to argue this result again from an even more generalized viewpoint. Our goal is to 'build' the formula $F(x) = a|x - h| + k$ from $f(x) = |x|$ in three stages, each corresponding to the role of one of the parameters a, h and k, and track the geometric changes that go along with each stage. We will revisit all of the ideas described below in complete generality in Section **??**.

The graph of $f(x) = |x|$ consists of the points $\{(c, |c|) \mid c \in \mathbb{R}\}$.[3] Consider $F_1(x) = |x - h|$. The graph of F_1 is the set of points $\{(x, |x - h|) \mid x \in \mathbb{R}\}$. If we relabel $x - h = c$, then $x = c + h$, and as x varies through all of the real numbers, so does c and vice-versa.[4]

[3]See the box on page **??**. Also, we use 'c' as our dummy variable to avoid the confusion that would arise by over-using 'x'.

[4]That is, every real number c can be written as $x - h$ for some x, and every real number x can be written as $c + h$ for some c.

Hence, we can write $\{(x, |x-h|) \mid x \in \mathbb{R}\} = \{(c+h, |c|) \mid c \in \mathbb{R}\}$. If we fix a y-coordinate, $|c|$, we see that the corresponding points on the graph of f and F_1, $(c, |c|)$ and $(c+h, |c|)$, respectively, differ only in that one is horizontally shifted by h. In other words, to get the graph of F_1, we simply take the graph of f and shift each point horizontally by adding h to the x-coordinate. Translating the graph in this manner preserves the '$\vee$' shape and symmetry, but moves the vertex from $(0, 0)$ to $(h, 0)$.

Next, we examine $F_2(x) = a|x-h|$ and compare its graph to that of $F_1(x) = |x-h|$. The graph of F_2 consists of the points $\{(x, a|x-h|) \mid x \in \mathbb{R}\}$ whereas the graph of F_1 consists of the points $\{(x, |x-h|) \mid x \in \mathbb{R}\}$. The only difference between the points $(x, |x-h|)$ and $(x, a|x-h|)$ is that the y-coordinate of one is a times the y-coordinate of the other. If $a > 0$, all we are doing is scaling the y-axis by a factor of a. As we've seen when plotting points and graphing functions, the scaling of the y-axis affects only the relative vertical displacement of points[5] and not the overall shape.

If $a < 0$, then in addition to scaling the vertical axis, we are reflecting the points across the x-axis.[6] Such a transformation doesn't change the '$\vee$' shape except for flipping it upside-down to make it a '$\wedge$'. In either case, the vertex $(h, 0)$ stays put at $(h, 0)$ because the y-value of the vertex is 0 and $a \cdot 0 = 0$ regardless if $a > 0$ or $a < 0$.

Last, we examine the graph of $F(x) = a|x-h| + k$ to see how it relates to the graph of $F_2(x) = a|x-h|$. The graph of F consists of the points $\{(x, a|x-h| + k) \mid x \in \mathbb{R}\}$ whereas the graph of F_2 consists of the points $\{(x, a|x-h|) \mid x \in \mathbb{R}\}$. The difference between the corresponding points $(x, a|x-h|)$ and $(x, a|x-h|+k)$ is the addition of k in the y-coordinate of the latter. Adding k to each of the y-values translates the graph of F_2 vertically by k units. The basic shape doesn't change but the vertex goes from $(h, 0)$ to (h, k).

In summary, the graph of $F(x) = a|x-h| + k$ can be obtained from the graph of $f(x) = |x|$ in three steps: first, add h to each of the x-coordinates; second, multiply each y-coordinate by a; and third, add k to each y-coordinate. Geometrically, these steps mean that we first move the graph left or right, then scale the y-axis by a factor of a (and reflect across the x-axis if $a < 0$), and then move the graph up or down. Throughout all of these **transformations**, the graph maintains its '$\vee$' or '$\wedge$' shape.

Of course, not every function involving absolute values can be written in the form given in Theorem 1. A good example of this is $G(x) = |x-2| - x$. However recognizing the ones that can be rewritten will greatly simplify the graphing process. In the next example, we graph four more absolute value functions, two using Theorem 1 and two using Definition 1.

Example 2.

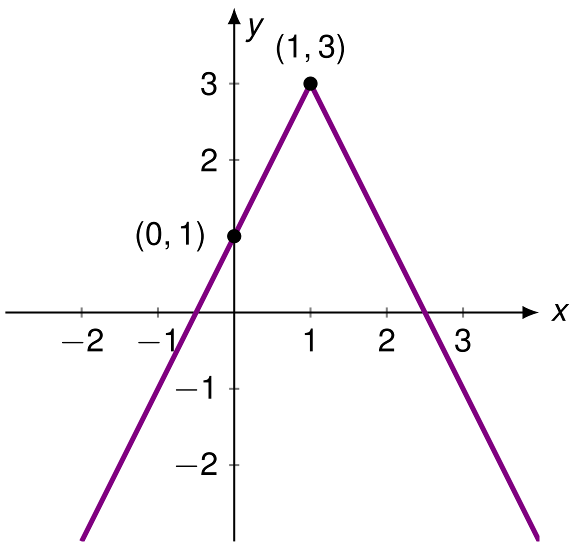
1. Graph each of the functions below using Theorem 1 or by rewriting it as a piecewise defined function using Definition 1. Find the zeros, axis-intercepts and the extrema (if any exist) and then list the intervals over which the function is increasing, decreasing or constant.

[5] See the discussion following Example **??** regarding the plot of Skippy's data.

[6] See the box on page **??** in Section **??**.

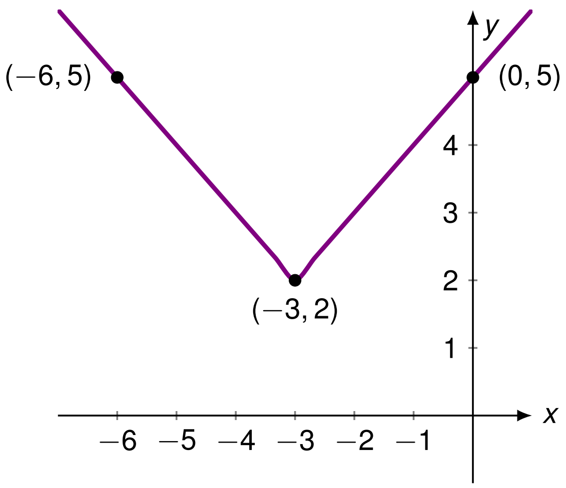
(i) $F(x) = |x+3| + 2$ (ii) $f(t) = \dfrac{4-|5-3t|}{2}$ (iii) $G(x) = |x-2| - x$ (iv) $g(t) = |t-2| - |t|$

2. Use Theorem 1 to write a possible formula for $H(x)$ whose graph is given below:



Solution.

1. (i) Rewriting $F(x) = |x+3|+2 = (1)|x-(-3)|+2$, we have $F(x)$ in the form stated in Theorem 1 with $a = 1$, $h = -3$ and $k = 2$. The vertex is $(-3, 2)$ and the graph will be a '$\vee$' shape. Seeing as the vertex is already above the x-axis and the graph opens upwards, there are no x-intercepts on the graph of F, hence there are no zeros.[7] With $F(0) = 5$, the y-intercept is $(0, 5)$. To get a third point, we can pick an arbitrary x-value to the left of the vertex or we could use symmetry: three units to the **right** of the vertex the y-value is 5, so the same must be true three units to the **left** of the vertex, at $x = -6$. Sure enough, $F(-6) = |-6+3|+2 = |-3|+2 = 5$. We get the graph below.



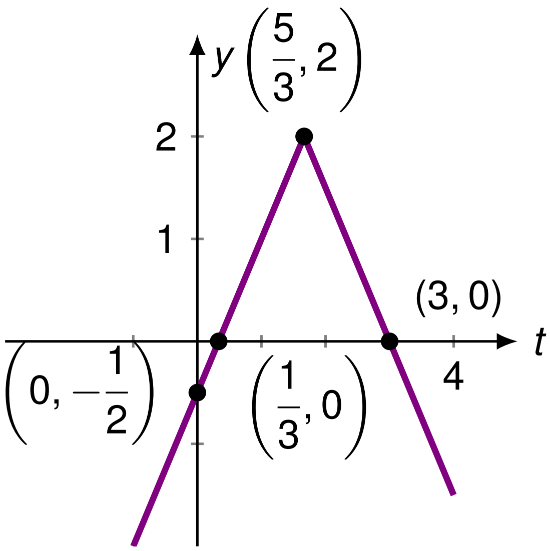
[7]Alternatively, setting $|x+3|+2 = 0$ gives $|x+3| = -2$. Absolute values are never negative thus we have no solution.

The range of F is $[2, \infty)$ with its minimum of 2 when $x = -3$ and F decreasing on $(-\infty, -3]$ then increasing on $[-3, \infty)$.

(ii) We see in the formula for $f(t)$ that t appears only once to the first power inside the absolute values, so we proceed to rewrite it in the form $a|t - h| + k$:

$$\begin{aligned} f(x) &= \frac{4 - |5 - 3t|}{2} \\ &= -\frac{|5 - 3t|}{2} + \frac{4}{2} \\ &= \left(-\frac{1}{2}\right)\left|(-3)\left(t - \frac{5}{3}\right)\right| + 2 \\ &= \left(-\frac{1}{2}\right)|-3|\left|t - \frac{5}{3}\right| + 2 \\ &= -\frac{3}{2}\left|t - \frac{5}{3}\right| + 2. \end{aligned}$$

Matching up the constants in the formula $f(t)$ to the parameters of $F(x)$ in Theorem 1, we identify $a = -\frac{3}{2}$, $h = \frac{5}{3}$ and $k = 2$. Hence the vertex is $\left(\frac{5}{3}, 2\right)$, and the graph is shaped like '$\wedge$' comprised of pieces of lines with slopes $\pm\frac{3}{2}$. To find the zeros of f, we set $f(t) = 0$. (We can use either expression here.) Solving $-\frac{3}{2}\left|t - \frac{5}{3}\right| + 2 = 0$, we get $\left|t - \frac{5}{3}\right| = \frac{4}{3}$, so $t - \frac{5}{3} = \pm\frac{4}{3}$. Hence our zeros are $t = \frac{1}{3}$ and $t = 3$, producing the t-intercepts $\left(\frac{1}{3}, 0\right)$ and $(3, 0)$. Using either formula gives $f(0) = -\frac{1}{2}$, so our y-intercept is $\left(0, -\frac{1}{2}\right)$. Plotting the vertex, along with the intercepts, gives us enough information to produce the graph below.



The range is $(-\infty, 2]$ with a maximum of 2 at $t = \frac{5}{3}$ and f is increasing on

$\left(-\infty, \frac{5}{3}\right]$ then decreasing on $\left[\frac{5}{3}, \infty\right)$.

(iii) We are unable to apply Theorem 1 to $G(x) = |x-2| - x$ because there is an x both inside and outside of the absolute value. We can, however, rewrite the function as a piecewise function using Definition 1. Our first step is to rewrite $|x-2|$ as a piecewise function:

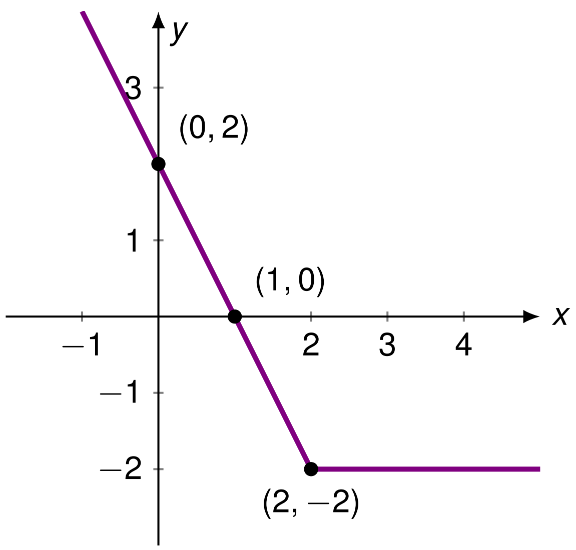
$$|x-2| = \begin{cases} -(x-2) & \text{if } x-2 < 0 \\ x-2 & \text{if } x-2 \geq 0 \end{cases} \longrightarrow \quad |x-2| = \begin{cases} -x+2 & \text{if } x < 2 \\ x-2 & \text{if } x \geq 2 \end{cases}$$

Hence, for $x < 2$, $|x-2| = -x+2$ so $G(x) = |x-2|-x = (-x+2)-x = -2x+2$. Likewise, for $x \geq 2$, $|x-2| = x-2$ so $G(x) = |x-2| - x = x-2-x = -2$.

$$G(x) = |x-2| - x = \begin{cases} (-x+2)-x & \text{if } x < 2 \\ (x-2)-x & \text{if } x \geq 2 \end{cases} \longrightarrow \quad G(x) = \begin{cases} -2x+2 & \text{if } x < 2 \\ -2 & \text{if } x \geq 2 \end{cases}$$

To find the zeros of G, we set $G(x) = 0$. Solving $|x-2| - x = 0$ can be problematic, given that x is both inside and outside of the absolute values.[8] We can, however, use the piecewise description of $G(x)$. With $G(x) = -2x+2$ for $x < 2$, we solve $-2x+2 = 0$ to get $x = 1$. This works because $1 < 2$, so we have $x = 1$ as the zero of G corresponding to the x-intercept $(1,0)$. The other piece of $G(x)$ is $G(x) = -2$ which is never 0. For the y-intercept, we find $G(0) = 2$, and get $(0,2)$.

To graph $y = G(x)$, we have the line $y = -2x+2$ which contains $(0,2)$ and $(1,0)$ and continues to a hole at $(2,-2)$. At this point, $G(x) = -2$ takes over and we have a horizontal line containing $(2,-2)$ extending indefinitely to the right. We get the graph below.



[8] We'll return to this momentarily.

The range of G is $[-2, \infty)$ with a minimum value of -2 attained for all $x \geq 2$. Moreover, G is decreasing on $(-\infty, 2]$ and then constant on $[2, \infty)$.

(iv) Once again we are unable to use Theorem 1 because $g(t) = |t-2| - |t|$ has two absolute values with no apparent way to combine them. Thus we proceed by re-writing the function g with two separate applications of Definition 1 to remove each instance of the absolute values. To start with we have:

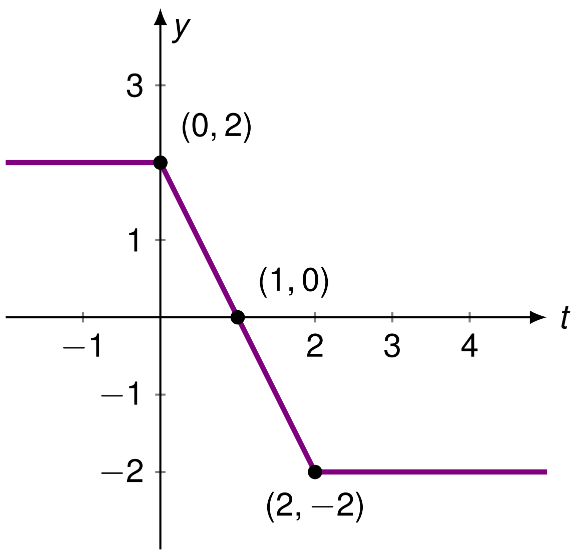
$$|t| = \begin{cases} -t & \text{if } t < 0 \\ t & \text{if } t \geq 0 \end{cases} \quad \text{and} \quad |t-2| = \begin{cases} -t+2 & \text{if } t < 2 \\ t-2 & \text{if } t \geq 2 \end{cases}$$

Taken together, these break the domain into **three** pieces: $t < 0$, $0 \leq t < 2$ and $t \geq 2$. For $t < 0$, $|t| = -t$ and $|t-2| = -t+2$. Therefore $g(t) = |t-2| - |t| = (-t+2) - (-t) = 2$ for $t < 0$. For $0 \leq t < 2$, $|t| = t$ and $|t-2| = -t+2$, so $g(t) = |t-2| - |t| = (-t+2) - t) = -2t+2$.

Last, for $t \geq 2$, $|t| = t$ and $|t-2| = t-2$, so $g(t) = |t-2| - |t| = (t-2) - (t) = -2$. Putting all three parts together yields:

$$g(t) = |t-2| - |t| = \begin{cases} (-t+2) - (-t) & \text{if } t < 0 \\ (-t+2) - (t) & \text{if } 0 \leq t < 2 \\ (t-2) - (t) & \text{if } t \geq 2 \end{cases} = \begin{cases} 2 & \text{if } t < 0 \\ -2t+2 & \text{if } 0 \leq t < 2 \\ -2 & \text{if } t \geq 2 \end{cases}$$

As with the previous example, we'll delay discussing the absolute value algebra needed to find the zeros of g and use the piecewise description instead. To graph g, we have the horizontal line $y = 2$ up to, but not including, the point $(0, 2)$. For $0 \leq t < 2$, we have the line $y = -2t+2$ which has a y-intercept at $(0, 2)$ (thus picking up where the first part left off) and a t-intercept at $(1, 0)$. This piece ends with a hole at $(2, -2)$ which is promptly plugged by the horizontal line $y = -2$ for $t \geq 2$. Hence the only zero of t is $t = 1$. Putting all this together gives us the graph below:



The range of g is $[-2, 2]$ with a minimum of -2 achieved for all $t \geq 2$, and a maximum of 2 for $t \leq 0$. We note that g is constant on $(-\infty, 0]$ and $[2, \infty)$, but with different values, and g is decreasing on $[0, 2]$.

2. We are told to use Theorem 1 to find a formula for $H(x)$ so we start with $H(x) = a|x - h| + k$ and look for real numbers a, h and k that make sense. The vertex is labeled as $(1, 3)$, meaning $h = 1$ and $k = 3$. Hence we know $H(x) = a|x - 1| + 3$, so all that is left for us to find is the value of a. The only other point labeled for us is $(0, 1)$, meaning $H(0) = 1$. Substituting $x = 0$ into our formula for $H(x)$ gives: $H(0) = a|0 - 1| + 3 = a + 3$. Given that $H(0) = 1$, we have $a + 3 = 1$, so $a = -2$. Our final answer is $H(x) = -2|x - 1| + 3$.

If nothing else, Example 2 demonstrates the value of **changing forms** of functions and the utility of the interplay between algebraic and graphical descriptions of functions. These themes resonate time and time again in this and later courses in Mathematics.

1.2.0.2 Graphical Solution Techniques for Equations and Inequalities

Consider the basic equation and related inequalities: $|x| = 3$, $|x| < 3$ and $|x| > 3$. At some point you learned how to solve these using properties of the absolute value inspired by the distance definition. (If not, see Section **??**.) While there is nothing wrong with this understanding, we wish to use these problems to motivate powerful graphical techniques which we'll use to solve more complicated equations and inequalities in this section, and in many other sections of the textbook.

To that end, let's call $f(x) = |x|$ and $g(x) = 3$. If we graph $y = f(x)$ and $y = g(x)$ on the same set of axes then, by looking for x values where $f(x) = g(x)$, we are looking for x-values which have the same y-value on both graphs. That is, the solutions to $f(x) = g(x)$ are the x-coordinates of the **intersection points** of the two graphs. We graph $y = f(x) = |x|$ (the characteristic '$\vee$') along with $y = g(x) = 3$ (the horizontal line) below on the far left. Indeed, the two graphs intersect at $(-3, 3)$ and $(3, 3)$ so our solutions to $f(x) = g(x)$ are the x-values of these points, $x = \pm 3$.

GeoGebra link: `https://www.geogebra.org/m/scjmphnq`

Likewise, if we wish to solve $|x| < 3$, we can view this as a functional inequality $f(x) < g(x)$ which means we are looking for the x-values where the $f(x)$ values are less than the corresponding $g(x)$ values. On the graphs, this means we'd be looking for the x-values where the y-values of $y = f(x)$ are less than, hence **below**, those on the graph of $y = g(x)$.

In the picture above we see that the graph of f is below the graph of g between $x = -3$ and $x = 3$, so our solution is $-3 < x < 3$, or in interval notation, $(-3, 3)$.

Finally, the inequality $|x| > 3$ is equivalent to $f(x) > g(x)$ so we are looking for the x-values where the graph of f is **above** the graph of g.[9] Once again, we refer to the picture above to

[9] Solving $f(x) > g(x)$ is equivalent to solving $g(x) < f(x)$ - that is, finding where the graph of g is below the graph of f.

see that this is true for all $x < -3$ or for all $x > 3$. In interval notation, the solution set is $(-\infty, -3) \cup (3, \infty)$.

The methodology and reasoning behind solving the above equation and inequalities extend to any pair of functions f and g, since when graphed on the same set of axes, function outputs are always the dependent variable or the ordinate (second coordinate) of the ordered pairs which comprise the graph. In general:

Graphical Interpretation of Equations and Inequalities

Suppose f and g are functions whose domains and ranges are sets of real numbers.

- The solutions to $f(x) = g(x)$ are the x-values where the graphs of f and g intersect.
- The solution to $f(x) < g(x)$ is the set of x-values where the graph of f is **below** the graph of g.
- The solution to $f(x) > g(x)$ is the set of x-values where the graph of f **above** the graph of g.

Let's return to Example 2 where were asked to find the zeros of the functions $G(x) = |x - 2| - x$ and $g(t) = |t - 2| - |t|$. In that Example, instead of tackling the algebra involving the absolute values head on we rewrote each function as a piecewise-defined function and obtained our solutions that way.

Let's see what this looks like graphically. Note that solving $|x - 2| - x = 0$ is equivalent to solving $|x - 2| = x$. Graphing $y = |x - 2|$ and $y = x$ on the same set of axes below, it certainly appears as if we have just one point of intersection, corresponding to just one solution.

GeoGebra link: `https://www.geogebra.org/m/sqwyzgzj`

Indeed, we can **show** that there is just one point of intersection. The graph of $y = |x - 2|$ is comprised of parts of two lines, $y = -(x - 2)$ and $y = x - 2$. The first line has a slope of -1 and the second has slope 1. The line $y = x$ also has a slope 1 meaning it and the 'right half' of $y = |x - 2|$ are parallel, so they never intersect. If our graphs are accurate enough, we may even be able to guess that the solution is $x = 1$, which we can verify by substituting $x = 1$ into $|x - 2| = x$ and seeing that it checks.

Likewise, solving $|t - 2| - |t| = 0$ is equivalent to solving $|t - 2| = |t|$. Graphing $y = |t - 2|$ and $y = |t|$ below, we see our only solution is $t = 1$ here as well.

GeoGebra link: `https://www.geogebra.org/m/db9r7h2t`

There is more to see here. Consider solving $|x - 2| - x = 0$ algebraically using the techniques from a previous Algebra course (or Section **??**). Our first step would be to isolate the absolute value quantity: $|x-2| = x$. We then 'drop' the absolute value by paying the price of a '$\pm$': $x-2=\pm x$. This gives us two equations: $x-2=x$ and $x-2=-x$. The first equation, $x-2=x$ reduces to $-2=0$ which has no solution. The second equation, $x-2=-x$, does have a solution, namely $x=1$.

How does the algebra tie into the graphs above? Instead of 'dropping' the absolute value and tagging the right hand side with a $\pm$, we can think about the piecewise definition of $|x-2|$ and write $|x-2| = \pm(x-2)$ depending on if $x<2$ or if $x \geq 2$. That is, $|x-2| = x$ is more precisely equivalent to the two equations: $-(x-2)=x$ which is valid for $x<2$ or $x-2=x$ which is valid for $x \geq 2$.

Graphically, the first equation is looking for intersection points between the 'left half' of the '$\vee$' of $y=|x-2|$ and the line $y=x$. Indeed, $-(x-2)=x$ is equivalent to $x-2=-x$ from which we obtain our solution $x=1$. Likewise, the second equation, $x-2=x$ is looking for intersection points of the 'right half' of the '$\vee$' and the line $y=x$, but there is none. The equation $-2=0$ is telling us that for us to have any solutions, the lines $y=x-2$ and $y=x$, which have the same slope, must also have the same y-intercepts: that is, -2 would have to equal 0 and that's just silly.

Similarly, when solving $|t-2|-|t|=0$ or $|t-2|=|t|$, we can use our graphs to prove that the only intersection point is when the 'left half' of $y=|t-2|$ intersects the 'right half' of $y=|t|$ - that is, when $-(t-2)=t$. The moral of the story is this: careful graphs can help us simplify the algebra, because we can narrow down the cases. This is especially useful in solving inequalities, as we'll see in our next example.

Example 3. Solve the following equations and inequalities.

1. $4-|x| = 0.9x-3.6$
2. $|t-3|-|t|=3$
3. $|x+1| \geq \dfrac{x+4}{2}$
4. $2 < |t-1| \leq 5$

Solution.

1. We begin by graphing $y=4-|x|$ and $y=0.9x-3.6$ to look for intersection points. Using Theorem 1, we know that the graph of $y=4-|x|=-|x|+4$ has a vertex at $(0,4)$ and is a '$\wedge$' shape, so there are x-intercepts to find. Solving $4-|x|=0$, we get $|x|=4$, or $x=\pm 4$. Hence, we have two x-intercepts: $(-4,0)$ and $(4,0)$.

   We know from Section **??** that the graph of $y=0.9x-3.6$ is a line with slope 0.9 and y-intercept $(0,-3.6)$. To find the x-intercept here we solve $0.9x-3.6=0$ and get $x=4$. Hence, $(4,0)$ is an x-intercept here as well, and we have stumbled upon one solution to $4-|x|=0.9x-3.6$, namely $x=4$. Our graph is below.

   GeoGebra link: `https://www.geogebra.org/m/ydcyxygn`

   The question is if there are any other solutions. Our graph certainly looks as if there is just one intersection point, but we know from Theorem 1 that the slopes of the linear parts of $y = 4 - |x|$ are ± 1. The slope of $y = 0.9x - 3.6$ is 0.9 and $0.9 \neq 1$ so we know that the left hand side of the'$\wedge$' must meet up with the graph of the line because they are not parallel.[10]

Definition 1 tells us that when $x < 0$, $|x| = -x$, so $4-|x| = 4-(-x) = 4+x$. Hence we set about solving $4 + x = 0.9x - 3.6$ and get $x = -76$. Both $x = -76$ and $x = 4$ check in our original equation, $4 - |x| = 0.9x - 3.6$, so we have found our two solutions. Returning to the graph above, we invite the reader to adjust the "zoom" slider to see the other intersection point, $(-76, -72)$ which corresponds to our second solution.[11]

2. While we could graph $y = |t - 3| - |t|$ and $y = 3$ to help us find solutions, we choose to rewrite the equation as $|t-3| = |t|+3$. This way, we have somewhat easier graphs to deal with, namely $y = |t - 3|$ and $y = |t| + 3$. The first graph, $y = |t - 3|$, has a vertex at $(3, 0)$ and is shaped like a '$\vee$' with slopes ± 1 and a y-intercept of $(0, 3)$. The second graph, $y = |t| + 3$, has a vertex at $(0, 3)$ and is also shaped like a '$\vee$,' with slopes ± 1, and has no t-intercepts.

   GeoGebra link: `https://www.geogebra.org/m/kkt2npqm`

   To our surprise and delight, the graphs appear to overlap for $t \leq 0$. Indeed, for $t \leq 0$, $|t - 3| = -(t - 3) = -t + 3$ and $|t| + 3 = -t + 3$. Since the formulas are **identical** for these values of t, our solutions are all values of t with $t \leq 0$. Using interval notation, we state our solution as $(-\infty, 0]$. (The other parts of the graphs are non-intersecting parallel lines so we ignored them.)

3. To solve $|x + 1| \geq \dfrac{x + 4}{2}$, we first graph $y = |x + 1|$ and $y = \dfrac{x + 4}{2} = \dfrac{1}{2}x + 2$. The former is '$\vee$' shaped with a vertex at $(-1, 0)$ and a y-intercept of $(0, 1)$. The latter is a line with y-intercept $(0, 2)$, slope $m = \dfrac{1}{2}$ and x-intercept $(-4, 0)$. Our graph of the situation is below.

   GeoGebra link: `https://www.geogebra.org/m/hgpajzzf`

   The graph suggests two points of intersection. To find these, we solve the equations: $-(x + 1) = \dfrac{x + 4}{2}$, obtaining $x = -2$, and $x + 1 = \dfrac{x + 4}{2}$ obtaining $x = 2$.

   Graphically, the inequality $|x+1| \geq \dfrac{x + 4}{2}$ is looking for where the graph of $y = |x+1|$, the '$\vee$,' intersects (=) or is above (>) the line $y = \dfrac{x + 4}{2}$. The graph shows this happening whenever $x \leq -2$ or $x \geq 2$. Using interval notation, our solution is $(-\infty, -2] \cup [2, \infty)$. While we cannot check every single x value individually, choosing test values $x < -2$, $x = 2$, $-2 < x < 2$, $x = 2$, and $x > 2$ to see if the original inequality $|x + 1| \geq \dfrac{x + 4}{2}$ holds would help us verify our solution.

[10] See Theorem **??**.

[11] Once again, we see that the technology *suggests* solutions whereas the algebra *proves* them.

4. Recall that the inequality $2 < |t-1| \leq 5$ is an example of a 'compound' inequality in that is two inequalities in one.[12] The values of t in the solution set need to satisfy $2 < |t-1|$ **and** $|t-1| \leq 5$. To help us sort through the cases, we graph the horizontal lines $y = 2$ and $y = 5$ along with the '$\vee$' shaped $y = |t-1|$ with vertex $(1, 0)$ and y-intercept $(0, 1)$ below.

GeoGebra link: https://www.geogebra.org/m/xpkc3jbj

Geometrically, we are looking for where $y = |t-1|$ is strictly **above** the line $y = 2$ but **below** (or meets) the line $y = 5$. Solving $|t-1| = 2$ gives $t = -1$ and $t = 3$ whereas solving $|t-1| = 5$ gives $t = -4$ or $t = 6$. Per the graph (below on the right), we see that $y = |t-1|$ lies between $y = 2$ and $y = 5$ when $-4 \leq t < -1$ and again when $3 < t \leq 6$.

In interval notation, our solution is $[-4, -1) \cup (3, 6]$. As with the previous example, it is impossible to check each and every one of these solutions, but choosing t values both in and around the solution intervals would give us some numerical confidence we have the correct and complete solution.

We will see the interplay of Algebra and Geometry throughout the rest of this course. In the Exercises, do not hesitate to use whatever mix of algebraic and graphical methods you deem necessary to solve the given equation or inequality. Indeed, there is great value in checking your algebraic answers graphically and vice-versa.

One of the classic applications of inequalities involving absolute values is the notion of tolerances.[13] Recall that for real numbers x and c, the quantity $|x-c|$ may be interpreted as the distance from x to c. Solving inequalities of the form $|x-c| \leq d$ for $d > 0$ can then be interpreted as finding all numbers x which lie within d units of c. We can think of the number d as a 'tolerance' and our solutions x as being within an accepted tolerance of c. We use this principle in the next example.

Example 4. Suppose a manufacturer needs to produce a 24 inch by 24 inch square piece of particle board as part of a home office desk kit. How close does the side of the piece of particle board need to be cut to 24 inches to guarantee that the area of the piece is within a tolerance of 0.25 square inches of the target area of 576 square inches?

Solution. Let x denote the length of the side of the square piece of particle board so that the area of the board is x^2 square inches. Our tolerance specifies that the area of the board, x^2, needs to be within 0.25 square inches of 576. Mathematically, this translates to $|x^2 - 576| \leq 0.25$. Rewriting, we get $-0.25 \leq x^2 - 576 \leq 0.25$, or $575.75 \leq x^2 \leq 576.25$. At this point, we take advantage of the fact that the square root is increasing.[14] Therefore,

[12] See Example **??** for examples of linear compound inequalities.

[13] The underlying concept of Calculus can be phrased in terms of tolerances, so this is well worth your attention.

[14] This means that for $a, b \geq 0$, if $a \leq b$, then $\sqrt{a} \leq \sqrt{b}$.

taking square roots preserves the inequality. When simplifying, we keep in mind that since x represents a length, $x > 0$.

$$\begin{array}{rcll}
575.75 \leq & x^2 & \leq 576.25 & \\
\sqrt{575.75} \leq & \sqrt{x^2} & \leq \sqrt{576.25} & \text{(take square roots.)} \\
\sqrt{575.75} \leq & |x| & \leq \sqrt{576.25} & (\sqrt{x^2} = |x|) \\
\sqrt{575.75} \leq & x & \leq \sqrt{576.25} & (|x| = x \text{ since } x > 0)
\end{array}$$

The side of the piece of particle board must be between $\sqrt{575.75} \approx 23.995$ and $\sqrt{576.25} \approx 24.005$ inches, a tolerance of (approximately) 0.005 inches of the target length of 24 inches, to ensure that the area is within 0.25 square inches of 576. ■

1.3 THE ALGEBRA OF DIFFERENCE QUOTIENTS

Definition 1. Given a function f, the **difference quotient** of f is the expression

$$\frac{f(x+h)-f(x)}{h}$$

We will revisit this concept in Chapter **??**, but for now, we use it as a way to practice function notation and function arithmetic. For reasons which will become clear in Calculus, ‘simplifying’ a difference quotient means rewriting it in a form where the ‘h’ in the definition of the difference quotient cancels from the denominator. Once that happens, we consider our work to be done.

Example 1. Find and simplify the difference quotients for the following functions

1. $f(x) = x^2 - x - 2$
2. $g(x) = \dfrac{3}{2x+1}$
3. $r(x) = \sqrt{x}$

Explanation. 1. To find $f(x+h)$, we replace every occurrence of x in the formula $f(x) = x^2 - x - 2$ with the quantity $(x+h)$ to get

$$\begin{aligned} f(x+h) &= (x+h)^2 - (x+h) - 2 \\ &= x^2 + 2xh + h^2 - x - h - 2. \end{aligned}$$

So the difference quotient is

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{\left(x^2+2xh+h^2-x-h-2\right)-\left(x^2-x-2\right)}{h} \\ &= \frac{x^2+2xh+h^2-x-h-2-x^2+x+2}{h} \\ &= \frac{2xh+h^2-h}{h} \\ &= \frac{h\,(2x+h-1)}{h} && \text{factor} \\ &= \frac{\cancel{h}\,(2x+h-1)}{\cancel{h}} && \text{cancel} \\ &= 2x+h-1. \end{aligned}$$

2. To find $g(x+h)$, we replace every occurrence of x in the formula $g(x) = \dfrac{3}{2x+1}$ with the quantity $(x+h)$ to get

$$\begin{aligned} g(x+h) &= \frac{3}{2(x+h)+1} \\ &= \frac{3}{2x+2h+1}, \end{aligned}$$

which yields

$$\begin{aligned} \frac{g(x+h)-g(x)}{h} &= \frac{\frac{3}{2x+2h+1}-\frac{3}{2x+1}}{h} \\ &= \frac{\frac{3}{2x+2h+1}-\frac{3}{2x+1}}{h} \cdot \frac{(2x+2h+1)(2x+1)}{(2x+2h+1)(2x+1)} \\ &= \frac{3(2x+1)-3(2x+2h+1)}{h(2x+2h+1)(2x+1)} \\ &= \frac{6x+3-6x-6h-3}{h(2x+2h+1)(2x+1)} \\ &= \frac{-6h}{h(2x+2h+1)(2x+1)} \\ &= \frac{-6\cancel{h}}{\cancel{h}(2x+2h+1)(2x+1)} \\ &= \frac{-6}{(2x+2h+1)(2x+1)}. \end{aligned}$$

Since we have managed to cancel the original 'h' from the denominator, we are done.

3. For $r(x) = \sqrt{x}$, we get $r(x+h) = \sqrt{x+h}$ so the difference quotient is

$$\frac{r(x+h)-r(x)}{h} = \frac{\sqrt{x+h}-\sqrt{x}}{h}$$

In order to cancel the 'h' from the denominator, we rationalize the *numerator* by multiplying by its conjugate.[15]

[15] Rationalizing the *numerator*!? How's that for a twist!

$$\begin{aligned}
\frac{r(x+h)-r(x)}{h} &= \frac{\sqrt{x+h}-\sqrt{x}}{h} \\
&= \frac{\left(\sqrt{x+h}-\sqrt{x}\right)}{h} \cdot \frac{\left(\sqrt{x+h}+\sqrt{x}\right)}{\left(\sqrt{x+h}+\sqrt{x}\right)} && \text{Multiply by the conjugate.} \\
&= \frac{\left(\sqrt{x+h}\right)^2 - \left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h}+\sqrt{x}\right)} && \text{Difference of Squares.} \\
&= \frac{(x+h)-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \\
&= \frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \\
&= \frac{\cancelto{1}{h}}{\cancel{h}\left(\sqrt{x+h}+\sqrt{x}\right)} \\
&= \frac{1}{\sqrt{x+h}+\sqrt{x}}
\end{aligned}$$

Since we have removed the original 'h' from the denominator, we are done.

As mentioned before, we will revisit difference quotients in Section 5 where we will explain them geometrically. For now, we want to move on to some classic applications of function arithmetic from Economics and for that, we need to think like an entrepreneur.[16]

Suppose you are a manufacturer making a certain product.[17] Let x be the **production level**, that is, the number of items produced in a given time period. It is customary to let $C(x)$ denote the function which calculates the total **cost** of producing the x items. The quantity $C(0)$, which represents the cost of producing no items, is called the **fixed** cost, and represents the amount of money required to begin production.

Associated with the total cost $C(x)$ is cost per item, or **average cost**, denoted $\overline{C}(x)$ and read 'C-bar' of x. To compute $\overline{C}(x)$, we take the total cost $C(x)$ and divide by the number of items produced x to get

$$\overline{C}(x) = \frac{C(x)}{x}$$

On the retail end, we have the **price** p charged per item. To simplify the dialog and computations in this text, we assume that *the number of items sold equals the number of items produced*. From a retail perspective, it seems natural to think of the number of items sold,

[16] Not really, but "entrepreneur" is the buzzword of the day and we're trying to be trendy.

[17] Poorly designed resin Sasquatch statues, for example. Feel free to choose your own entrepreneurial fantasy.

x, as a function of the price charged, p. After all, the retailer can easily adjust the price to sell more product.

In the language of functions, x would be the *dependent* variable and p would be the *independent* variable or, using function notation, we have a function $x(p)$. While we adopt this convention elsewhere in the text,[18] we will hold with tradition at this point and consider the price p as a function of the number of items sold, x. That is, we regard x as the independent variable and p as the dependent variable and speak of the **price-demand** function, $p(x)$. Hence, $p(x)$ returns the price charged per item when x items are produced and sold.

Our next function to consider is the **revenue** function, $R(x)$. The function $R(x)$ computes the amount of money collected as a result of selling x items. Since $p(x)$ is the price charged per item, we have $R(x) = xp(x)$. Finally, the **profit** function, $P(x)$ calculates how much money is earned after the costs are paid. That is, $P(x) = (R - C)(x) = R(x) - C(x)$. We summarize all of these functions below.

Summary of Common Economic Functions

Suppose x represents the quantity of items produced and sold.

- The price-demand function $p(x)$ calculates the price per item.
- The revenue function $R(x)$ calculates the total money collected by selling x items at a price $p(x)$, $R(x) = x\,p(x)$.
- The cost function $C(x)$ calculates the cost to produce x items. The value $C(0)$ is called the fixed cost or start-up cost.
- The average cost function $\overline{C}(x) = \dfrac{C(x)}{x}$ calculates the cost per item when making x items. Here, we necessarily assume $x > 0$.
- The profit function $P(x)$ calculates the money earned after costs are paid when x items are produced and sold, $P(x) = (R - C)(x) = R(x) - C(x)$.

It is high time for an example.

Example 2. Let x represent the number of dOpi media players ('dOpis'[19]) produced and sold in a typical week. Suppose the cost, in dollars, to produce x dOpis is given by $C(x) = 100x + 2000$, for $x \geq 0$, and the price, in dollars per dOpi, is given by $p(x) = 450 - 15x$ for $0 \leq x \leq 30$.

1. Find and interpret $C(0)$.
2. Find and interpret $\overline{C}(10)$.
3. Find and interpret $p(0)$ and $p(20)$.

[18] See Example **??** in Section **??**.

[19] Pronounced 'dopeys' . . .

4. Solve $p(x) = 0$ and interpret the result.

5. Find and simplify expressions for the revenue function $R(x)$ and the profit function $P(x)$.

6. Find and interpret $R(0)$ and $P(0)$.

7. Solve $P(x) = 0$ and interpret the result.

Explanation. 1. We substitute $x = 0$ into the formula for $C(x)$ and get $C(0) = 100(0) + 2000 = 2000$. This means to produce 0 dOpis, it costs \$2000. In other words, the fixed (or start-up) costs are \$2000. The reader is encouraged to contemplate what sorts of expenses these might be.

2. Since $\overline{C}(x) = \dfrac{C(x)}{x}$, $\overline{C}(10) = \dfrac{C(10)}{10} = \dfrac{3000}{10} = 300$. This means when 10 dOpis are produced, the cost to manufacture them amounts to \$300 per dOpi.

3. Plugging $x = 0$ into the expression for $p(x)$ gives $p(0) = 450 - 15(0) = 450$. This means no dOpis are sold if the price is \$450 per dOpi. On the other hand, $p(20) = 450 - 15(20) = 150$ which means to sell 20 dOpis in a typical week, the price should be set at \$150 per dOpi.

4. Setting $p(x) = 0$ gives $450 - 15x = 0$. Solving gives $x = 30$. This means in order to sell 30 dOpis in a typical week, the price needs to be set to \$0. What's more, this means that even if dOpis were given away for free, the retailer would only be able to move 30 of them.[20]

5. To find the revenue, we compute $R(x) = xp(x) = x(450 - 15x) = 450x - 15x^2$. Since the formula for $p(x)$ is valid only for $0 \leq x \leq 30$, our formula $R(x)$ is also restricted to $0 \leq x \leq 30$. For the profit, $P(x) = (R - C)(x) = R(x) - C(x)$. Using the given formula for $C(x)$ and the derived formula for $R(x)$, we get $P(x) = \left(450x - 15x^2\right) - (100x + 2000) = -15x^2 + 350x - 2000$. As before, the validity of this formula is for $0 \leq x \leq 30$ only.

6. We find $R(0) = 0$ which means if no dOpis are sold, we have no revenue, which makes sense. Turning to profit, $P(0) = -2000$ since $P(x) = R(x) - C(x)$ and $P(0) = R(0) - C(0) = -2000$. This means that if no dOpis are sold, more money (\$2000 to be exact!) was put into producing the dOpis than was recouped in sales. In number 1, we found the fixed costs to be \$2000, so it makes sense that if we sell no dOpis, we are out those start-up costs.

7. Setting $P(x) = 0$ gives $-15x^2+350x-2000 = 0$. Factoring gives $-5(x-10)(3x-40) = 0$ so $x = 10$ or $x = \dfrac{40}{3}$. What do these values mean in the context of the problem? Since $P(x) = R(x) - C(x)$, solving $P(x) = 0$ is the same as solving $R(x) = C(x)$. This means that the solutions to $P(x) = 0$ are the production (and sales) figures for which

[20] Imagine that! Giving something away for free and hardly anyone taking advantage of it . . .

the sales revenue exactly balances the total production costs. These are the so-called '**break even**' points. The solution $x = 10$ means 10 dOpis should be produced (and sold) during the week to recoup the cost of production. For $x = \frac{40}{3} = 13.\overline{3}$, things are a bit more complicated. Even though $x = 13.\overline{3}$ satisfies $0 \leq x \leq 30$, and hence is in the domain of P, it doesn't make sense in the context of this problem to produce a fractional part of a dOpi.[21] Evaluating $P(13) = 15$ and $P(14) = -40$, we see that producing and selling 13 dOpis per week makes a (slight) profit, whereas producing just one more puts us back into the red.[22]

[21] We've seen this sort of thing before in Section **??**.

[22] While breaking even is nice, we ultimately would like to find what production level (and price) will result in the largest profit. We invite the reader to revisit the tools in Section **??** and find the answer.

1.3.A EXERCISES FOR THE ALGEBRA OF DIFFERENCE QUOTIENTS

Question 1 *In Exercises 5 - 5, find and simplify the difference quotient* $\dfrac{f(x+h)-f(x)}{h}$ *for the given function.*

Problem 1.1 $f(x) = 2x - 5$

2

Problem 1.2 $f(x) = -3x + 5$

-3

Problem 1.3 $f(x) = 6$

0

Problem 1.4 $f(x) = 3x^2 - x$

$6x + 3h - 1$

Problem 1.5 $f(x) = -x^2 + 2x - 1$

$-2x - h + 2$

Problem 1.6 $f(x) = 4x^2$

$8x + 4h$

Problem 1.7 $f(x) = x - x^2$

$-2x - h + 1$

Problem 1.8 $f(x) = x^3 + 1$

$3x^2 + 3xh + h^2$

Problem 1.9 $f(x) = mx + b$ *where* $m \neq 0$

m

Problem 1.10 $f(x) = ax^2 + bx + c$ *where* $a \neq 0$

$2ax + ah + b$

Problem 1.11 $f(x) = \dfrac{2}{x}$

$$\frac{-2}{x(x+h)}$$

Problem 1.12 $f(x) = \dfrac{3}{1-x}$

$$\frac{3}{(1-x-h)(1-x)}$$

1.3.A EXERCISES FOR THE ALGEBRA OF DIFFERENCE QUOTIENTS

Problem 1.13 $f(x) = \dfrac{1}{x^2}$

$\dfrac{-(2x+h)}{x^2(x+h)^2}$

Problem 1.14 $f(x) = \dfrac{2}{x+5}$

$\dfrac{-2}{(x+5)(x+h+5)}$

Problem 1.15 $f(x) = \dfrac{1}{4x-3}$

$\dfrac{-4}{(4x-3)(4x+4h-3)}$

Problem 1.16 $f(x) = \dfrac{3x}{x+1}$

$\dfrac{3}{(x+1)(x+h+1)}$

Problem 1.17 $f(x) = \dfrac{x}{x-9}$

$\dfrac{-9}{(x-9)(x+h-9)}$

Problem 1.18 $f(x) = \dfrac{x^2}{2x+1}$

$\dfrac{2x^2+2xh+2x+h}{(2x+1)(2x+2h+1)}$

Problem 1.19 $f(x) = \sqrt{x-9}$

$\dfrac{1}{\sqrt{x+h-9}+\sqrt{x-9}}$

Problem 1.20 $f(x) = \sqrt{2x+1}$

$\dfrac{2}{\sqrt{2x+2h+1}+\sqrt{2x+1}}$

Problem 1.21 $f(x) = \sqrt{-4x+5}$

$\dfrac{-4}{\sqrt{-4x-4h+5}+\sqrt{-4x+5}}$

Problem 1.22 $f(x) = \sqrt{4-x}$

$$\frac{-1}{\sqrt{4-x-h}+\sqrt{4-x}}$$

Problem 1.23 $f(x) = \sqrt{ax+b}$, *where* $a \neq 0$.

$$\frac{a}{\sqrt{ax+ah+b}+\sqrt{ax+b}}$$

Problem 1.24 $f(x) = x\sqrt{x}$

$$\frac{3x^2+3xh+h^2}{(x+h)^{3/2}+x^{3/2}}$$

Problem 1.25 $f(x) = \sqrt[3]{x}$

Hint: $(a-b)\left(a^2+ab+b^2\right) = a^3-b^3$

$$\frac{1}{(x+h)^{2/3}+(x+h)^{1/3}x^{1/3}+x^{2/3}}$$

Question 2 *In Exercises 5 - 5, $C(x)$ denotes the cost to produce x items and $p(x)$ denotes the price-demand function in the given economic scenario. In each Exercise, do the following:*

- *Find and interpret* $C(0)$.
- *Find and interpret* $\overline{C}(10)$.
- *Find and interpret* $p(5)$
- *Find and simplify* $R(x)$.
- *Find and simplify* $P(x)$.
- *Solve* $P(x) = 0$ *and interpret.*

Problem 2.1 *The cost, in dollars, to produce x "I'd rather be a Sasquatch" T-Shirts is $C(x) = 2x + 26$, $x \geq 0$ and the price-demand function, in dollars per shirt, is $p(x) = 30 - 2x$, $0 \leq x \leq 15$.*

- *Find and interpret* $C(0)$.

  $C(0) = 26$, *so the fixed costs are* \$26.

- *Find and interpret* $\overline{C}(10)$.

  $\overline{C}(10) = 4.6$, *so when 10 shirts are produced, the cost per shirt is* \$4.60.

- *Find and interpret* $p(5)$

  $p(5) = 20$*, so to sell* 5 *shirts, set the price at* \$20 *per shirt.*

- *Find and simplify* $R(x)$*.*

  $R(x) = -2x^2 + 30x$, $0 \leq x \leq 15$

- *Find and simplify* $P(x)$*.*

  $P(x) = -2x^2 + 28x - 26$, $0 \leq x \leq 15$

- *Solve* $P(x) = 0$ *and interpret.*

  $P(x) = 0$ *when* $x = 1$ *and* $x = 13$*. These are the 'break even' points, so selling* 1 *shirt or* 13 *shirts will guarantee the revenue earned exactly recoups the cost of production.*

Problem 2.2 *The cost, in dollars, to produce x bottles of* 100% *All-Natural Certified Free-Trade Organic Sasquatch Tonic is* $C(x) = 10x + 100$*,* $x \geq 0$ *and the price-demand function, in dollars per bottle, is* $p(x) = 35 - x$*,* $0 \leq x \leq 35$*.*

- *Find and interpret* $C(0)$*.*

  $C(0) = 100$*, so the fixed costs are* \$100*.*

- *Find and interpret* $\overline{C}(10)$*.*

  $\overline{C}(10) = 20$*, so when 10 bottles of tonic are produced, the cost per bottle is* \$20*.*

- *Find and interpret* $p(5)$

  $p(5) = 30$*, so to sell* 5 *bottles of tonic, set the price at* \$30 *per bottle.*

- *Find and simplify* $R(x)$*.*

  $R(x) = -x^2 + 35x$, $0 \leq x \leq 35$

- *Find and simplify* $P(x)$*.*

  $P(x) = -x^2 + 25x - 100$, $0 \leq x \leq 35$

- *Solve* $P(x) = 0$ *and interpret.*

  $P(x) = 0$ *when* $x = 5$ *and* $x = 20$*. These are the 'break even' points, so selling* 5 *bottles of tonic or* 20 *bottles of tonic will guarantee the revenue earned exactly recoups the cost of production.*

Problem 2.3 *The cost, in cents, to produce x cups of Mountain Thunder Lemonade at Junior's Lemonade Stand is* $C(x) = 18x + 240$*,* $x \geq 0$ *and the price-demand function, in cents per cup, is* $p(x) = 90 - 3x$*,* $0 \leq x \leq 30$*.*

- *Find and interpret* $C(0)$*.*

  $C(0) = 240$*, so the fixed costs are* 240¢ *or* \$2.40*.*

Exercises for The Algebra of Difference Quotients

- *Find and interpret* $\overline{C}(10)$*.*

  $\overline{C}(10) = 42$*, so when 10 cups of lemonade are made, the cost per cup is* 42¢*.*

- *Find and interpret* $p(5)$

  $p(5) = 75$*, so to sell* 5 *cups of lemonade, set the price at* 75¢ *per cup.*

- *Find and simplify* $R(x)$*.*

  $R(x) = -3x^2 + 90x$*,* $0 \leq x \leq 30$

- *Find and simplify* $P(x)$*.*

  $P(x) = -3x^2 + 72x - 240$*,* $0 \leq x \leq 30$

- *Solve* $P(x) = 0$ *and interpret.*

  $P(x) = 0$ *when* $x = 4$ *and* $x = 20$*. These are the 'break even' points, so selling* 4 *cups of lemonade or* 20 *cups of lemonade will guarantee the revenue earned exactly recoups the cost of production.*

Problem 2.4 *The daily cost, in dollars, to produce x Sasquatch Berry Pies* $C(x) = 3x+36$*,* $x \geq 0$ *and the price-demand function, in dollars per pie, is* $p(x) = 12 - 0.5x$*,* $0 \leq x \leq 24$*.*

- *Find and interpret* $C(0)$*.*

  $C(0) = 36$*, so the daily fixed costs are* \$36*.*

- *Find and interpret* $\overline{C}(10)$*.*

  $\overline{C}(10) = 6.6$*, so when 10 pies are made, the cost per pie is* \$6.60*.*

- *Find and interpret* $p(5)$

  $p(5) = 9.5$*, so to sell* 5 *pies a day, set the price at* \$9.50 *per pie.*

- *Find and simplify* $R(x)$*.* $R(x) = -0.5x^2 + 12x$*,* $0 \leq x \leq 24$

- *Find and simplify* $P(x)$*.*

  $P(x) = -0.5x^2 + 9x - 36$*,* $0 \leq x \leq 24$

- *Solve* $P(x) = 0$ *and interpret.*

  $P(x) = 0$ *when* $x = 6$ *and* $x = 12$*. These are the 'break even' points, so selling* 6 *pies or* 12 *pies a day will guarantee the revenue earned exactly recoups the cost of production.*

Problem 2.5 *The monthly cost, in hundreds of dollars, to produce x custom built electric scooters is* $C(x) = 20x + 1000$*,* $x \geq 0$ *and the price-demand function, in hundreds of dollars per scooter, is* $p(x) = 140 - 2x$*,* $0 \leq x \leq 70$*.*

- *Find and interpret* $C(0)$*.*

  $C(0) = 1000$*, so the monthly fixed costs are* 1000 *hundred dollars, or* \$100,000*.*

- *Find and interpret* $\overline{C}(10)$*.*

  $\overline{C}(10) = 120$*, so when 10 scooters are made, the cost per scooter is* 120 *hundred dollars, or* \$12,000*.*

- *Find and interpret* $p(5)$

  $p(5) = 130$*, so to sell* 5 *scooters a month, set the price at* 130 *hundred dollars, or* \$13,000 *per scooter.*

- *Find and simplify* $R(x)$*.*

  $R(x) = -2x^2 + 140x$*,* $0 \leq x \leq 70$

- *Find and simplify* $P(x)$*.*

  $P(x) = -2x^2 + 120x - 1000$*,* $0 \leq x \leq 70$

- *Solve* $P(x) = 0$ *and interpret.*

  $P(x) = 0$ *when* $x = 10$ *and* $x = 50$*. These are the 'break even' points, so selling* 10 *scooters or* 50 *scooters a month will guarantee the revenue earned exactly recoups the cost of production.*

Problem 2.6 *Let us return to Example* **??** *where* $C(x) = .03x^3 - 4.5x^2 + 225x + 250$ *denotes the cost, in dollars, of producing x PortaBoy game systems. Recall the* ***average cost***[23] *is defined as* $\overline{C}(x) = \dfrac{C(x)}{x}$*,* $x > 0$*, is the cost per item.*

1. *Find and interpret* $\overline{C}(75)$*.*
2. *Define the* ***marginal cost*** $MC(x) = C(x+1) - C(x)$*. Find and interpret* $MC(75)$*.*
3. *How do your answers to parts* **??** *and* **??** *compare?*
4. *Graph* $y = \overline{C}(x)$ *with help from a graphing utility. What is happening graphically near* $x = 75$*?*
5. *Use Theorem* **??** *to show that, in general,* $ARoC[\overline{C}(x)] = 0$ *when* $MC(x) = \overline{C}(x)$*.*

   HINT: *Note that, by definition,* $MC(x) = C(x+1) - C(x) = \Delta[C(x)]$ *when* $\Delta x = 1$*.*

   Hence, $ARoC[C(x)] = \dfrac{\Delta[C(x)]}{\Delta x} = \dfrac{\Delta[C(x)]}{1} = \Delta[C(x)] = MC(x)$ *in this case ...*

[23] First mentioned in Definition **??** in Section **??**.

2.1 EXERCISES FOR RATIONAL INEQUALITIES

Question 3 *(Review of Solving Equations).*[24]

In Exercises 60 - 60, solve the rational equation. Be sure to check for extraneous solutions.

Problem 3.1 $\dfrac{x}{5x+4} = 3$

$x = -\dfrac{6}{7}$

Problem 3.2 $\dfrac{3x-1}{x^2+1} = 1$

Select All Correct Answers:

1. $x = -2$
2. $x = -1$
3. $x = 0$
4. $x = 1$ ✓
5. $x = 2$ ✓
6. $x = 3$

Problem 3.3 $\dfrac{1}{t+3} + \dfrac{1}{t-3} = \dfrac{t^2-3}{t^2-9}$

$t = -1$

Problem 3.4 $\dfrac{2t+17}{t+1} = t+5$

Select All Correct Answers:

1. $t = -6$ ✓
2. $t = -4$
3. $t = -2$
4. $t = 2$ ✓
5. $t = 4$
6. $t = 6$

[24] For more review, see Section **??**.

2.1 EXERCISES FOR RATIONAL INEQUALITIES

Problem 3.5 $\dfrac{z^2 - 2z + 1}{z^3 + z^2 - 2z} = 1$

No solution

Problem 3.6 $\dfrac{4z - z^3}{z^2 - 9} = 4z$

Select All Correct Answers:

1. $t = -4\sqrt{2}$
2. $t = -2\sqrt{2}$ ✓
3. $t = 0$ ✓
4. $t = 2\sqrt{2}$ ✓
5. $t = 4\sqrt{2}$

Question 4 *In Exercises 60 - 60, solve the rational inequality. Express your answer using interval notation.*

Problem 4.1 $\dfrac{1}{x+2} \geq 0$

For what value is the rational function undefined?

$x = -2$

Exercise 4.1.1 *Use test points to determine all intervals which are part of the solution set.*

Select All Correct Answers:

1. $(-\infty, -2)$
2. $(-2, \infty)$ ✓

Problem 4.2 $\dfrac{5}{x+2} \geq 1$

$(-2, 3]$

Problem 4.3 $\dfrac{x}{x^2 - 1} < 0$

For which value is the rational function equal to 0? $x = 0$

For which values is the rational function undefined? (Give your answers in increasing order.)

$x = -1$, $x = 1$

Exercise 4.3.1 *Use test points to determine all intervals which are part of the solution set.*

Select All Correct Answers:

1. $(-\infty, -1)$ ✓
2. $(-1, 0)$
3. $(0, 1)$ ✓
4. $(1, \infty)$

Problem 4.4 $\dfrac{4t}{t^2+4} \geq 0$

For what value is the rational function equal to 0*?*

$x = 0$

Exercise 4.4.1 *Use test points to determine all intervals which are part of the solution set.*

Select All Correct Answers:

1. $(-\infty, 0]$
2. $[(]0, \infty)$ ✓

Problem 4.5 $\dfrac{2t+6}{t^2+t-6} < 1$

$(-\infty, -3) \cup (-3, 2) \cup (4, \infty)$

Problem 4.6 $\dfrac{5}{t-3} + 9 < \dfrac{20}{t+3}$

$\left(-3, -\dfrac{1}{3}\right) \cup (2, 3)$

Problem 4.7 $\dfrac{6z+6}{2+z-z^2} \leq z+3$

$(-1, 0] \cup (2, \infty)$

Problem 4.8 $\dfrac{6}{z-1} + 1 > \dfrac{1}{z+1}$

$(-\infty, -3) \cup (-2, -1) \cup (1, \infty)$

Problem 4.9 $\dfrac{3z-1}{z^2+1} \leq 1$

$(-\infty, 1] \cup [2, \infty)$

Problem 4.10 $(2x+17)(x+1)^{-1} > x+5$

$(-\infty, -6) \cup (-1, 2)$

Problem 4.11 $(4x - x^3)(x^2-9)^{-1} \geq 4x$

$(-\infty, -3) \cup \left[-2\sqrt{2}, 0\right] \cup \left[2\sqrt{2}, 3\right)$

Problem 4.12 $(x^2+1)^{-1} < 0$

No solution.

Problem 4.13 $(2t-8)(t+1)^{-1} \leq (t^2-8t)(t+1)^{-2}$

$[-4, -1) \cup (-1, 2]$

Problem 4.14 $(t-3)(2t+7)(t^2+7t+6)^{-2} \geq (t^2+7t+6)^{-1}$

$(-\infty, -6) \cup (-6, -3] \cup [9, \infty)$

Problem 4.15 $60z^{-2} + 23z^{-1} \geq 7(z-4)^{-1}$

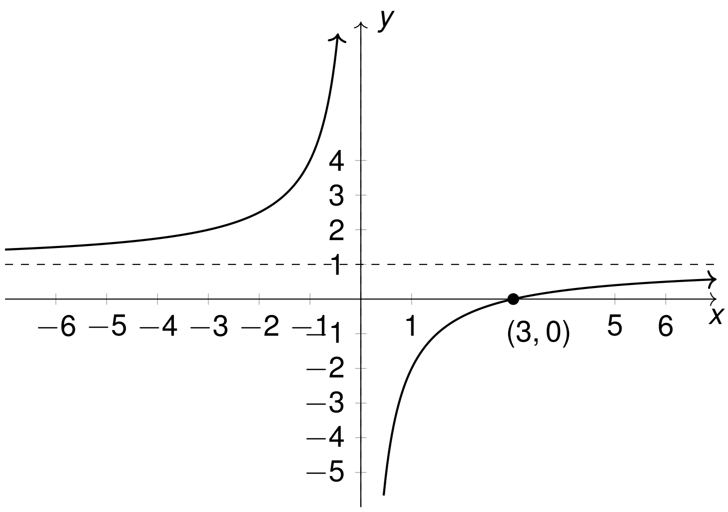
$[-3, 0) \cup (0, 4) \cup [5, \infty)$

Problem 4.16 $2z + 6(z-1)^{-1} \geq 11 - 8(z+1)^{-1}$

$\left(-1, -\dfrac{1}{2}\right] \cup (1, \infty)$

Question 5 *In Exercises 60 - 60, use the the graph of the given rational function to solve the stated inequality.*

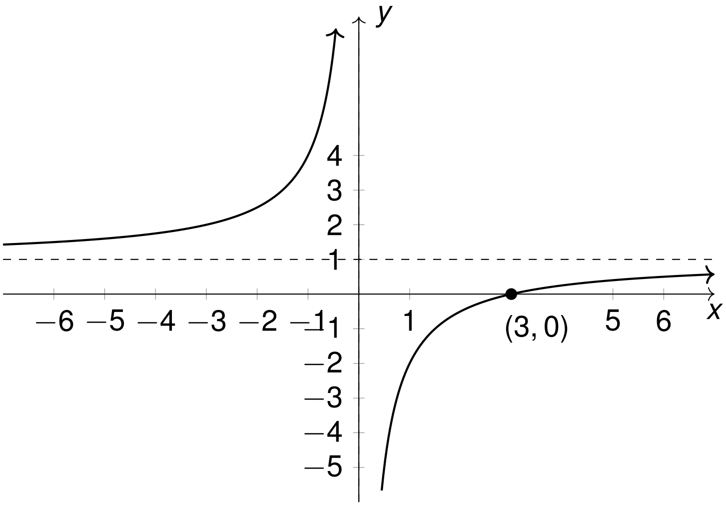
Problem 5.1 *Solve $f(x) \geq 0$.*



$y = f(x)$, asymptotes: $x = 0$, $y = 1$

$f(x) \geq 0$ *on* $(-\infty, 0) \cup [3, \infty)$.

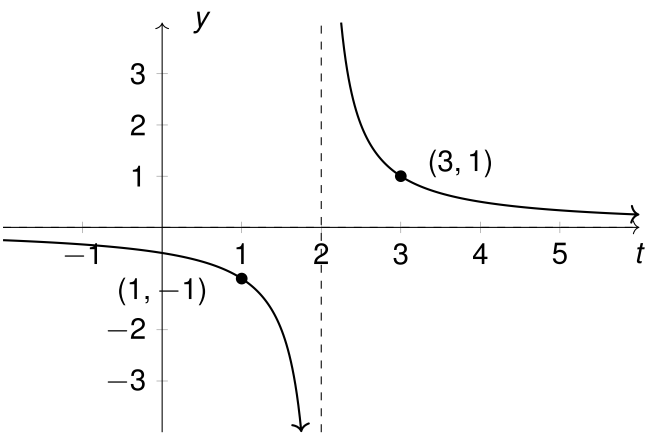
Problem 5.2 *Solve* $f(x) < 1$.



$y = f(x)$, asymptotes: $x = 0$, $y = 1$

$f(x) < 1$ *on* $(0, \infty)$.

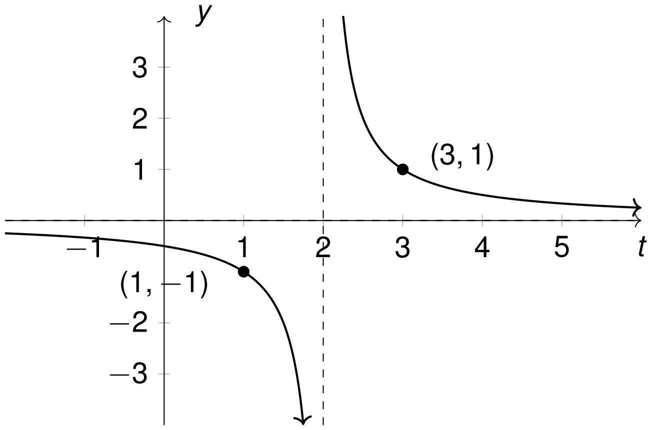
Problem 5.3 *Solve* $g(t) \geq -1$.



$y = g(t)$, asymptotes: $t = 2$, $y = 0$

$g(t) \geq -1$ *on* $(-\infty, 1] \cup (2, \infty)$.

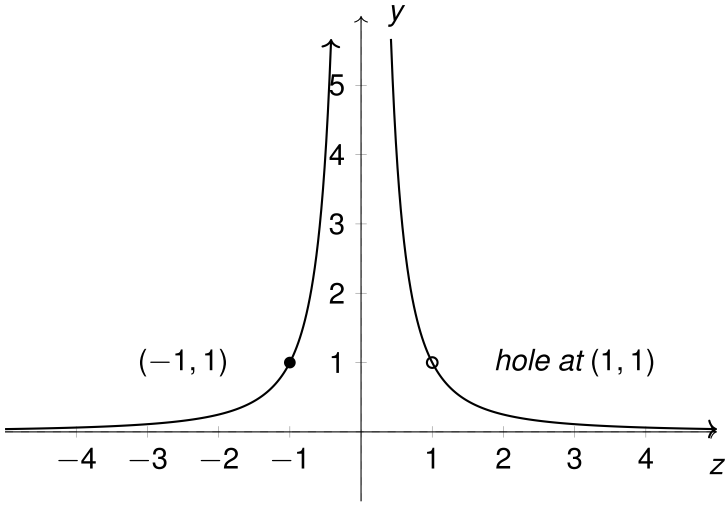
Problem 5.4 *Solve* $-1 \leq g(t) < 1$.



$y = g(t)$, asymptotes: $t = 2$, $y = 0$

$-1 \leq g(t) < 1$ *on* $(-\infty, 1] \cup (3, \infty)$.

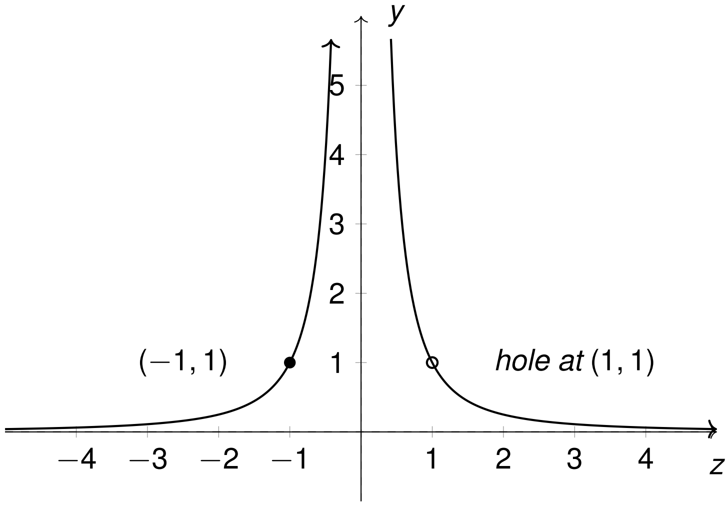
Problem 5.5 *Solve* $r(z) \leq 1$



$y = r(z)$, asymptotes: $z = 0$, $y = 0$

$r(z) \leq 1$ *on* $(-\infty, -1] \cup (1, \infty)$.

Problem 5.6 *Solve* $r(z) > 0$.



$y = r(z)$, asymptotes: $z = 0$, $y = 0$

$r(z) > 0$ *on* $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$.

Problem 6 *In Exercise* **??** *in Section* **??**, *the function* $C(x) = .03x^3 - 4.5x^2 + 225x + 250$, *for* $x \geq 0$ *was used to model the cost (in dollars) to produce x PortaBoy game systems. Using this cost function, find the number of PortaBoys which should be produced to minimize the average cost* $\overline{C}$. *Round your answer to the nearest number of systems.*

The absolute minimum of $y = \overline{C}(x)$ *occurs at* $\approx (75.73, 59.57)$. *Since x represents the number of game systems, we check* $\overline{C}(75) \approx 59.58$ *and* $\overline{C}(76) \approx 59.57$. *Hence, to minimize the average cost,* 76 *systems should be produced at an average cost of* \$59.57 *per system.*

Problem 7 *Suppose we are in the same situation as Example* **??**. *If the volume of the box is to be* 500 *cubic centimeters, use a graphing utility to find the dimensions of the box which minimize the surface area. What is the minimum surface area? Round your answers to two decimal places.*

The width (and depth) should be 10.00 *centimeters, the height should be* 5.00 *centimeters. The minimum surface area is* 300.00 *square centimeters.*

Problem 8 *The box for the new Sasquatch-themed cereal, 'Crypt-Os', is to have a volume of* 140 *cubic inches. For aesthetic reasons, the height of the box needs to be* 1.62 *times the width of the base of the box.*[25] *Find the dimensions of the box which will minimize the surface area of the box. What is the minimum surface area? Round your answers to two decimal places.*

The width of the base of the box should be approximately 4.12 *inches, the height of the box should be approximately* 6.67 *inches, and the depth of the base of the box should be approximately* 5.09 *inches. The minimum surface area is approximately* 164.91 *square inches.*

[25] 1.62 is a crude approximation of the so-called 'Golden Ratio' $\phi = \frac{1+\sqrt{5}}{2}$.

Problem 9 *Sally is Skippy's neighbor from Exercise* **??** *in Section* **??**. *Sally also wants to plant a vegetable garden along the side of her home. She doesn't have any fencing, but wants to keep the size of the garden to 100 square feet. What are the dimensions of the garden which will minimize the amount of fencing she needs to buy? What is the minimum amount of fencing she needs to buy? Round your answers to the nearest foot. (Note: Since one side of the garden will border the house, Sally doesn't need fencing along that side.)*

The dimensions are approximately 7 *feet by* 14 *feet. Hence, the minimum amount of fencing required is approximately* 28 *feet.*

Problem 10 *Another Classic Problem: A can is made in the shape of a right circular cylinder and is to hold one pint. (For dry goods, one pint is equal to* 33.6 *cubic inches.)*[26]

1. *Find an expression for the volume* V *of the can in terms of the height* h *and the base radius* r.

   $V = \pi r^2 h$

2. *Find an expression for the surface area* S *of the can in terms of the height* h *and the base radius* r. *(Hint: The top and bottom of the can are circles of radius* r *and the side of the can is really just a rectangle that has been bent into a cylinder.)*

   $S = 2\pi r^2 + 2\pi rh$

3. *Using the fact that* $V = 33.6$, *write* S *as a function of* r *and state its applied domain.*

   $S(r) = 2\pi r^2 + \dfrac{67.2}{r}$, *Domain* $r > 0$

4. *Use your graphing calculator to find the dimensions of the can which has minimal surface area.*

   $r \approx 1.749$ *in. and* $h \approx 3.498$ *in*

Problem 11 *A right cylindrical drum is to hold 7.35 cubic feet of liquid. Find the dimensions (radius of the base and height) of the drum which would minimize the surface area. What is the minimum surface area? Round your answers to two decimal places.*

The radius of the drum should be approximately 1.05 *feet and the height of the drum should be approximately* 2.12 *feet. The minimum surface area of the drum is approximately* 20.93 *cubic feet.*

Problem 12 *In Exercise* **??** *in Section* **??**, *the population of Sasquatch in Portage County is modeled by*

$$P(t) = \frac{150t}{t+15}, \quad t \geq 0,$$

where $t = 0$ *corresponds to the year 1803. According to this model, when were there fewer than 100 Sasquatch in Portage County?*

[26] According to www.dictionary.com, there are different values given for this conversion. We use 33.6in^3 for this problem.

$P(t) < 100$ *on* $(-15, 30)$*, and the portion of this which lies in the applied domain is* $[0, 30)$*. Since* $t = 0$ *corresponds to the year 1803, from 1803 through the end of 1832, there were fewer than 100 Sasquatch in Portage County.*

2.2 POWER FUNCTIONS

Monomial, and, more generally, Laurent monomial functions are specific examples of a much larger class of functions called **power functions**, as defined below.

Definition 1. Let a and p be nonzero real numbers. A **power function** is either a constant function or a function of the form $f(x) = ax^p$.

Definition 1 broadens our scope of functions to include non-integer exponents such as $f(x) = 2x^{4/3}$, $g(t) = t^{0.4}$ and $h(w) = w^{\sqrt{2}}$. Our primary aim in this section is to ascribe meaning to these quantities.

2.2.0.1 Rational Number Exponents

The road to real number exponents starts by defining rational number exponents.

Definition 2. Let r be a rational number where in lowest terms $r = \dfrac{m}{n}$ where m is an integer and n is a natural number.[1] If $n = 1$, then $x^r = x^m$. If $n > 1$, then

$$x^r = x^{\frac{m}{n}} = \left(\sqrt[n]{x}\right)^m = \sqrt[n]{x^m},$$

whenever $\left(\sqrt[n]{x}\right)^m$ is defined.[2]

There are quite a few items worthy of note which are consequences of Definition 2. First off, if m is an integer, then $x^{\frac{m}{1}} = x^m$ so expressions like $x^{\frac{3}{1}}$ are synonymous with x^3, as we would expect.[3] Second, the definition of $x^{\frac{m}{n}}$ can be taken as just $\left(\sqrt[n]{x}\right)^m$ and shown to be equal to $\sqrt[n]{x^m}$ (or vice-versa) courtesy of properties of radicals. We state both in Definition 2 to allow for the reader to choose whichever form is more convenient in a given situation. The critical point to remember is no matter which representation you choose, keep in mind the restrictions if n is even, $x \geq 0$ and if $m < 0$, $x \neq 0$.

Moreover, per this definition, $x^{\frac{1}{n}} = \sqrt[n]{x^1} = \sqrt[n]{x}$, so we may rewrite principal roots as exponents: $\sqrt{x} = x^{\frac{1}{2}}$ and $\sqrt[5]{x} = x^{\frac{1}{5}}$. This makes sense from an algebraic standpoint since per Theorem **??**, $\left(\sqrt[n]{x}\right)^n = x$. Hence if we were to assign an exponent notation to $\sqrt[n]{x}$, say $\sqrt[n]{x} = x^r$, then $\left(\sqrt[n]{x}\right)^n = (x^r)^n = x$. If the properties of exponents are to hold, then, necessarily, $(x^r)^n = x^{rn} = x = x^1$, so $rn = 1$ or $r = \dfrac{1}{n}$. While this argument helps *motivate* the notation, as we shall see shortly, great care must be exercised in applying exponent properties in these cases. The long and short of this is that root functions as defined in Section **??** are all members of the 'power functions' family.

Another important item worthy of note in Definition 2 is that it is absolutely essential we express the rational number r in *lowest terms* before applying the root-power definition. For example, consider $x^{0.4}$. Expressing r in lowest terms, we get: $r = 0.4 = \dfrac{4}{10} = \dfrac{2}{5}$. Hence,

[1] Recall 'lowest terms' means m and n have no common factors other than 1.

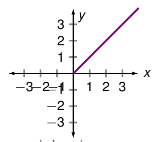
[2] That is, if n is even, $x \geq 0$ and if $m < 0$, $x \neq 0$.

[3] Either $n = 1$ is a special case in Definition 2 or we need to define what is meant by $\sqrt[1]{x}$. The authors chose the former.

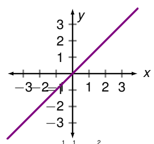
$x^{0.4} = x^{2/5} = (\sqrt[5]{x})^2$ or $\sqrt[5]{x^2}$, either of which is defined for all real numbers x. In contrast, consider the equivalence $r = 0.4 = \dfrac{4}{10}$. Here, the expression $(\sqrt[10]{x})^4$ is defined only for $x \geq 0$ owing to the presence of the even indexed root, $\sqrt[10]{x}$. Hence, $(\sqrt[10]{x})^4 \neq x^{\frac{4}{10}} = x^{\frac{2}{5}}$ unless $x \geq 0$. On the other hand, the expression $\sqrt[10]{x^4}$ *is* defined for all numbers, x, since $x^4 \geq 0$ for all x. In fact, it can be shown that $\sqrt[10]{x^4} = \sqrt[5]{x^2}$ for all real numbers. This means $\sqrt[10]{x^4} = \sqrt[5]{x^2} = x^{\frac{2}{5}} = x^{\frac{4}{10}}$. So, to review, in general we have: $x^{\frac{4}{10}} = \sqrt[10]{x^4}$, but $x^{\frac{4}{10}} \neq \left(\sqrt[10]{x}\right)^4$ unless $x \geq 0$. Once again the easiest way to avoid confusion here is to *reduce* the exponent to *lowest terms* before converting it to root-power notation.

Likewise, we have to be careful about the properties of exponents when it comes to rational exponents. Consider, for instance, the product rule for integer exponents: $x^m x^n = x^{m+n}$. Consider $f(x) = x^{\frac{1}{2}}x^{\frac{1}{2}}$ and $g(x) = x^{\frac{1}{2}+\frac{1}{2}}$. In the first case, $f(x) = x^{\frac{1}{2}}x^{\frac{1}{2}} = \sqrt{x}\sqrt{x} = (\sqrt{x})^2 = x$ only for $x \geq 0$. In the second case, $g(x) = x^{\frac{1}{2}+\frac{1}{2}} = x^{\frac{2}{2}} = x^1 = x$ for all real numbers x. Even though $f(x) = g(x)$ for $x \geq 0$, f and g are *different functions* since they have *different domains*.

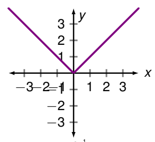
Similarly, the power rule for integer exponents: $(x^n)^m = x^{nm}$ does not hold in general for rational exponents. To see this, consider the three functions: $f(x) = (x^{\frac{1}{2}})^2$, $g(x) = x^{\frac{2}{2}}$, and $h(x) = (x^2)^{\frac{1}{2}}$. In the first case, $f(x) = (x^{\frac{1}{2}})^2 = (\sqrt{x})^2 = x$ for $x \geq 0$ only (this is the same function f above.) In the second case, the rational number $r = \dfrac{2}{2} = 1$, so $g(x) = x^{\frac{2}{2}} = x^{\frac{1}{1}} = x^1 = x$ for *all* real numbers, x (this is the same function g from above.) In the last case, $h(x) = (x^2)^{\frac{1}{2}} = \sqrt{x^2} = |x|$ for all real numbers, x. Once again, despite $f(x) = g(x) = h(x)$ for all $x \geq 0$, f, g and h and are *three different functions*. We graph f, g, and h below.



$f(x) = x^{\frac{1}{2}}x^{\frac{1}{2}} = (x^{\frac{1}{2}})^2 = x,\ x \geq 0$



$g(x) = x^{\frac{1}{2}+\frac{1}{2}} = x^{\frac{2}{2}} = x$



$h(x) = (x^2)^{\frac{1}{2}} = |x|$

In general, the properties of integer exponents *do not extend* to rational exponents *unless* the bases involved represent non-negative real numbers *or* the roots involved are *odd*. We have the following:

Theorem 1. Let r and s are rational numbers. The following properties hold provided none of the computations results in division by 0 and **either** r and s have odd denominators **or** $x \geq 0$ **and** $y \geq 0$:

- **Product Rules:** $x^r x^s = x^{r+s}$ and $(xy)^r = x^r y^r$.
- **Quotient Rules:** $\dfrac{x^r}{x^s} = x^{r-s}$ and $\left(\dfrac{x}{y}\right)^r = \dfrac{x^r}{y^r}$
- **Power Rule:** $(x^r)^s = x^{rs}$

Next, we turn our attention to the graphs of $f(x) = x^r = x^{\frac{m}{n}}$ for varying values of m and n. When n is even, the domain is restricted owing to the presence of the even indexed root to $[0, \infty)$. The range is likewise $[0, \infty)$, a fact leave to the reader. All of the functions below are increasing on their domains, and it turns out this is always the case provided $r > 0$. There is, however, is a difference in *how* the functions are increasing - and this is the concept of *concavity*. As with many concepts we've encountered so far in the text, concavity is most precisely defined using Calculus terminology, but we can nevertheless get a sense of concavity geometrically. For us, a curve is **concave up** over an interval if it resembles a portion of a '$\smile$' shape. Similarly, a curve is called **concave down** over an interval if resembles part of a '$\frown$' shape. When $0 < r < 1$, the graphs of $f(x) = x^r$ resemble the left half of $\frown$ and so are concave down; when $r > 1$, the graphs resemble the right half of a '$\smile$' and are hence described as 'concave up.'

Desmos link: `https://www.desmos.com/calculator/fni0liyn5n`

Below we graph several examples of $f(x) = x^r = x^{\frac{m}{n}}$ where n is odd. Here, the domain is $(-\infty, \infty)$ since the index on the root here is odd. Note that when m is even, the graphs appear to be symmetric about the y-axis and the range looks to be $[0, \infty)$. When m is odd, the graphs appear to be symmetric about the origin with range $(-\infty, \infty)$. We leave verification of these facts to the reader. Note here also that for $x \geq 0$, the graphs are down for $0 < r < 1$ and concave up for $r > 1$.

Desmos link: `https://www.desmos.com/calculator/gyv8qfxhjs`

When $r < 0$, we have variables appear in the denominator which open the opportunities for vertical and horizontal asymptotes. Below are graphed two examples

Desmos link: `https://www.desmos.com/calculator/t2bfjpr8ar`

Unsurprisingly, Theorem **??**, which, as stated, applied to root functions, generalizes to all rational powers.

Theorem 2. For real numbers a, b, h, and k and rational number r with $a, b, r \neq 0$, the graph of $F(x) = a(bx - h)^r + k$ can be obtained from the graph of $f(x) = x^r$ by performing the following operations, in sequence:

1. add h to each of the x-coordinates of the points on the graph of f. This results in a horizontal shift to the right if $h > 0$ or left if $h < 0$.

   NOTE: This transforms the graph of $y = x^r$ to $y = (x - h)^r$.

2. divide the x-coordinates of the points on the graph obtained in Step 1 by b. This results in a horizontal scaling, but may also include a reflection about the y-axis if $b < 0$.

   NOTE: This transforms the graph of $y = (x - h)^r$ to $y = (bx - h)^r$.

3. multiply the y-coordinates of the points on the graph obtained in Step 2 by a. This results in a vertical scaling, but may also include a reflection about the x-axis if $a < 0$.

   NOTE: This transforms the graph of $y = (bx - h)^r$ to $y = a(bx - h)^r$.

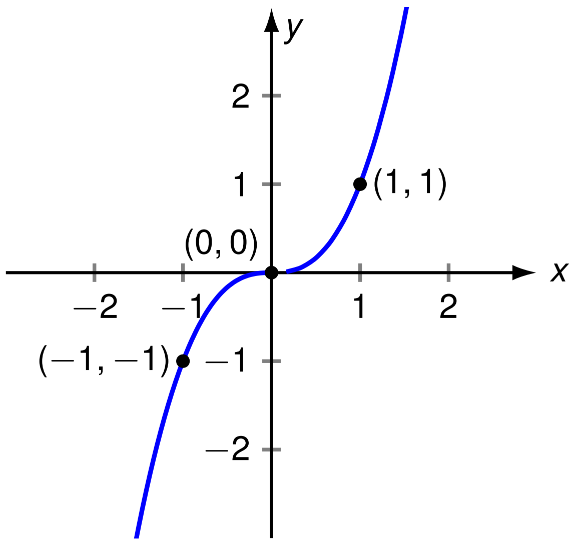
4. add k to each of the y-coordinates of the points on the graph obtained in Step 3. This results in a vertical shift up if $k > 0$ or down if $k < 0$.

   NOTE: This transforms the graph of $y = a(bx - h)^r$ to $y = a(bx - h)^r + k$.

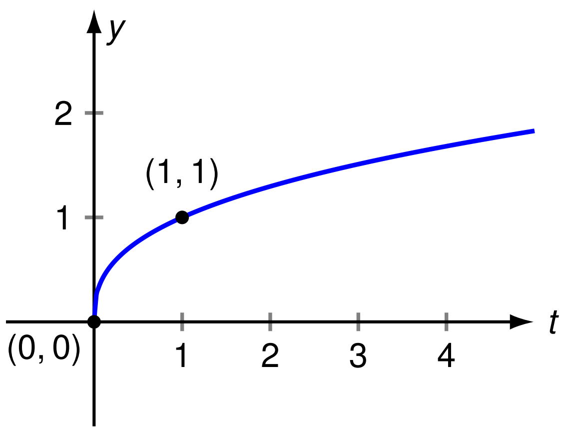
The proof of Theorem 2 is identical to that of Theorem **??**, and we suggest the reader work through the details. We give Theorem 2 a test run in the following example.

Example 1. Use the given graphs of f and g below long with Theorem 2 to graph F and G. State the domain and range of F and G using interval notation.

1. Graph $F(x) = (2x - 1)^{2.6}$.



2. Graph $G(t) = 1 - 2(t + 3)^{\frac{3}{8}}$.



Solution.

1. The expression $F(x) = (2x-1)^{2.6}$ is given to us in the form prescribed by Theorem 2, and we identify $r = 2.6$, $a = 1$, $b = 2$, $h = 1$, and $k = 0$. Even though the graph of $f(x) = x^{2.6}$ is given to us, it's worth taking a moment to reinforce some concepts. Since, in lowest terms, $2.6 = \frac{26}{10} = \frac{13}{5}$, it makes sense the domain and range of $f(x) = x^{2.6}$ are both all real numbers and the graph is symmetric about the origin.[4] Moreover, since $2.6 > 1$, the concavity matches what we would expect, too. We proceed as we have several times in the past, beginning with the horizontal shift.

   GeoGebra link: `https://www.geogebra.org/m/eazjuzbg`

   We get the domain and range here are both $(-\infty, \infty)$.

2. We first need to rewrite $G(t) = 1 - 2(t+3)^{\frac{3}{8}}$ in the form required by Theorem 2: $G(t) = -2(t+3)^{\frac{3}{8}} + 1$. We identify $r = \frac{3}{8}$, $a = -2$, $b = 1$, $h = -3$, and $k = 1$. Since $\frac{3}{8}$ is in lowest terms and has an even denominator, it makes sense the domain and range of $g(t) = t^{\frac{3}{8}}$ is $[0, \infty)$, since the root here, 8 is even. Also, since $0 < \frac{3}{8} < 1$, the graph of $y = t^{\frac{3}{8}}$ is concave down, as we would expect. As usual, we start with the horizontal shift.

   GeoGebra link: `https://www.geogebra.org/m/hr42fgrf`

   From the graph, we get the domain is $[-3, \infty)$ and the range is $(-\infty, 1]$.

We now turn our attention to more complicated functions involving rational exponents.

Example 2. For the following functions:

- Analytically:
  - find the domain.
  - find the axis intercepts.
  - analyze the end behavior.
- Graph the function with help from a graphing utility and determine:
  - the range.
  - the local extrema, if they exist.

[4]The domain is all real numbers since the denominator (root) 5 is odd; the range is all real numbers since the numerator (power) 13 is odd. Since both power and root are odd, the function itself is an odd function, hence the symmetry about the origin.

– intervals of increase.

– intervals of decrease.

- Construct a sign diagram for each function using the intercepts and graph.

1. $f(x) = 3x^2(x^3 - 8)^{-\frac{2}{3}}$

2. $g(t) = \dfrac{(t^2-4)^{\frac{3}{2}}}{t^2-36}$

Solution.

1. We first note that, owing to the negative exponent, the quantity $(x^3 - 8)^{\frac{2}{3}}$ is in the denominator, alerting us to a potential domain issue. Rewriting $(x^3 - 8)^{\frac{2}{3}}$ we set about solving $\sqrt[3]{(x^3-8)^2} = 0$. Cubing both sides and extracting square roots gives $x^3 - 8 = 0$ or $x = 2$. Hence, $x = 2$ is excluded from the domain.[5] Since the root involved here is odd (3), the only issue we have is with the denominator, hence our domain is $\{x \in \mathbb{R} \mid x \neq 2\}$ or $(-\infty, 2) \cup (2, \infty)$.

   While not required to do so, we analyze the behavior of f near $x = 2$. As $x \to 2^-$, $3x^2 \approx 12$ and $x^3 - 8 \approx$ small $(-)$. Hence, $(x^3-8)^{\frac{2}{3}} = \sqrt[3]{(x^3-8)^2} \approx \sqrt[3]{(\text{small }(-))^2} \approx \sqrt[3]{\text{small }(+)} \approx$ small(+). As such, $f(x) \approx \dfrac{12}{\text{small}(+)} \approx$ big (+). We conclude $\lim\limits_{x\to 2^-} f(x) = \infty$. As $x \to 2^+$, $3x^2 \approx 12$ and $x^3 - 8 \approx$ small (+), and we likewise get $\lim\limits_{x\to 2^+} f(x) = \infty$. Since the unbounded behavior agrees from both directions as $x \to 2$, we write $\lim\limits_{x\to 2} f(x) = \infty$. Our analysis suggests $x = 2$ is a vertical asymptote to the graph.

   To find the x-intercepts, we set $f(x) = 3x^2(x^3 - 8)^{-\frac{2}{3}} = 0$, so that $3x^2 = 0$ or $x = 0$. We get $(0, 0)$ is our only x- (and y-)intercept.

   For end behavior, note that in the denominator the x^3 term dominates so as $x \to -\infty$ and $x \to \infty$,

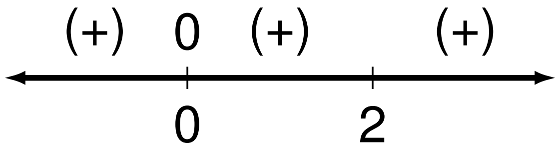
$$f(x) = 3x^2(x^3-8)^{-\frac{2}{3}} = \frac{3x^2}{(x^3-8)^{\frac{2}{3}}} \approx \frac{3x^2}{(x^3)^{\frac{2}{3}}} = \frac{3x^2}{\sqrt[3]{(x^3)^2}} = \frac{3x^2}{\sqrt[3]{x^6}} = \frac{3x^2}{x^2} = 3.$$

   This suggests $\lim\limits_{x\to -\infty} f(x) = 3$ and $\lim\limits_{x\to\infty} f(x) = 3$ so $y = 3$ is a horizontal asymptote to the graph.

   Graphing $y = f(x)$ below on the right bears out our analysis regarding zeros and asymptotes. The range appears to be $[0, \infty)$, with the graph of $y = f(x)$ crossing its horizontal asymptote between $x = 1$ and $x = 2$. We see we have a single local minimum at $(0, 0)$ with f is decreasing on $(-\infty, 0]$ and $(2, \infty)$ and increasing on $[0, 2)$.

   For the sign diagram, we note that f has only one zero, $x = 0$ and is undefined at $x = 2$. For all x values between these two numbers, $f(x) > 0$ or (+). Our sign diagram for $f(x)$ is below on the right.

[5] In general if $u^p = 0$ where $p > 0$, then $u = 0$.



Desmos link: https://www.desmos.com/calculator/nto7sdaomz

2. To find the domain of $g(t) = \dfrac{(t^2-4)^{\frac{3}{2}}}{t^2-36}$, we have two issues to address: the denominator and an even (square) root. Solving $t^2 - 36 = 0$ gives two excluded values, $t = \pm 6$. For the numerator, we may rewrite $(t^2-4)^{\frac{3}{2}} = (\sqrt{t^2-4})^3$, so we require $t^2 - 4 \geq 0$, or $t^2 \geq 4$. Extracting square roots, we have $\sqrt{t^2} \geq \sqrt{4}$ or $|t| \geq 2$ which means $t \leq -2$ or $t \geq 2$. Taking into account our excluded values $t = \pm 6$, we get the domain of g is $(-\infty, -6) \cup (-6, -2] \cup [2, 6) \cup (6, \infty)$.

   Looking near $t = -6$, we note that as $t \to -6$, $(t^2-4)^{\frac{3}{2}} \approx 32^{\frac{3}{2}} = 32^{1.5}$, a positive number. As $t \to -6^-$, $t^2 - 36 \approx \text{small}\,(+)$, so $g(t) \approx \dfrac{32^{1.5}}{\text{small}(+)} \approx \text{big}\,(+)$. This suggests $\displaystyle\lim_{t \to -6^-} g(t) = \infty$. On the other hand, as $t \to -6^+$, $t^2 - 36 \approx \text{small}\,(-)$, so $g(t) \approx \dfrac{32^{1.5}}{\text{small}(-)} \approx \text{big}\,(-)$, suggesting $\displaystyle\lim_{t \to -6^+} g(t) = -\infty$. Similarly, we find as $\displaystyle\lim_{t \to 6^-} g(t) = -\infty$ and as $\displaystyle\lim_{t \to 6^+} g(t) = \infty$. This suggests we have two vertical asymptotes to the graph of $y = g(t)$: $t = -6$ and $t = 6$.

   To find the t-intercepts, we set $g(t) = 0$ and solve $(t^2-4)^{\frac{3}{2}} = 0$. This reduces to $t^2 - 4 = 0$ or $t = \pm 2$. As these are (just barely!) in the domain of g, we have two t-intercepts, $(-2, 0)$ and $(2, 0)$. The graph of g has no y-intercepts, since 0 is not in the domain of g, so $g(0)$ is undefined.

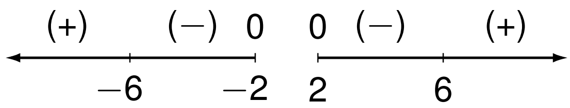
   Regarding end behavior, as $t \to -\infty$ and $t \to \infty$, the t^2 in both numerator and denominator dominate the constant terms, so we have

$$g(t) = \frac{(t^2-4)^{\frac{3}{2}}}{t^2-36} \approx \frac{\left(t^2\right)^{\frac{3}{2}}}{t^2} = \frac{\left(\sqrt{t^2}\right)^3}{t^2} = \frac{|t|^3}{t^2} = \frac{|t||t|^2}{t^2} = \frac{|t|t^2}{t^2} = |t|.$$

   This suggests that as $t \to -\infty$ and $t \to \infty$, the graph of $y = g(t)$ resembles $y = |t|$. Hence, $\displaystyle\lim_{t \to -\infty} g(t) = \infty$ and $\displaystyle\lim_{t \to \infty} g(t) = \infty$. Using the piecewise definition of $|t|$, we have that as $t \to -\infty$, $g(t) \approx -t$ and as $t \to \infty$, $g(t) \approx t$. In other words, the graph of $y = g(t)$ has *two* slant asymptotes with slopes ± 1.

   Graphing $y = g(t)$ below on the left verifies our analysis. From the graph, the range appears to be $(-\infty, 0] \cup [14.697, \infty)$. The points $(-10, 14.697)$ and $(10, 14.697)$ are local minimums. g appears to be decreasing on $(-\infty, -10]$, $[2, 6)$, and $(6, 10]$. Likewise, g is increasing on $[-10, -6)$, $(-6, -2]$ and $[10, \infty)$. The graph of $y = g(t)$ certainly appears to be symmetric about the y-axis. We leave it to the reader to show g is, indeed, an even function.

   For the sign diagram for $g(t)$, we note that g has zeros $t = \pm 2$ and is undefined at $t = \pm 6$. Moreover, there is a gap in the domain for all values in the interval $(-2, 2)$, so we excise that portion of the real number line for our discussion. We find $g(t) > 0$ or (+) on the intervals $(-\infty - 6)$ and $(6, \infty)$ while $g(t) < 0$ or $(-)$ on $(-6, -2)$ and $(2, 6)$. Our sign diagram for $g(t)$ is below on the right.



Desmos link: https://www.desmos.com/calculator/m2hzmagth8

■

2.2.0.2 Real Number Exponents

We wish now to extend the concept of 'exponent' from rational to all real numbers which means we need to discuss how to interpret an irrational exponent. Once again, the notions presented here are best discussed using the language of Calculus or Analysis, but we nevertheless do what we can with the notions we have.

Consider the wildly famous irrational number π. The number π is defined geometrically as the ratio of the circumference of a circle to that circle's diameter.[6] The reason we use the *symbol* 'π' instead of any numerical expression is that π is an irrational number, and, as such, its decimal representation neither terminates nor repeats. Hence we *approximate* π as $\pi \approx 3.14$ or $\pi \approx 3.14159265$. No matter how many digits we write, however, what we have is a *rational number* approximation of π.

The good news is we can approximate π to any desired accuracy using rational numbers by taking enough digits, so while we'll never 'reach' the *exact* value of π with rational numbers, we can get *as close as we like* to π using rational numbers. That being said, we assume π exists on the real number line, despite the fact the list of digits to pinpoint its location is, in some sense, infinite.

We take this tack when defining the value of a number raised to an irrational exponent. Consider, for instance, 2^{π}. We can compute $2^3 = 8$, $2^{3.1} = 2^{\frac{31}{10}} = \sqrt[10]{2^{31}} \approx 8.574$, $2^{3.14} = 2^{\frac{314}{100}} = 2^{\frac{157}{50}} = \sqrt[50]{2^{157}} \approx 8.8512$, and so on, so one way to *define* 2^{π} as the unique real number we obtain as the exponents 'approach' π.

It is with this understanding that we present the notion of a 'power function,' as described in Definition 1: $f(x) = ax^p$ where a and p are nonzero real number parameters. Here the exponent p is open to any (nonzero) real number. Because of how we define real number exponents, if p is irrational, then $x \geq 0$ to avoid having negatives under even-indexed roots as we go through the approximation process.[7]

In general, real number exponents inherit their properties from rational number exponents. For instance, Theorem 1 also holds for all real number exponents and the graphs of power functions inherit their behavior from graphs of rational exponent functions. More specifically, the graphs of functions of the form $f(x) = x^p$ where $p > 0$ all contain the points $(0,0)$ and $(1,1)$. Moreover, these functions are increasing and their graphs are concave down if $0 < p < 1$ and concave up if $p > 1$.

[6] This works for each and every circle, by the way, regardless of how large or small the circle is!
[7] or $x > 0$ if p is negative.

Desmos link: https://www.desmos.com/calculator/sjuhbguk78

Theorem 2 generalizes to real number power functions, so, for instance to graph $F(x) = (x-2)^{\pi}$, one need only start with $y = x^{\pi}$ and shift horizontally two units to the right. (See the Exercises.)

We close this section with an application to economics. According to the US Census, Table 2, the share of money income (2014-2015) is given in the table below on the left. From these data, we can create a cumulative distribution, $y = L(x)$ called the **Lorenz Curve**.

The number $L(x)$ gives the percentage of the total national income earned by the bottom x percent of wage earners, ranked from lowest income to highest income. Since the population here is separated into 'quintiles,' each data point corresponds to 20% of the population. So, for example, $L(20)$ is the percentage of money income earned by the lowest 20% of wage earners. In this case, we see $L(20) = 3.1$. The number $L(40)$ is the percentage of the money income earned by the bottom 40% of wage earners - so this includes not only the money from the Second Quintile, but also the Lowest Quintile: $L(40) = 8.2 + L(20) = 8.2 + 3.1 = 11.3$. Likewise, $L(60)$ is the total income share of the bottom 60% of wage earners which includes the income from the Middle, Second, and Lowest Quintiles: $L(60) = 14.3 + L(40) = 14.3 + (8.2 + 3.1) = 25.6$.

Continuing in this manner, we get $L(80) = 48.8$ and $L(100) = 100$, which is what we would expect: 100% of the income is earned by 100% of the population. We summarize these findings below on the right.

Portion of Population	Percent of Money Income
Lowest Quintile	3.1
Second Quintile	8.2
Middle Quintile	14.3
Fourth Quintile	23.2
Highest Quintile	51.2

percent wage earners, x	percent income, $L(x)$
20	3.1
40	11.3
60	25.6
80	48.8
100	100

Example 3.

1. Find power function to model the Lorenz Curve: $L(x) = ax^p$. Comment on the goodness of fit.
2. Find and interpret $L(90)$.

Solution.

1. Using desmos, we get $L(x) = 0.00027901x^{2.7738}$ with $R^2 = 0.993$, indicating a pretty good fit.

   Desmos link: https://www.desmos.com/calculator/nqjvl92ytp

2. We compute $L(90) = 0.00027901(90)^{2.7738} \approx 73.5$ meaning that the bottom 90% of the wage earners brought home 73.5% of the total income. Said differently, the top 10% of wage earners made over 25% of the total national income.

2.3 EXERCISES FOR POWER FUNCTIONS

Question 13 *In Exercises 60 - 60, sketch the graph of the new function by starting with the graph of the given function and using Theorem 2. Track at least two points and state the domain and range using interval notation.*

Problem 13.1 $F(x) = (x-2)^{\frac{2}{3}} - 1$

Use the Desmos graph below with the settings $f(x) = x^{\frac{2}{3}}$, $h = 2$, $k = -1$, $a = 1$, $b = 1$

Desmos link: `https://www.desmos.com/calculator/otto03uikp`

Domain: $(-\infty, \infty)$*, Range:* $[-1, \infty)$

Problem 13.2 $G(t) = (t+3)^{\pi} + 1$

Use the Desmos graph below with the settings $g(t) = t^{\pi}$, $h = -3$, $k = 1$, $a = 1$, $b = 1$

Desmos link: `https://www.desmos.com/calculator/ayc738smcf`

Domain: $[-3, \infty)$*, Range:* $[1, \infty)$

Problem 13.3 $F(x) = 3 - x^{\frac{2}{3}}$

Use the Desmos graph below with the settings $f(x) = x^{\frac{2}{3}}$, $h = 0$, $k = 3$, $a = -1$, $b = 1$

Desmos link: `https://www.desmos.com/calculator/otto03uikp`

Domain: $(-\infty, \infty)$*, Range:* $(-\infty, 3]$

Problem 13.4 $G(t) = (1-t)^{\pi} - 2$

Use the Desmos graph below with the settings $g(t) = t^{\pi}$, $h = -1$, $k = -2$, $a = 1$, $b = -1$

Desmos link: `https://www.desmos.com/calculator/ayc738smcf`

Domain: $(-\infty, 1]$*, Range:* $[-2, \infty)$

Problem 13.5 $F(x) = (2x+5)^{\frac{2}{3}} + 1$ *Use the Desmos graph below with the settings* $f(x) = x^{\frac{2}{3}}$, $h = -5$, $k = 1$, $a = -1$, $b = 2$

Desmos link: `https://www.desmos.com/calculator/otto03uikp`

Domain: $(-\infty, \infty)$*, Range:* $[1, \infty)$

2.3 EXERCISES FOR POWER FUNCTIONS

Problem 13.6 $G(t) = \left(\frac{t+3}{2}\right)^{\pi} - 1$

Use the Desmos graph below with the settings $g(t) = t^{\pi}$, $h = -\frac{3}{2}$, $k = -1$, $a = 1$, $b = \frac{1}{2}$

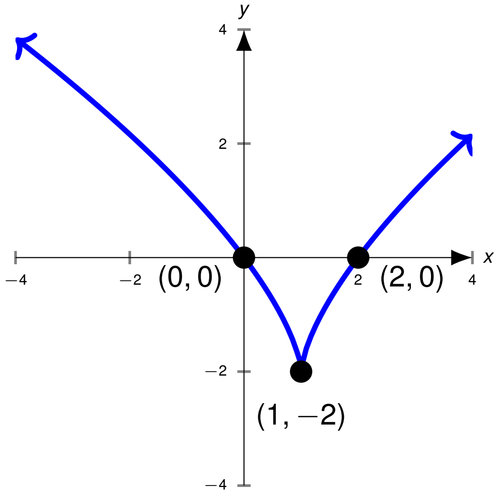
Desmos link: https://www.desmos.com/calculator/ayc738smcf

Domain: $[-3, \infty)$, *Range:* $[-1, \infty)$

Question 14 *In Exercises* **??** *-* **??***, find a formula for each function below in the form* $F(x) = a(bx - h)^{\frac{2}{3}} + k$.

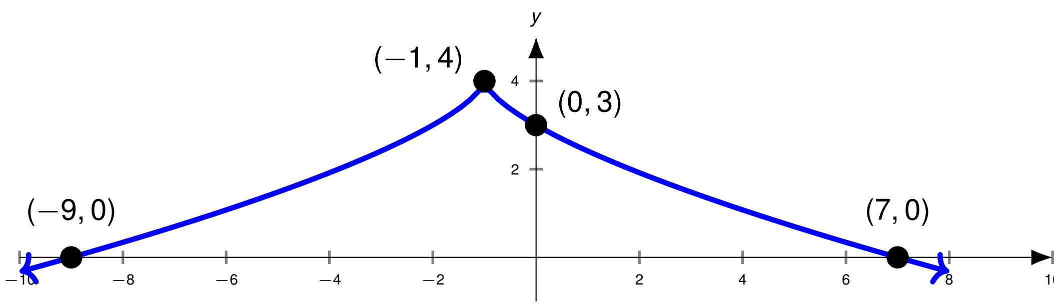
NOTE: *There may be more than one solution!*

Problem 14.1 $y = F(x)$



One solution is: $F(x) = 2(x-1)^{\frac{2}{3}} - 2$

Problem 14.2 $y = F(x)$



One solution is: $F(x) = -(x+1)^{\frac{2}{3}} + 4$

Question 15 *For each function in Exercises 60 - 60 below*

1. *Analytically:*

   - *find the domain.*
   - *find the axis intercepts.*
   - *analyze the end behavior.*

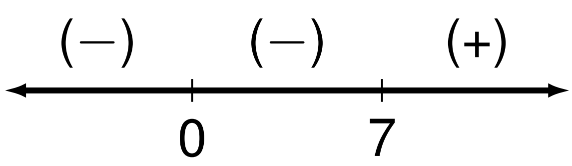
2. *Graph the function with help from a graphing utility and determine:*
   - *the range.*
   - *the local extrema, if they exist.*
   - *intervals of increase/decrease.*
   - *any 'unusual steepness' or 'local' verticality.*
   - *vertical asymptotes.*
   - *horizontal / slant asymptotes.*
3. *Construct a sign diagram for each function using the intercepts and graph.*
4. *Comment on any observed symmetry.*

Problem 15.1 $f(x) = x^{\frac{2}{3}}(x-7)^{\frac{1}{3}}$

$f(x) = x^{\frac{2}{3}}(x-7)^{\frac{1}{3}}$
Domain: $(-\infty, \infty)$
Intercepts: $(0,0)$, $(7,0)$
Graph:

Graph of $f(x) = x^{2/3} * (x-7)^{1/3}$

As $x \to -\infty$, $f(x) \to -\infty$
As $x \to \infty$, $f(x) \to \infty$
Range: $(-\infty, \infty)$
Local minimum: $\approx (4.667, -3.704)$
Local maximum: $(0,0)$ *(this is a cusp)*
Increasing: $(-\infty, 0]$, $\approx [4.667, \infty)$
Decreasing: $[0, 4.667]$
Unusual steepness at $x = 7$
Using Calculus it can be shown that $y = x - \dfrac{7}{3}$ *is a slant asymptote of this graph.*
Sign Diagram:

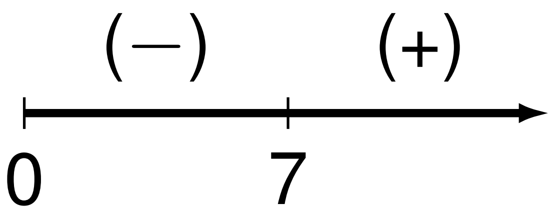


Problem 15.2 $f(x) = x^{\frac{3}{2}}(x-7)^{\frac{1}{3}}$

$f(x) = x^{\frac{3}{2}}(x-7)^{\frac{1}{3}}$
Domain: $[0, \infty)$
Intercepts: $(0, 0)$, $(7, 0)$
Graph:

Graph of $f(x) = x^{3/2} * (x-7)^{1/3}$

As $x \to \infty$, $f(x) \to \infty$
Range: $\approx [-14.854, \infty)$
Local minimum: $\approx (5.727, -14.854)$
Increasing: $\approx [5.727, \infty)$
Decreasing: $\approx [0, 5.727]$
Unusual steepness at $x = 7$
Sign Diagram:

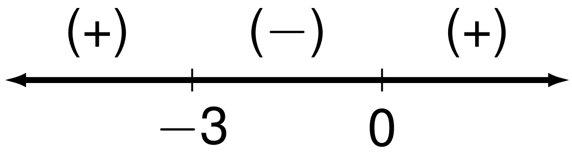


Problem 15.3 $g(t) = 2t(t+3)^{-\frac{1}{3}}$

$g(t) = 2t(t+3)^{-\frac{1}{3}}$
Graph:

Graph of $g(t) = 2t(t+3)^{-1/3}$

Domain: $(-\infty, -3) \cup (-3, \infty)$
Intercept: $(0, 0)$
As $t \to \pm\infty$, $g(t) \to \infty$
Range: $(-\infty, \infty)$
Local minimum: $\approx (-4.5, 7.862)$
Increasing: $\approx [-4.5, -3)$, $(-3, \infty)$
Decreasing: $\approx (-\infty, -4.5]$
Vertical Asymptote: $t = -3$
Sign Diagram:

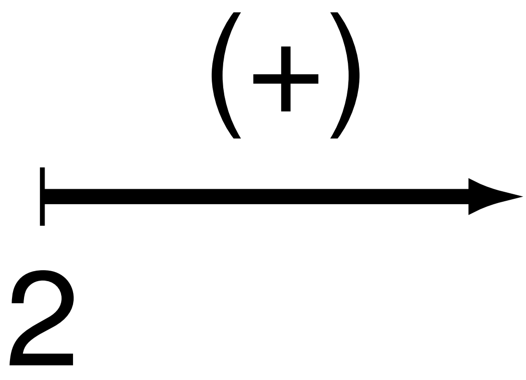


Problem 15.4 $g(t) = t^{\frac{3}{2}}(t-2)^{-\frac{1}{2}}$

$g(t) = t^{\frac{3}{2}}(t-2)^{-\frac{1}{2}}$
Domain: $(2, \infty)$
As $t \to \infty$, $g(t) \to \infty$
Graph:

Graph of $g(t) = t^{3/2}(t-2)^{-\frac{1}{2}}$

Range: $\approx [5.196, \infty)$
Local minimum: $\approx (3, 5.196)$
Increasing: $\approx [3, \infty)$
Decreasing: $\approx (2, 3]$
Vertical asymptote: $t = 2$
Using Calculus it can be shown that $y = t + 1$ *is a slant asymptote of this graph.*
Sign Diagram:

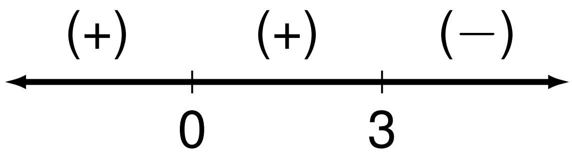


Problem 15.5 $f(x) = x^{0.4}(3-x)^{0.6}$

$f(x) = x^{0.4}(3-x)^{0.6}$
Domain: $(-\infty, \infty)$
Intercepts: $(0, 0)$, $(3, 0)$
Graph:

Graph of $f(x) = x^{0.4}(3-x)^{0.6}$

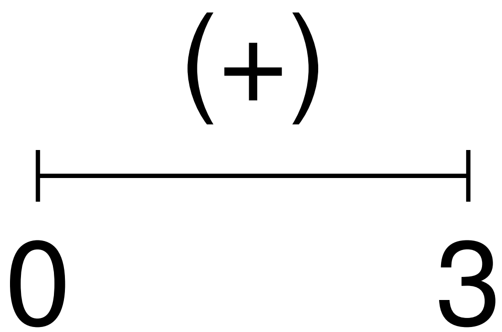
As $x \to -\infty$, $f(x) \to \infty$
As $x \to \infty$, $f(x) \to -\infty$
Range: $(-\infty, \infty)$
Local minimum: $(0, 0)$ *(this is a cusp)*
Local maximum: $\approx (1.2, 1.531)$
Increasing: $\approx [0, 1.2]$
Decreasing: $\approx (-\infty, 0], [1.2, \infty)$
Unusual Steepness: $x = 3$
Sign Diagram:



Problem 15.6 $f(x) = x^{0.5}(3 - x)^{0.5}$
$f(x) = x^{0.5}(3 - x)^{0.5}$
Graph:

Graph of $f(x) = x^{0.5}(3 - x)^{0.5}$

Domain: $[0, 3]$
Intercepts: $(0, 0)$, $(3, 0)$
Range: $\approx [0, 1.5]$
Increasing: $\approx [0, 1.5]$
Decreasing: $\approx [1.5, 3]$
Unusual Steepness:[8] $x = 0$, $x = 3$



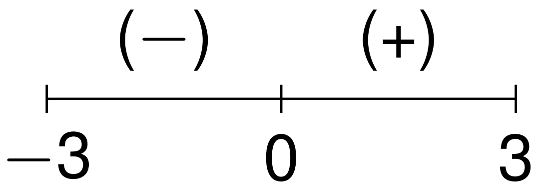
Problem 15.7 $g(t) = 4t(9 - t^2)^{-\sqrt{2}}$

[8] Note you may need to zoom in to see this.

$g(t) = 4t(9 - t^2)^{-\sqrt{2}}$
Graph:

Graph of $g(t) = 4t(9 - t^2)^{-\sqrt{2}}$

Domain: $(-3, 3)$
Intercepts: $(0, 0)$
Range: $(-\infty, \infty)$
Increasing: $(-3, 3)$
Sign Diagram:

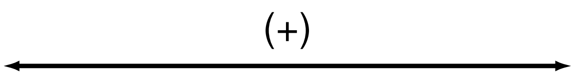


Problem 15.8 $g(t) = 3(t^2 + 1)^{-\pi}$

$g(t) = 3(t^2 + 1)^{-\pi}$
Domain: $(-\infty, \infty)$
Graph:

Graph of $g(t) = 3(t^2 + 1)^{-\pi}$

Intercept: $(0, 3)$
As $t \to \pm\infty$*,* $g(t) \to 0$
Range: $(0, 3]$
Increasing: $(-\infty, 0]$
Decreasing: $[0, \infty)$
Horizontal asymptote: $y = 0$
Sign Diagram:



Problem 16 *For each function $f(x)$ listed below, compute the average rate of change over the indicated interval.*[9] *What trends do you observe? How do your answers manifest*

[9]See Definition 5 in Section 5 for a review of this concept, as needed.

themselves graphically? Compare the results of this exercise with those of Exercise **??** *in Section* **??** *and Exercise* **??** *in Section* **??**

$f(x)$	[0.9, 1.1]	[0.99, 1.01]	[0.999, 1.001]	[0.9999, 1.0001]
$x^{\frac{1}{2}}$				
$x^{\frac{2}{3}}$				
$x^{-0.23}$				
x^{π}				

As in Exercise **??** *in Section* **??** *and Exercise* **??** *in Section* **??**, *the slopes of these curves near* $x = 1$ *approach the value of the exponent on* x.

$f(x)$	[0.9, 1.1]	[0.99, 1.01]	[0.999, 1.001]	[0.9999, 1.0001]
$x^{\frac{1}{2}}$	0.5006	$\approx \frac{1}{2}$	$\approx \frac{1}{2}$	$\approx \frac{1}{2}$
$x^{\frac{2}{3}}$	0.6672	0.6667	$\approx \frac{2}{3}$	$\approx \frac{2}{3}$
$x^{-0.23}$	-0.2310	≈ -0.23	≈ -0.23	≈ -0.23
x^{π}	3.1544	3.1417	$\approx \pi$	$\approx \pi$

Problem 17 *The National Weather Service uses the following formula to calculate the wind chill:*

$$W = 35.74 + 0.6215\,T_a - 35.75\,V^{0.16} + 0.4275\,T_a\,V^{0.16}$$

where W *is the wind chill temperature in* $^\circ F$, T_a *is the air temperature in* $^\circ F$, *and* V *is the wind speed in miles per hour. Note that* W *is defined only for air temperatures at or lower than* $50^\circ F$ *and wind speeds above* 3 *miles per hour.*

- *Suppose the air temperature is* 42° *and the wind speed is* 7 *miles per hour. Find the wind chill temperature. Round your answer to two decimal places.*

  $W \approx 37.55^\circ F$.

- *Suppose the air temperature is* $37^\circ F$ *and the wind chill temperature is* $30^\circ F$. *Find the wind speed. Round your answer to two decimal places.*

  $V \approx 9.84$ *miles per hour.*

Problem 18 *As a follow-up to Exercise* **??**, *suppose the air temperature is* $28^\circ F$.

- *Use the formula from Exercise* **??** *to find an expression for the wind chill temperature as a function of the wind speed,* $W(V)$.

  $W(V) = 53.142 - 23.78V^{0.16}$. *Since we are told in Exercise* **??** *that wind chill is only effect for wind speeds of more than 3 miles per hour, we restrict the domain to* $V > 3$.

- *Solve $W(V) = 0$, round your answer to two decimal places, and interpret.*
- *Graph the function W using a graphing utility and check your answer to part* **??**.

Problem 19 *Suppose Fritzy the Fox, positioned at a point (x, y) in the first quadrant, spots Chewbacca the Bunny at $(0, 0)$. Chewbacca begins to run along a fence (the positive y-axis) towards his warren. Fritzy, of course, takes chase and constantly adjusts his direction so that he is always running directly at Chewbacca. If Chewbacca's speed is v_1 and Fritzy's speed is v_2, the path Fritzy will take to intercept Chewbacca, provided v_2 is directly proportional to, but not equal to, v_1 is modeled by*

$$y = \frac{1}{2}\left(\frac{x^{1+v_1/v_2}}{1 + v_1/v_2} - \frac{x^{1-v_1/v_2}}{1 - v_1/v_2}\right) + \frac{v_1 v_2}{v_2^2 - v_1^2}$$

- *Determine the path that Fritzy will take if he runs exactly twice as fast as Chewbacca; that is, $v_2 = 2v_1$. Use your calculator to graph this path for $x \geq 0$. What is the significance of the y-intercept of the graph?*
- *Determine the path Fritzy will take if Chewbacca runs exactly twice as fast as he does; that is, $v_1 = 2v_2$. Use a graphing utility to graph this path for $x > 0$. Describe the behavior of y as $x \to 0^+$ and interpret this physically.*
- *With the help of your classmates, generalize parts (a) and (b) to two cases: $v_2 > v_1$ and $v_2 < v_1$. We will discuss the case of $v_1 = v_2$ in Exercise* **??** *in Section* **??**.