MTH 255H

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Contents

Part I

Notes

1 Polar coordinates

Polar coordinates are great for describing:

- Circles centered at the origin.
- Lines passing through the origin.
- Circles passing through the origin.

To go from rectangular coordinates to polar coordinates, we use the formulas

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \arctan(y/x)$$

To go from polar coordinates to rectangular coordinates, we use the formulas

$$x = r\cos\theta$$
$$y = r\sin\theta$$

The Jacobian for this coordinate system is given by

$$\det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r$$

1.1 Circle centered at the origin

A circle of radius c centered at the origin has equation $r = c$. Hence the interior is described as

$$0 \le r \le c$$
$$0 \le \theta \le 2\pi$$

Note: θ goes from 0 to 2π, but any interval of length 2π would work. For example $-\pi \le \theta \le \pi$.

1.2 Angular sections

A line passing through the origin has rectangular equation $y = mx$, where m is the slope. In polar coordinates, its equation is

$$\theta = \arctan(m)$$

The space between the lines with slopes m_1 and m_2 with $m_1 < m_2$, is given by

$$\arctan(m_1) \leq \theta \leq \arctan(m_2)$$

Note: the function arctan is not actually a function because there are pairs of angles with the same tangent. To successfully apply the above formulas, double check that your result coincides with what you actually want to describe. If not, you may need to replace arctan by $\arctan + \pi$.

1.3 Circles passing through the origin

A circle passing through the origin has equation

$$(x - a)^2 + (y - b)^2 = c^2$$

where (a, b) is the center of the circle, c is the radius, and $a^2 + b^2 = c^2$ to guarantee it passes through the origin. Expanding this, we get

$$x^2 + y^2 - 2ax - 2by + a^2 + b^2 = c^2$$

Cancelling $a^2 + b^2 = c^2$, and writing x and y in terms of r and θ, we get

$$r^2 - 2ar\cos\theta - 2br\sin\theta = 0$$

Dividing over r, we get

$$r = 2a\cos\theta + 2b\sin\theta$$

Therefore, the description of the interior of the circle in polar coordinates is given by

$$\arctan(b/a) - \pi/2 \leq \theta \leq \arctan(b/a) + \pi/2$$
$$0 \leq r \leq 2a\cos\theta + 2b\sin\theta$$

2 Cylindrical coordinates

Cylindrical coordinates are great for describing:

- Vertical cylinders centered at the origin.
- Vertical planes passing through the origin.

To go from rectangular coordinates to cylindrical coordinates, we use the same formulas as in polar coordinates

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \theta &= \arctan(y/x) \\ z &= z \end{aligned}$$

To go from cylindrical coordinates to rectangular coordinates, we use the formulas

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

The Jacobian for this coordinate system is given by

$$\det \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{pmatrix} = \det \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = r$$

2.1 Vertical cylinder centered at the origin

A vertical cylinder of radius c centered at the origin has equation $r = c$. Hence the interior is described as

$$\begin{aligned} 0 \le r \le c \\ 0 \le \theta \le 2\pi \\ -\infty < z < \infty \end{aligned}$$

2.2 Vertical plane passing through the origin

In rectangular coordinates, a vertical plane passing through the origin has equation $y = mx$. In cylindrical coordinates, its equation is

$$\theta = \arctan(m)$$

3 Spherical coordinates

Spherical coordinates are great for describing:

- Spheres centered at the origin.
- Vertical cones with tip at the origin.
- Spheres with center in the z-axis and passing through the origin.

To go from rectangular coordinates to spherical coordinates, we use the following formulas

$$\rho = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \arctan(y/x)$$
$$\phi = \arctan\left(\left(\sqrt{x^2 + z^2}\right) /z\right)$$

To go from spherical coordinates to rectangular coordinates, we use the formulas

$$x = \rho \cos\theta \sin\phi$$
$$y = \rho \sin\theta \sin\phi$$
$$z = \rho \cos\phi$$

To get the Jacobian, we compute

$$\det \begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix} = \det \begin{pmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{pmatrix} = -\rho^2 \sin\phi$$

Since the Jacobian is non-negative, we just consider the absolute value. This means the Jacobian of the spherical change of coordinates is

$$\rho^2 \sin\phi$$

3.1 Sphere centered at the origin

A sphere of radius c centered at the origin has equation $\rho = c$. Hence the interior is described as

$$0 \leq \rho \leq c$$
$$0 \leq \theta \leq 2\pi$$
$$0 \leq \phi \leq \phi$$

3.2 Vertical cone with tip at the origin

In rectangular coordinates, a vertical cone with tip at the origin has equation $z = m\sqrt{x^2 + y^2}$. In spherical coordinates, this equation becomes

$$\phi = \arctan(1/m)$$

The region above the cone is given by

$$0 \leq \phi \leq \arctan(1/m)$$

The region below the cone is given by

$$\arctan(1/m) \leq \phi \leq \pi$$

3.3 Sphere centered in the z-axis and passsing through the origin

A sphere passing through the origin and centered in the z-axis has equation

$$x^2 + y^2 + (z - c)^2 = c^2$$

Expanding this, we get

$$x^2 + y^2 + z^2 = 2cz$$

Putting this in terms of ρ and ϕ, we get

$$\rho^2 = 2c\rho\cos\phi$$

Dividing over ρ, we get

$$\rho = 2c\cos\phi$$

Therefore, the description of the interior of the sphere is

$$0 \leq \theta \leq 2\pi$$
$$0 \leq \phi \leq \pi/2$$
$$0 \leq \rho \leq 2c\cos\phi$$

if $c > 0$, and

$$0 \leq \theta \leq 2\pi$$
$$\pi/2 \leq \phi \leq \pi$$
$$0 \leq \rho \leq 2c\cos\phi$$

if $c < 0$.

4 Exercises on polar, cylindrical, and spherical coordinates

Exercise 1 *Compute the following integral*

$$\int_0^3 \int_0^{\sqrt{9-x^2}} (x^2+y^2)\, dydx$$

The restrictions

$$0 \le x \le 3$$
$$0 \le y \le \sqrt{9-x^2}$$

represent the region in the first quadrant and inside the circle of radius three. Those restrictions in polar coordinates become

$$0 \le \theta \le \pi/2$$
$$0 \le r \le 3$$

The integrand is x^2+y^2, which in polar coordinates becomes r^2. When we pass to polar coordinates, we multiply the integrand by the Jacobian, which is r. The integral becomes

$$\int_0^{\pi/2} \int_0^3 r^3\, drd\theta$$

Computing it, we get $[\pi/2]\left[3^4/4\right] = 81\pi/8$.

Exercise 2 *Compute the following integral*

$$\int_{-2}^2 \int_0^{\sqrt{4-y^2}} \int_0^{x^2+y^2} z\, dzdxdy$$

The restrictions

$$-2 \le y \le 2$$
$$0 \le x \le \sqrt{4-y^2}$$
$$0 \le z \le x^2+y^2$$

represent the region above the xy-plane, below the paraboloid $z = x^2+y^2$, on the side of the yz-plane with positive x-coordinate, and inside the cylinder $x^2+y^2=2$ of radius 2. In cylindrical coordinates, these restrictions become

$$-\pi/2 \le \theta \le \pi/2$$
$$0 \le r \le 2$$
$$0 \le z \le r^2$$

The integrand is already in terms of the coordinates r, θ, z. By passing to cylindrical coordinates, we multiply the integrand by the Jacobian, which is r. The integral becomes

$$\int_{-\pi/2}^{\pi/2} \int_0^2 \int_0^{r^2} zr \, dzdrd\theta$$

Solving it, we get

$$= \int_{-\pi/2}^{\pi/2} \int_0^2 r^3 \, drd\theta = [\pi] \left[2^4/4\right] = 4\pi$$

Exercise 3 *Compute the following integral*

$$\int_0^2 \int_x^{\sqrt{4-x^2}} \int_{\sqrt{3x^2+3y^2}}^{\sqrt{16-x^2-y^2}} z(x^2+y^2) \, dzdydx$$

The restrictions

$$0 \leq x \leq 2$$
$$x \leq y \leq \sqrt{4-x^2}$$
$$\sqrt{3x^2+3y^2} \leq z \leq \sqrt{16-x^2-y^2}$$

describe the region..... maybe it is too difficult to see directly, so first look at the restrictions on x and y. They represent the region in the first quadrant above the line $y = x$ and inside the circle of radius 2. This means

$$0 \leq r \leq 2$$
$$\pi/4 \leq \theta \leq \pi/2$$

The restrictions on z correspond to the region above the cone $z = \sqrt{3}\sqrt{x^2+y^2}$ and inside the sphere $x^2+y^2+z^2 = 16$. In sphereical coordinates, this becomes

$$0 \leq \rho \leq 4$$
$$0 \leq \phi \leq \pi/6$$

The integrand was $z(x^2+y^2)$, which in spherical coordinates becomes $\rho^3 \cos\phi \sin^2\phi$. When we pass to spherical coordinates, we multiply the integrand by the Jacobian, which is $\rho^2 \sin\phi$. Then the integral becomes

$$\int_{\pi/4}^{\pi/2} \int_0^2 \int_0^{\pi/6} \rho^5 \cos\phi \sin^3\phi \, d\phi d\rho d\theta$$

Solving it we get

$$= [\pi/4] \left[2^6/6\right] \left[\sin^4(\pi/6)/4\right] = 3\pi/8.$$

Exercise 4 *Compute the following integral*

$$\iiint_B \frac{z}{x^2+y^2+z^2}\,dxdydz$$

where B is the interior of the ball $x^2+y^2+(z-2)^2=4$.

The restrictions corresponding to B in spherical coordinates are

$$0 \le \theta \le 2\pi$$
$$0 \le \phi \le \pi/2$$
$$0 \le \rho \le 4\cos\phi$$

The integrand in terms of spherical coordinates becomes $\rho\cos\phi/\rho^2 = \cos\phi/\rho$. When we pass to spherical coordinates, we multiply the integrand by the Jacobian, which is $\rho^2\sin\phi$. Then the integral becomes

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^{4\cos\phi} \rho\cos\phi\sin\phi\, d\rho d\phi d\theta$$

Solving, we get

$$= [2\pi]\int_0^{\pi/2}\left[(4\cos\phi)^2/2\right]\cos\phi\sin\phi\, d\phi = 4\pi\left[\cos^4(0)-\cos^4(\pi/2)\right] = 4\pi$$

5 uv-substitution

Just like we use polar, cylindrical, and spherical integrals to solve integrals, we can use any other coordinate system. For simplicity, we restrict ourselves to dimension 2, but the concepts make sense in any dimensions.

If we write x and y in terms of the coordinates u and v, and vice-versa, the Jacobian of this change of coordinates is given by the absolute value of the determinant of the matrix of partial derivatives.

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

Exercise 5 *Compute*

$$\iint_R \frac{3x}{4} dxdy$$

where R is the region in the first quadrant between the hyperbolas $y = 1/x$ and $y = 5/x$, and between the lines $y = x/3$ and $y = 2x$.

We can rewrite the restrictions as

$$1 \leq xy \leq 5$$
$$\frac{1}{3} \leq \frac{y}{x} \leq 2$$

Hence if we introduce the variables

$$u := xy$$
$$v := y/x$$

the restrictions become

$$1 \leq u \leq 5$$
$$\frac{1}{3} \leq v \leq 2$$

Then we need to write x and y in terms of the new variables:

$$x = \sqrt{u/v}$$
$$y = \sqrt{uv}$$

To get the Jacobian, we need the partial derivatives

$$\frac{\partial x}{\partial u} = \frac{1}{2} \cdot \frac{1}{\sqrt{uv}}$$
$$\frac{\partial x}{\partial v} = -\frac{1}{2} \cdot \frac{\sqrt{u}}{\sqrt{v}^3}$$
$$\frac{\partial y}{\partial u} = \frac{1}{2} \cdot \frac{\sqrt{v}}{\sqrt{u}}$$
$$\frac{\partial y}{\partial v} = \frac{1}{2} \cdot \frac{\sqrt{u}}{\sqrt{v}}$$

Then the Jacobian is the absolute value of

$$\det \begin{pmatrix} \frac{1}{2} \cdot \frac{1}{\sqrt{uv}} & -\frac{1}{2} \cdot \frac{\sqrt{u}}{\sqrt{v}^3} \\ \frac{1}{2} \cdot \frac{\sqrt{v}}{\sqrt{u}} & \frac{1}{2} \cdot \frac{\sqrt{u}}{\sqrt{v}} \end{pmatrix} = \frac{1}{4} \left[\frac{1}{v} + \frac{1}{v} \right] = \frac{1}{2v}$$

The integrand in terms of the new variables is

$$\frac{3x}{4} = \frac{3\sqrt{u}}{4\sqrt{v}}$$

Then the integral becomes

$$\int_1^5 \int_{1/3}^2 \frac{3\sqrt{u}}{4\sqrt{v}} \cdot \frac{1}{2v} \, dvdu$$

Solving, we get

$$\begin{aligned} \int_1^5 \int_{1/3}^2 \frac{3\sqrt{u}}{4\sqrt{v}} \cdot \frac{1}{2v} \, dvdu &= \frac{3}{8} \int_1^5 \sqrt{u} \, du \int_{1/3}^2 \frac{1}{\sqrt{v}^3} \, dv \\ &= \frac{3}{8} \left[\frac{2}{3} (\sqrt{5}^3 - 1) \right] \left[2 \left(\frac{1}{\sqrt{1/3}} - \frac{1}{\sqrt{2}} \right) \right] \\ &= \frac{1}{4} \left[\sqrt{125} - 1 \right] \left[\sqrt{3} - \frac{1}{\sqrt{2}} \right] \end{aligned}$$

Exercise 6 *Compute*

$$\iint_R 9y \, dxdy$$

where $R \subset \mathbb{R}^2$ *is the parallelogram with vertices* $(-3, 1)$, $(0, -1)$, $(3, 0)$, $(0, 2)$.

The sides of the parallelogram are the lines

$$y = \frac{x}{3} - 1$$
$$y = \frac{x}{3} + 2$$
$$y = -\frac{2x}{3} - 1$$
$$y = -\frac{2x}{3} + 2$$

Then the parallelogram can be described as

$$-1 \leq y - \frac{x}{3} \leq 2$$
$$-1 \leq y + \frac{2x}{3} \leq 2$$

If we introduce the variables

$$u := 3y - x$$
$$v := 3y + 2x$$

the bounds above become

$$-3 \leq u \leq 6$$
$$-3 \leq v \leq 6$$

Then we need to write x and y in terms of u and v. This is done by solving a linear system.

$$x = \frac{v - u}{3}$$
$$y = \frac{v + 2u}{9}$$

The Jacobian is the absolute value of the determinant

$$\det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \det \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ \\ \frac{2}{9} & \frac{1}{9} \end{pmatrix} = \frac{1}{27}(-1 - 2) = -\frac{1}{9}$$

The integrand written in terms of the new variables is $9y = v + 2u$. Then the integral becomes

$$\int_{-3}^{6} \int_{-3}^{6} (v + 2u) \left[\frac{1}{9}\right] \, dudv$$

Solving, we get

$$= \int_{-3}^{6} vdv + 2 \int_{-3}^{6} udu = \frac{1}{2}[6^2 - 3^2 + 2(6^2 - 3^2)] = \frac{81}{2}$$

6 Curves

A curve is a continuous function $\gamma : [a, b] \to \mathbb{R}^3$ from an interval to the plane or space. It also corresponds to three continuous functions

$$\gamma(t) = (x(t), y(t), z(t))$$

which can be interpreted as a change of variables. It can be used to model:

- A curved object in the plane or space like a wire or a fence.
- A particle moving around. The coordinates $(x(t), y(t), z(t))$ represent the position at time t.

Sometimes, we call the image of γ the curve (the object in the plane or space), and call γ the parametrization.

The derivative

$$\gamma'(t) = (x'(t), y'(t), z'(t))$$

is called the velocity. Under the second interpretation, the direction of $\gamma'(t)$ is the direction in which the particle is moving at time t. The length $|\gamma'(t)|$ is the speed in which the particle is moving at time t.

The Jacobian of this change of variables is

$$\text{Jacobian}(\gamma) = |\gamma'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$

Using this, the length of a curve is defined as

$$\text{length}(\gamma) := \int_a^b |\gamma'(t)|\, dt$$

7 Line integrals of scalar functions

Definition 1. Let $C \subset \mathbb{R}^3$ be a curve, $\gamma : [a, b] \to \mathbb{R}^3$ a parametrization of C, and $f : \mathbb{R}^3 \to \mathbb{R}$ a continuous function. The integral of f along C is defined as

$$\int_C f\, ds := \int_a^b f(\gamma(t))|\gamma'(t)|\, dt$$

If $f(\gamma(t)) > 0$, the integral above can be interpreted as the mass of a wire with shape C and density f.

Note: the above integral is independent of the parametrization, and in particular does not depend on the direction in which the curve is travelled by the parametrization. Later we will consider line integrals of vector fields. They will change sign if we travel the curve in the opposite direction.

8 Exercises on curves and line integrals

Exercise 7 *Find the curve of intersection of the surfaces $y = x^2$ and $z = x^3$.*

We are looking for a curve

$$\gamma(t) = (x(t), y(t), z(t))$$

The condition $y = x^2$ establishes that whatever we use for $x(t)$, then $y(t)$ is going to be the square of that. The condition $z = x^3$ establishes that whatever we use for $x(t)$, then $z(t)$ is going to be the cube of that. Using these restrictions, we can write

$$x(t) = t$$
$$y(t) = t^2$$
$$z(t) = t^3$$

This yields

$$\gamma(t) = (t, t^2, t^3)$$

with $-\infty \leq t \leq \infty$

Exercise 8 *Find the curve of intersection of the surfaces $(x-2)^2+(y-3)^2 = 4$ and $x + y + z = 0$.*

We are looking for a curve

$$\gamma(t) = (x(t), y(t), z(t))$$

that goes around the intersection of the surfaces, which is an ellipse because it is the intersection of a cylinder with a plane. In the equation of the plane, we can isolate each variable in terms of the others. This means that once two of $x(t)$, $y(t)$, and $z(t)$ are defined, the other will be defined in terms of the other two. The cylinder $(x-2)^2 + (y-3)^2 = 4$ has center $(2, 3)$ and radius 2, so to guarantee a curve winds around we can define

$$x(t) = 2 + 2\cos t$$
$$y(t) = 3 + 2\sin t$$

From the equation of the plane we get $z = -x - y$, so

$$z(t) = -x(t) - y(t) = -5 - 2\cos t - 2\sin t$$

Then the curve is

$$\gamma(t) = (2 + 2\cos t, 3 + 2\sin t, -5 - 2\cos t - 2\sin t)$$

with $0 \leq t \leq 2\pi$

Note: using similar logic, a good idea would be to instead take the curve

$$\gamma(t) = (2+t, 3+\sqrt{4-t^2}, -5-t-\sqrt{4-t^2})$$

with $-2 \leq t \leq 2$. However, this curve would only cover half of the ellipse.

Exercise 9 *Find the tangent line to the curve $\gamma(t) = (t\cos t, t\sin t, t)$ at the point $(-\pi, 0, \pi)$. Find the speed of the curve when it is passing through that point.*

To answer both questions, we need to identify the velocity

$$\gamma'(t) = (\cos t - t\sin t, \sin t + t\cos t, 1)$$

We also need to identify when the curve passes through that point. By looking at the third coordinate, we see that it is at $t = \pi$. At that time, the velocity is

$$\gamma'(\pi) = (-1-0, 0-\pi, 1) = (-1, -\pi, 1)$$

Then the line tangent to the curve at the point $(-\pi, 0, \pi)$ is

$$\alpha(t) = (-\pi, 0, \pi) + t(-1, -\pi, 1)$$

The speed at time $t = \pi$ is given by $|\gamma'(\pi)| = \sqrt{\pi^2+2}$

Exercise 10 *Find the tangent line to the curve $\gamma(t) = (t^2+1, 3t+1, t-t^2)$ at the point $(5, 7, -2)$. Find the speed of the curve when it is passing through that point.*

To answer both questions, we need to identify the velocity

$$\gamma'(t) = (2t, 3, 1-2t)$$

We also need to identify when the curve passes through that point. By looking at the third coordinate, we see that it is at $t = 2$. At that time, the velocity is

$$\gamma'(2) = (4, 3, -3)$$

Then the line tangent to the curve at the point $(5, 7, -2)$ is

$$\alpha(t) = (5, 7, -2) + t(4, 3, -3)$$

The speed at time $t = 2$ is given by $|\gamma'(2)| = \sqrt{16+9+9} = \sqrt{34}$

Exercise 11 *Compute the length of the curve $\gamma(t) = (\cos t, \sin t, t)$ with $0 \leq t \leq 4\pi$*

The length is the integral of the speed. We compute

$$\gamma'(t) = (-\sin(t), \cos(t), 1)$$
$$|\gamma'(t)| = \sqrt{2}$$

Then

$$\text{length}(\gamma) = \int_0^{4\pi} |\gamma'(t)|\,dt = \int_0^{4\pi} \sqrt{2}\,dt = 4\pi\sqrt{2}$$

Exercise 12 *Compute the length of the curve* $\gamma(t) = (2+t-t^3/3, t^2+3)$ *with* $0 \le t \le 2$

The length is the integral of the speed. We compute

$$\gamma'(t) = (1-t^2, 2t)$$
$$|\gamma'(t)| = \sqrt{(1-t^2)^2+(2t)^2} = \sqrt{1-2t^2+t^4+4t^2} = \sqrt{(1+t^2)^2} = 1+t^2$$

Then

$$\text{length}(\gamma) = \int_0^{2} |\gamma'(t)|\,dt = \int_0^{2} (1+t^2)\,dt = 2+2^3/3 = 14/3$$

Exercise 13 *Assume a wire has the shape of a curve* $C \subset \mathbb{R}^3$ *with parametrization* $\gamma : [0,1] \to \mathbb{R}^3$ *given by*

$$\gamma(t) = (t^2, 2t, t^3)$$

Further assume it has density given by $\rho(x,y,z) = 16z + 10xy + 4y$. *Find its mass.*

The mass is the integral of the density. To find the Jacobian, we compute

$$\gamma'(t) = (2t, 2, 3t^2)$$
$$|\gamma'(t)| = \sqrt{4+4t^2+9t^4}$$

The integrand $\rho = 16z + 10xy + 4y$ under the substitution

$$x = t^2$$
$$y = 2t$$
$$z = t^3$$

becomes

$$\rho(t) = 16t^3 + 20t^3 + 8t = 36t^3 + 8t$$

The integral then becomes,

$$\begin{aligned}\int_C \rho\, ds &= \int_0^1 \rho(t)|\gamma'(t)|\, dt \\ &= \int_0^1 (8t+36t^3)\sqrt{4+4t^2+9t^4} \\ &= \int_4^{17} \sqrt{u}\, du \\ &= \frac{2}{3}\left[\sqrt{17}^3 - 2^3\right]\end{aligned}$$

where we used the substitution $u = 4 + 4t^2 + 9t^3$.

Exercise 14 *We are building a fence whose shape is the curve $C \subset \mathbb{R}^2$ with parametrization $\gamma(t) = (2\sin t, 3\cos t)$ with $\pi/4 \leq t \leq \pi/2$. Assume the cost of building at the point with coordinates (x, y) is given by $f(x, y) = 200xy$ dollars per meter. What is the cost of building the fence?*

The total cost is the accumulation of f along the trajectory γ. In other words, the integral

$$\int_C f\, ds$$

To compute it, we need the Jacobian, which we obtan by

$$\begin{aligned}\gamma'(t) &= (2\cos t, -3\sin t) \\ |\gamma'(t)| &= \sqrt{4\cos^2 t + 9\sin^2 t} = \sqrt{4 + 5\sin^2 t}\end{aligned}$$

The integrand $f(x, y) = 200xy$ in terms of t becomes

$$f(t) = 1200\cos t \sin t$$

Then the integral becomes

$$\int_C f\, ds = \int_{\pi/4}^{\pi/2} (1200\cos t\sin t)\sqrt{4 + 5\sin^2 t}\, dt$$

With the change of variables $u = 4 + 5\sin^2 t$, we get $u' = 10\sin t\cos t$, and

$$\begin{aligned}\int_C f\, ds &= \int_{4+\frac{5}{2}}^{4+5} 120\sqrt{u}\, du \\ &= 80\int_{13/2}^{9} \frac{3}{2}\sqrt{u} du \\ &= 80\left[27 - \sqrt{13/2}^3\right]\end{aligned}$$

9 Surfaces

A parametrized surface is a map $\varphi : U \to \mathbb{R}^3$ with $U \subset \mathbb{R}^2$ a region in the plane.

Sometimes, we call the image of φ the surface (the object in the space) and call φ the parametrization.

Example 1. The graph of a function $f(x, y)$ of two variables, is the set $\{(x, y, z) \in \mathbb{R}^3 | z = f(x, y)\}$. It admits the parametrization $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\varphi(u, v) = (u, v, f(u, v))$$

Example 2. The cylinder $x^2+y^2 = 1$ is a surface that admits the parametrization $\varphi : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}^3$ given by

$$\varphi(u, v) = (\cos u, \sin u, v)$$

Example 3. The sphere $x^2+y^2+z^2 = 1$ is a surface that admits the parametrization $\varphi : [0, 2\pi] \times [0, \pi] \to \mathbb{R}^3$ given by

$$\varphi(u, v) = (\cos u \sin v, \sin u \sin v, \cos v)$$

Example 4. Assume a plane curve $\gamma : [a, b] \to \mathbb{R}^2$ given by $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ satisfies $\gamma_1 > 0$. We can rotate it around the vertical axis to get a surface of revolution. This surface admits the parametrization $\varphi : [0, 2\pi] \times [a, b] \to \mathbb{R}^3$ given by

$$\varphi(u, v) = (\gamma_1(v) \cos u, \gamma_1(v) \sin u, \gamma_2(v))$$

The Jacobian of the parametrization of a surface $\varphi : U \to \mathbb{R}^3$ is given by

$$\text{Jacobian}(\varphi) = \Big|\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial u}\Big|$$

Using this, the area of the surface is given by

$$\text{area}(\varphi) = \iint_U \Big|\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial u}\Big| \, dudv$$

10 Surface integrals of scalar functions

Definition 2. Let $\Sigma \subset \mathbb{R}^3$ be a surface, $\varphi : U \to \mathbb{R}^3$ a parametrization of Σ, and $f : \mathbb{R}^3 \to \mathbb{R}$ a continuous function. The integral of f over Σ is defined as

$$\iint_\Sigma f \, dS := \iint_U f(\varphi(u, v))\Big|\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial u}\Big| \, dudv$$

If $f(\varphi(u,v)) > 0$, the integral above can be interpreted as the mass of a thin metal sheet with shape Σ and density f.

Note: the above integral is independent of the parametrization. Later we will consider surface integrals of vector fields. They will change sign if we change the orientation of the parametrization we use.

Exercise 15 *Let $\Sigma \subset \mathbb{R}^3$ be the portion of the paraboloid $z = 4 - x^2 - y^2$ above the xy-plane. Find the area of Σ*

Inspired by cylindrical coordinates, we can use the parametrization $\varphi : [0,2] \times [0, 2\pi] \to \mathbb{R}^3$ given by

$$\varphi(r,\theta) = (r\cos\theta, r\sin\theta, 4 - r^2)$$

Then

$$\frac{\partial \varphi}{\partial r} = (\cos\theta, \sin\theta, -2r)$$
$$\frac{\partial \varphi}{\partial \theta} = (-r\sin\theta, r\cos\theta, 0)$$

Taking cross product,

$$\frac{\partial \varphi}{\partial r} \times \frac{\partial \varphi}{\partial \theta} = (2r^2\cos\theta, 2r^2\sin\theta, r)$$

Then the Jacobian is

$$\text{Jacobian}(\varphi) = \left|\frac{\partial \varphi}{\partial r} \times \frac{\partial \varphi}{\partial \theta}\right| = r\sqrt{4r^2+1}$$

Then the area is

$$\begin{aligned}
\text{area}(\varphi) &= \int_0^2 \int_0^{2\pi} r\sqrt{4r^2+1}\, d\theta dr \\
&= 2\pi \int_1^{17} \frac{1}{8}\sqrt{u}\, du \\
&= \frac{\pi}{4}\left[\frac{2}{3}(\sqrt{17}^3 - 1)\right] \\
&= \frac{\pi}{6}(\sqrt{17}^3 - 1)
\end{aligned}$$

where we used the substitution $u = 4r^2 + 1$ with $u' = 8r$ and $1 \leq u \leq 17$

Exercise 16 *Let $\Sigma \subset \mathbb{R}^3$ be the portion of the sphere $x^2 + y^2 + z^2 = 1$ above the cone $z = \frac{1}{\sqrt{3}}\sqrt{x^2+y^2}$. Find the area of Σ and its average height.*

Inspired by spherical coordinates, we can use the parametrization $\varphi : [0, 2\pi] \times [0, \pi/3] \to \mathbb{R}^3$ given by

$$\varphi(u, v) = (\cos u \sin v, \sin u \sin v, \cos v)$$

Then

$$\frac{\partial \varphi}{\partial u} = (-\sin u \sin v, \cos u \sin v, 0)$$
$$\frac{\partial \varphi}{\partial v} = (\cos u \cos v, \sin u \cos v, -\sin v)$$

Taking cross product,

$$\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} = (-\cos u \sin^2 v, -\sin u \sin^2 v, -\sin v \cos v)$$

Then the Jacobian is

$$\text{Jacobian}(\varphi) = \left| \frac{\partial \varphi}{\partial r} \times \frac{\partial \varphi}{\partial \theta} \right| = \sin v$$

Then the area is

$$\begin{aligned} \text{area}(\varphi) &= \int_0^{2\pi} \int_0^{\pi/3} \sin v \, dv du \\ &= 2\pi(\cos(0) - \cos(\pi/3)) \\ &= \pi \end{aligned}$$

Note that the height of the point $\varphi(u, v)$ is given by the third coordinate, which is $\cos v$. To compute the average, we integrate this quantity and divide over the area:

$$\begin{aligned} \iint_\Sigma z \, dS &= \int_0^{2\pi} \int_0^{\pi/3} \cos v \sin v \, dv du \\ &= 2\pi \left[\frac{1}{2} (\sin^2(\pi/3) - \sin^2(0)) \right] \\ &= 3\pi/4. \end{aligned}$$

Then the average height is given by

$$\left[\iint_\Sigma z \, dS \right] / \text{area}(\varphi) = 3/4$$

Exercise 17 *Let $\Sigma \subset \mathbb{R}^3$ be the portion of the cone $z = \sqrt{x^2 + y^2}$ between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$. Find the area of Σ. Assume a metal sheet with this shape has density $\rho(x, y, z) = e^{z^2}$. Find its mass.*

Inspired by spherical coordinates, we can use the parametrization $\varphi : [1, 3] \times [0, 2\pi] \to \mathbb{R}^3$ given by

$$\varphi(u, v) = (u \cos v, u \sin v, u)$$

Then

$$\begin{aligned} \frac{\partial \varphi}{\partial u} &= (\cos v, \sin v, 1) \\ \frac{\partial \varphi}{\partial v} &= (-u \sin v, u \cos v, 0) \end{aligned}$$

Taking cross product,

$$\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} = (-u \cos v, -u \sin v, u)$$

Then the Jacobian is

$$\text{Jacobian}(\varphi) = \left| \frac{\partial \varphi}{\partial r} \times \frac{\partial \varphi}{\partial \theta} \right| = u\sqrt{2}$$

Then the area is

$$\begin{aligned} \text{area}(\varphi) &= \int_1^3 \int_0^{2\pi} u\sqrt{2}\, dv du \\ &= 2\pi(\frac{1}{\sqrt{2}}(3^2 - 1)) \\ &= 8\pi\sqrt{2} \end{aligned}$$

The mass is given by the integral

$$\begin{aligned} \iint_\Sigma \rho\, dS &= \int_1^3 \int_0^{2\pi} [e^{u^2}][u\sqrt{2}]\, dv du \\ &= 2\pi \frac{1}{\sqrt{2}}(e^9 - e) \\ &= \pi\sqrt{2}(e^9 - e) \end{aligned}$$

11 Surfaces with multiple pieces

Some surfaces cannot be nicely covered by a single parametrization. It is common to need more than one parametrization.

Exercise 18 *A metal sheet has shape Σ, where Σ consists of the portion of the cylinder $x^2 + y^2 = 4$ between the planes $x + y + z = 0$ and $z = 3$, and the portion of the sphere $x^2 + y^2 + (z - 3)^2 = 4$ above the plane $z = 3$. It has density $\rho(x, y, z) = 5 - z$. Find its mass.*

Inspired by cylindrical coordinates, for the cylinder part we can use the parametrization $\varphi : U \to \mathbb{R}^3$ given by

$$\varphi(u, v) = (2\cos u, 2\sin u, v)$$

with U given by the restrictions

$$\begin{gathered} 0 \le u \le 2\pi \\ -2\cos u - 2\sin u \le v \le 3 \end{gathered}$$

The derivatives of φ are given by

$$\begin{aligned} \frac{\partial \varphi}{\partial u} &= (-2\sin u, 2\cos u, 0) \\ \frac{\partial \varphi}{\partial v} &= (0, 0, 1) \end{aligned}$$

Then the Jacobian is given by

$$\left| \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right| = 2$$

Then the integral over this piece becomes

$$\begin{aligned} &\int_0^{2\pi} \int_{-2\cos u - 2\sin u}^{3} (5 - v)2 \; dv du \\ &= \int_0^{2\pi} [5(3 + 2\cos u + 2\sin u) - 9 + (2\cos u + 2\sin u)^2] \, du \\ &= 30\pi - 18\pi + 4 \int_0^{2\pi} (\cos^2 u + \sin^2 u + 2\cos u \sin u) \, du \\ &= 20\pi \end{aligned}$$

Inspired by spherical coordinates, for the sphere part we can use the parametrization $\psi : [0, 2\pi] \times [0, \pi/2] \to \mathbb{R}^3$ given by

$$\psi(u, v) = (2\cos u \sin v, 2\sin u \sin v, 3 + 2\cos v)$$

The derivatives of ψ are given by

$$\begin{aligned} \frac{\partial \psi}{\partial u} &= (-2\sin u \sin v, 2\cos u \sin v, 0) \\ \frac{\partial \psi}{\partial v} &= (2\cos u \cos v, 2\sin u \cos v, -2\sin v) \end{aligned}$$

Then the Jacobian is given by

$$\left| \frac{\partial \psi}{\partial u} \times \frac{\partial \psi}{\partial v} \right| = 4\sin v$$

Then the integral over this piece becomes

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{\pi/2}(5-3-2\cos v)(4\sin v)\ dvdu \\
&= 16\pi\int_0^{\pi/2}(1-\cos v)\sin v\,dv \\
&= 8\pi
\end{aligned}$$

Then the total mass is

$$20\pi + 8\pi = 28\pi$$

12 Vector fields

A planar vector field is a function $F : D \to \mathbb{R}^2$ with $D \subset \mathbb{R}^2$ a region we call the domain of F.

A space vector field is a function $F : D \to \mathbb{R}^3$ with $D \subset \mathbb{R}^3$ a region we call the domain of F.

A vector field in general corresponds to having a vector at each point. These are really good to model:

- Force fields.
- Electromagnetic fields.
- Motions of fluids and wind.
- Ordinary differential equations.

You can draw vector fields on Desmos:

https://www.desmos.com/calculator/eijhparfmd

13 Flows

Given a planar vector field $F : D \to \mathbb{R}^2$, a flow line is a differentiable curve $\gamma : [a, b] \to D$ with

$$\gamma'(t) = F(\gamma(t))$$

for all t. This means that γ models a particle that at each time, its velocity is the vector provided by F at its position.

You can create animations of flows of planar vector fields using the following:

- Clic on the following link:
  https://drive.google.com/file/d/1e6GniqFvsR_vx6HhRw0y3NhjV37WHVMF/view?usp=sharing

  Copy that code in your cliboard
- Go to the following website:

  https://animg.app/playground
- Paste the code in the text box and render (can take a couple of minutes).
- Edit the vector fields on lines 24-28 and render again.
- The free version of Animg only has three free renders per day, so do this with your friends.

Flows of space vector fields are defined analogously.

Theorem 1. Let $F : D \to \mathbb{R}^2$ be a smooth vector field. Then for any point $p \in D$ there is a unique flow line of F that starts at p. The same is true for space vector fields.

14 Curl

The curl of a vector field, measures how much rotation (swirl) is generated by its flow. This is measured in very distinct ways in the plane and in the space because angular momentum is encoded in the plane by a scalar and in the space by a vector.

14.1 2D curl

In the plane, rotation is simply either clockwise or counterclockwise.

Definition 3. The curl of a planar vector field

$$F(x, y) = \langle P(x, y), Q(x, y)\rangle$$

is given by

$$\mathrm{curl}(F) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

- The curl of a vector field F at a point p is positive if its flow generates counterclockwise swirl near p.
- The curl of a vector field F at a point p is negative if its flow generates clockwise swirl near p.

Example 5. The vector field $F(x, y) = \langle -y/2, x/2\rangle$ generates a strong counterclockwise swirl and its curl is given by

$$\mathrm{curl}(F) = 1$$

Example 6. The vector field $F(x, y) = \langle x/2, y/2\rangle$ generates a strong flow, but does not generate any rotation. Hence its curl is

$$\mathrm{curl}(F) = 0$$

14.2 3D curl

In the space, rotation (angular momentum) is encoded by a vector. The angular momentum of a rotation is a vector v such that:

- the direction of v is the axis of rotation.
- the length of v is the speed of rotation.
- looking back at the object from the tip of v, we see it spinning counter-clockwise.

Definition 4. The curl of a space vector field

$$F(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$$

is given by

$$\text{Curl}(F) = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

This formula is somewhat hard to remember, but if we consider the abstract vector

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle,$$

then

$$\text{Curl}(F) = \nabla \times F$$

becomes just a cross product.

Example 7. The vector field $F(x, y, z) = \langle -y/2, x/2, 0\rangle$ generates a strong swirl around the z-axis. Its curl is given by

$$\text{Curl}(F) = \langle 0, 0, 1\rangle$$

Example 8. The vector field $F(x, y, z) = \langle 0, -z/2, y/2\rangle$ generates a strong swirl around the x-axis. Its curl is given by

$$\text{Curl}(F) = \langle 1, 0, 0\rangle$$

Example 9. The vector field $F(x, y, z) = \langle x/2, y/2, z/2\rangle$ generates a strong flow, but does not generate any rotation. Hence its curl is

$$\text{Curl}(F) = \langle 0, 0, 0\rangle$$

15 Divergence

The divergence of a vector field at a point measures how much the vector field is pointing away from that point.

Definition 5. The divergence of a planar vector field

$$F(x, y) = \langle P(x, y), Q(x, y)\rangle$$

is given by

$$\text{div}(F) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

Definition 6. The divergence of a space vector field

$$F(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$$

is given by

$$\operatorname{div}(F) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Even though the divergence formula is quite simple, it can be simplified more as as dot product:

$$\operatorname{div}(F) = \nabla \cdot F$$

Example 10. The vector field $F(x, y) = \langle -y/2, x/2\rangle$ generates a strong counterclockwise swirl, but the flow lines are just rotating around. Its divergence is

$$\operatorname{div}(F) = 1$$

Example 11. The vector field $F(x, y) = \langle x/2, y/2\rangle$ generates a strong flow that is blowing-up away from each point. Its divergence is

$$\operatorname{div}(F) = 1$$

16 Conservative vector fields

Definition 7. A vector field $F : D \to \mathbb{R}^3$ is conservative if it is the gradient of a function $f : D \to \mathbb{R}$. In such a case, the function f is called a potential of F.

Theorem 2. A smooth conservative vector field F has no curl.

Proof In the plane, a conservative vector field is

$$F(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

Then its curl is

$$\begin{aligned} \operatorname{curl}(F) &= \frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial y}\frac{\partial f}{\partial x} \\ &= \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \\ &= 0 \end{aligned}$$

In the space, a conservative vector field is

$$F(x, y, z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

Then its curl is

$$\begin{aligned}\text{Curl}(F) &= \left\langle \frac{\partial}{\partial y}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial y}, \frac{\partial}{\partial z}\frac{\partial f}{\partial x} - \frac{\partial}{\partial x}\frac{\partial f}{\partial z}, \frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial y}\frac{\partial f}{\partial x} \right\rangle \\ &= \left\langle \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}, \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right\rangle \\ &= \langle 0, 0, 0 \rangle\end{aligned}$$

■

Remark 1. There are non-conservative vector fields with no curl. For example,

$$F(x, y) = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle$$

has zero curl, but it is not conservative. The main issue is that $(0, 0)$ is not in the domain of F.

Theorem 3. If $F : D \to \mathbb{R}^2$ is a planar smooth vector field with

$$\text{curl}(F) = 0$$

and D has no holes, then F is conservative.

Technically, "having no holes" can be defined in terms of curves: any closed curve in D can be continuously deformed within D to a single point within D. For example, the domain of

$$F(x, y) = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle$$

is $\mathbb{R}^2 \backslash \{(0, 0)\}$, which contains the unit circle, a curve that cannot be continuously deformed to a single point within D. It is like having a rubber band stuck around a pole.

17 Exercises on vector fields

Exercise 19 *Consider the planar vector field*

$$F(x, y) = \langle e^x \cos y, \sin y + x^2 \rangle$$

Find its curl, divergence, and determine whether it is conservative or not. If it is conservative, find a potential function.

The curl can be computed as

$$\begin{aligned}\text{curl}(F) &= \frac{\partial}{\partial x}(\sin y + x^2) - \frac{\partial}{\partial y}(e^x \cos y) \\ &= 2x + e^x \sin y\end{aligned}$$

For the divergence,

$$\begin{aligned}\operatorname{div}(F) &= \frac{\partial}{\partial x}(e^x \cos y) + \frac{\partial}{\partial y}(\sin y + x^2) \\ &= e^x \cos y + \cos y\end{aligned}$$

Since the curl is not zero, F is not conservative.

Exercise 20 *Consider the planar vector field*

$$F(x, y) = \langle y^2 - x^2, 2xy \rangle$$

Find its curl, divergence, and determine whether it is conservative or not. If it is conservative, find a potential function.

The curl can be computed as

$$\begin{aligned}\operatorname{curl}(F) &= \frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(y^2 - x^2) \\ &= 2y - 2y \\ &= 0\end{aligned}$$

For the divergence,

$$\begin{aligned}\operatorname{div}(F) &= \frac{\partial}{\partial x}(y^2 - x^2) + \frac{\partial}{\partial y}(2xy) \\ &= -2x + 2x \\ &= 0\end{aligned}$$

Since the curl is zero, and F is defined everywhere, it is conservative. To find the potential function, we perform some "partial integration":

$$\begin{aligned}\frac{\partial f}{\partial x} &= y^2 - x^2 \\ \frac{\partial f}{\partial x} &= 2xy\end{aligned}$$

We get

$$\begin{aligned}f(x, y) &= xy^2 - x^3/3 + a(y) \\ f(x, y) &= xy^2 + b(x)\end{aligned}$$

Matching terms, we get

$$f(x, y) = xy^2 - x^3/3 + C$$

Exercise 21 *Consider the space vector field*

$$F(x,y,z) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, z \right\rangle$$

Find its curl, divergence, and determine whether it is conservative or not. If it is conservative, find a potential function.

Before we compute the curl and divergence, we find the partial derivatives

$$\begin{aligned}
\frac{\partial P}{\partial x} &= \frac{2xy}{(x^2+y^2)^2} \\
\frac{\partial P}{\partial y} &= \frac{y^2-x^2}{x^2+y^2} \\
\frac{\partial P}{\partial z} &= 0 \\
\frac{\partial Q}{\partial x} &= \frac{y^2-x^2}{x^2+y^2} \\
\frac{\partial Q}{\partial y} &= \frac{-2xy}{(x^2+y^2)^2} \\
\frac{\partial Q}{\partial z} &= 0 \\
\frac{\partial R}{\partial x} &= 0 \\
\frac{\partial R}{\partial y} &= 0 \\
\frac{\partial R}{\partial z} &= 1
\end{aligned}$$

The curl is

$$\begin{aligned}
\mathrm{Curl}(F) &= \left\langle 0-0, 0-0, \frac{y^2-x^2}{x^2+y^2} - \frac{y^2-x^2}{x^2+y^2} \right\rangle \\
&= \langle 0,0,0 \rangle
\end{aligned}$$

The divergence is

$$\begin{aligned}
\mathrm{div}(F) &= \frac{2xy}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2} + 1 \\
&= 1
\end{aligned}$$

The curl is zero, but F is not defined along the z-axis, where $x^2+y^2=0$. We will see later that F is not conservative.

18 Oriented curves

An oriented curve is a curve with a choice of direction. Each curve has two orientations: forward and backward. A parametrization of an oriented curve is a parametrization that travels the curve in the correct direction.

Note: the words "forward" and "backward" are subjective.

When a plane curve $\gamma : [a, b] \to \mathbb{R}^2$ is closed and simple ($\gamma(a) = \gamma(b)$, and the curve doesn't self intersect), then we say it is positively oriented if it is oriented with counterclockwise direction and negatively oriented if it is oriented with clockwise direction.

For example, the curve

$$\gamma(t) = (\cos t, \sin t)$$

with $0 \leq t \leq 2\pi$ is positively oriented, while the curve

$$\sigma(t) = (\sin t, \cos t)$$

with $0 \leq t \leq 2\pi$ is negatively oriented.

19 Line integrals of vector fields

Definition 8. Let C be an oriented curve, $\gamma : [a, b] \to \mathbb{R}^3$ a parametrization, and

$$F(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$$

a vector field. The integral of F along C is given by

$$\int_C F \cdot ds := \int_C (P\,dx + Q\,dy + R\,dz) := \int_a^b F(\gamma(t)) \cdot \gamma'(t)\,dt.$$

Note: if you use a parametrization in the opposite direction, the integral changes sign.

If we let

$$T(t) := \frac{\gamma'(t)}{|\gamma'(t)|}$$

be the unit tangent vector, then the line integral above becomes

$$\begin{aligned}\int_C F \cdot ds &= \int_a^b F(\gamma(t)) \cdot \gamma'(t)\,dt \\ &= \int_a^b F(\gamma(t)) \cdot \frac{\gamma'(t)}{|\gamma'(t)|}\,|\gamma'(t)|\,dt \\ &= \int_C [F \cdot T]\,ds\end{aligned}$$

The dot product $F \cdot T$ represents "how much is F pointing in the direction in which γ is moving". Therefore the integral can be interpreted as "how much did the force field F help γ perform its trajectory".

Example 12. Let $F = \langle 1, 0 \rangle$ and $\alpha : [0,1] \to \mathbb{R}^2$, $\beta : [0,1] \to \mathbb{R}^2$, $\gamma : [0,1] \to \mathbb{R}^2$ be given by

$$\alpha(t) = (t, 0)$$
$$\beta(t) = (0, t)$$
$$\gamma(t) = (-t, 0)$$

This means:

α is moving right

β is moving up

γ is moving left

Since F is pointing right, it is helping α perform its trajectory, it is not helping nor preventing β from performing its trajectory, and is pushing γ against its trajectory. From here we intuitively deduce that

$$\int_\alpha F\, ds \text{ is positive}$$

$$\int_\beta F\, ds \text{ is zero}$$

$$\int_\gamma F\, ds \text{ is negative}$$

This can be easily computed from the dot products:

$$F(\alpha(t)) \cdot \alpha'(t) = \langle 1, 0 \rangle \cdot \langle 1, 0 \rangle = 1$$
$$F(\beta(t)) \cdot \beta'(t) = \langle 1, 0 \rangle \cdot \langle 0, 1 \rangle = 0$$
$$F(\gamma(t)) \cdot \gamma'(t) = \langle 1, 0 \rangle \cdot \langle -1, 0 \rangle = -1$$

Basically:

- if α goes with the flow of F, the integral $\displaystyle\int_\alpha F \cdot ds$ is positive.
- if γ is swimming against the current, the integral $\displaystyle\int_\gamma F \cdot ds$ is negative.

The integral $\displaystyle\int_C F \cdot ds$ is also called the work of F along the trajectory.

20 Exercises on line integrals

Exercise 22 *Let* $F(x, y, z) = \langle -2y, z^2 + 3x, x - 1\rangle$ *and* C *the curve with parametrization* $\gamma : [0, 2] \to \mathbb{R}^3$ *given by*

$$\gamma(t) = (t^2 - 3, 2t, t^3).$$

Find

$$\int_C F \cdot ds$$

Using the change of variables

$$\begin{aligned} x(t) &= t^2 - 3 \\ y(t) &= 2t \\ z(t) &= t^3 \end{aligned}$$

we get

$$F(\gamma(t)) = \langle -4t, t^6 + 3t^2 - 9, t^2 - 4\rangle$$

We also need

$$\gamma'(t) = \langle 2t, 2, 3t^2\rangle$$

Then

$$\begin{aligned} \int_C F \cdot ds &= \int_0^2 \langle -4t, t^6 + 3t^2 - 9, t^2 - 4\rangle \cdot \langle 2t, 2, 3t^2\rangle \, dt \\ &= \int_0^2 \left[-8t^2 + 2t^6 + 6t^2 - 18 + 3t^4 - 12t^2\right] \, dt \\ &= -\frac{64}{3} + \frac{256}{7} + \frac{48}{3} - 36 + \frac{96}{5} - 32 \end{aligned}$$

Exercise 23 *Let* $F(x, y, z) = \langle z + 1, x, y\rangle$ *and* C *the curve with parametrization* $\gamma : [0, 3] \to \mathbb{R}^3$ *given by*

$$\gamma(t) = (e^t, -t^2, t).$$

Find

$$\int_C F \cdot ds$$

Using the change of variables

$$\begin{aligned} x(t) &= e^t \\ y(t) &= -t^2 \\ z(t) &= t \end{aligned}$$

we get

$$F(\gamma(t)) = \langle t+1, e^t, -t^2 \rangle$$

We also need

$$\gamma'(t) = \langle e^t, -2t, 1 \rangle$$

Then

$$\begin{aligned}
\int_C F \cdot ds &= \int_0^3 \langle t+1, e^t, -t^2 \rangle \cdot \langle e^t, -2t, 1 \rangle \, dt \\
&= \int_0^3 \left[te^t + e^t - 2te^t - t^2 \right] \, dt \\
&= \left[-te^t + 2e^t - t^3/3 \right] |_{t=0}^3 \\
&= -3e^3 + 2e^3 - 9 - 2 \\
&= -e^3 - 11
\end{aligned}$$

21 Fundamental Theorem of Calculus II

The classic Fundamental Theorem of Calculus says that the integral of the derivative of a function $F(x)$ is the function F itself:

$$\int_a^b F'(x) \, dx = F(b) - F(a)$$

Something similar happens when we take the line integral of a gradient. Consider a differentiable function $f : D \to \mathbb{R}$ with $D \subset \mathbb{R}^3$ and its gradient vector field ∇f. Also take an oriented curve $C \subset \mathbb{R}^3$ and a parametrization $\gamma : [a, b] \to \mathbb{R}^3$. Recall that by the chain rule, we have

$$\frac{d}{dt}(f(\gamma(t))) = \nabla f(\gamma(t)) \cdot \gamma'(t).$$

Therefore

$$\begin{aligned}
\int_C \nabla f \cdot ds &= \int_a^b \nabla f(\gamma(t)) \cdot \gamma'(t) \, dt \\
&= \int_a^b \frac{d}{dt}(f(\gamma(t))) \, dt \\
&= f(\gamma(b)) - f(\gamma(a))
\end{aligned}$$

Theorem 4. Let $D \subset \mathbb{R}^3$ be a region, $f : D \to \mathbb{R}^3$ a differentiable function, $C \subset \mathbb{R}^3$ an oriented curve with parametrization $\gamma : [a, b] \to \mathbb{R}^3$. Then

$$\int_C \nabla f \cdot ds = f(\gamma(b)) - f(\gamma(a))$$

In particular, the integral $\int_C \nabla f \cdot ds$ does only depend on the endpoints of C and not on the trajectory.

Exercise 24 *Let $F(x, y, z) = \langle x, \cos y, e^z\rangle$ and C the curve with parametrization $\gamma : [0, \pi] \to \mathbb{R}^3$ given by*

$$\gamma(t) = (t^2\sqrt{t^2+1}, e^{t^2}, t^2 \cos t).$$

Find

$$\int_C F \cdot ds$$

Note that

$$\operatorname{curl}(F) = \langle 0, 0, 0\rangle,$$

and the domain of F is $\mathbb{R}^3$, so F is conservative. To find the potential function, we do partial integration,

$$\begin{aligned} f(x, y, z) &= x^2/2 + g_1(y, z) \\ f(x, y, z) &= \sin y + g_2(x, z) \\ f(x, y, z) &= e^z + g_3(x, y) \end{aligned}$$

Matching terms, we get

$$f(x, y, z) = x^2/2 + \sin y + e^z + C$$

On the other hand, the endpoints of C are

$$\begin{aligned} \gamma(0) &= (0, 1, 0) \\ \gamma(\pi) &= (\pi^2\sqrt{\pi^2+1}, e^{\pi^2}, -\pi^2) \end{aligned}$$

Therefore,

$$\begin{aligned} \int_C F \cdot ds &= f(\pi^2\sqrt{\pi^2+1}, e^{\pi^2}, -\pi^2) - f(0, 1, 0) \\ &= \pi^4(\pi^2+1)/2 + \sin(e^{\pi^2}) + e^{-\pi^2} - \sin(1) - 1 \end{aligned}$$

Exercise 25 *Show that the vector field*

$$F(x, y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

is not conservative, even though it has zero curl.

Let $C \subset \mathbb{R}^2$ be the unit circle oriented counterclockwise. Take the parametrization $\gamma(t) = (\cos t, \sin t)$ with $0 \le t \le 2\pi$. Then

$$\begin{aligned} F(\gamma(t)) &= \left\langle \frac{-\sin t}{\cos^2 t + \sin^2 t}, \frac{\cos t}{\cos^2 t + \sin^2 t} \right\rangle \\ &= \langle -\sin t, \cos t\rangle \end{aligned}$$

Also,

$$\gamma'(t) = \langle -\sin t, \cos t \rangle$$

Therefore,

$$\begin{aligned}
\int_C F \cdot ds &= \int_0^{2\pi} F(\gamma(t)) \cdot \gamma'(t)\, dt \\
&= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt \\
&= \int_0^{2\pi} 1\, dt \\
&= 2\pi \neq 0
\end{aligned}$$

If F was conservative, we would have $F = \nabla f$ for some scalar function $f(x, y)$. By the Fundamental Theorem of Calculus, we would have

$$\begin{aligned}
\int_C F \cdot ds &= f(\gamma(2\pi)) - f(\gamma(0)) \\
&= f(1, 0) - f(1, 0) \\
&= 0
\end{aligned}$$

Part II

Midterm Guide

22 Polar, cylindrical, and spherical coordinates

Exercise 26 *Using polar coordinates, compute*

$$\int_{-4}^{0}\int_{-\sqrt{16-x^2}}^{0} y\, dydx$$

Hint: *This is a quarter of the disk of radius 4*

Exercise 27 *Using polar coordinates, compute*

$$\int_{0}^{3/\sqrt{2}}\int_{y}^{\sqrt{9-y^2}} x^2 y\, dxdy$$

Hint: *This is a sector with angle $\pi/4$ of the circle of radius 3*

Exercise 28 *Using polar coordinates, compute*

$$\iint_B x\, dxdy$$

where $B \subset \mathbb{R}^2$ is the region in the first quadrant inside the circle $x^2+(y-3)^2 = 9$

Hint: *This is a quarter of the unit disk*

Exercise 29 *Using polar coordinates, show that*

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx = \sqrt{2\pi}$$

Hint: *The square of the integral on the left is $\iint_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2}}\, dA$.*

Exercise 30 *Let $D \subset \mathbb{R}^2$ be the interior of the circle $(x-1)^2 + (y-1)^2 = 2$. Using polar coordinates, find*

$$\iint_D \frac{x}{x^2+y^2} dA$$

Hint: *The notes have a section on how to describe a region like this in polar coordinates.*

Exercise 31 *Using cylindrical coordinates, compute*

$$\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_y^4 x^2 \, dzdxdy$$

Hint: *Deal separately with the x, y variables and the z variable. For cylindrical coordinates, most of the time you don't need to do anything to the z variable*

Exercise 32 *Using cylindrical coordinates, compute*

$$\int_0^1 \int_{-1}^1 \int_z^{\sqrt{1-y^2}} y^2 \, dxdydz$$

Hint: *This is the region inside the cylinder $x^2 + y^2 = 1$, above the xy-plane, and below the plane $z = x$. Verify this!!!*

Exercise 33 *Using spherical coordinates, compute*

$$\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} y^2 \, dzdydx$$

Hint: *This is a portion of the ball of radius 2*

Exercise 34 *Using spherical coordinates, compute*

$$\iiint_E zy^2 \, dV$$

where E is the region above the cone $z = -\sqrt{x^2 + y^2}$ and inside the sphere $x^2 + y^2 + z^2 = 25$

Hint: *The equation of the cone is $\phi = 3\pi/4$ and the equation of the sphere is $\rho = 5$*

Exercise 35 *Using spherical coordinates, compute*

$$\iiint_E \frac{zx^2}{x^2 + y^2} \, dV$$

where E is the region below the plane $z = 3$ and above the cone $z = \sqrt{x^2 + y^2}/\sqrt{3}$

Hint: *The equation of the plane is $\rho = 3/\cos\phi$*

Exercise 36 *Using spherical coordinates, compute*

$$\iiint_E z \, dV$$

where E is the region inside the sphere $x^2 + y^2 + (z-3)^2 = 9$

Hint: *The notes have a section on how to describe a region like this in polar coordinates.*

23 General changes of coordinates

Exercise 37 *Using an appropriate change of variables, compute*

$$\iint_D x^2 y \, dA$$

where $D \subset \mathbb{R}^2$ is the region between the hyperbolas $y = 2/x$ and $y = 4/x$ and the lines $y = 1$ and $y = 5$

Hint: *The region can be described as*

$$2 \leq xy \leq 4$$
$$1 \leq y \leq 5$$

Exercise 38 *Using an appropriate change of variables, compute*

$$\iint_D x^2 \, dA$$

where $D \subset \mathbb{R}^2$ is the region between the hyperbolas $y = 2/x$ and $y = 3/x$ and the lines $y = x$ and $y = 4x$

Hint: *The region can be described as*

$$2 \leq xy \leq 3$$
$$1 \leq y/x \leq 4$$

Exercise 39 *Using an appropriate change of variables, compute*

$$\iint_D 3x \, dA$$

where $D \subset \mathbb{R}^2$ is the parallelogram with vertices $(2,0)$, $(4,3)$, $(3,4)$, $(1,1)$.

Hint: *The sides of the parallogram have slopes $3/2$ and -1*

Exercise 40 *Using an appropriate change of variables, compute*

$$\iint_D \frac{y}{3y + x} \, dA$$

where $D \subset \mathbb{R}^2$ is the parallelogram with vertices $(-1,2)$, $(2,1)$, $(3,3)$, $(0,4)$.

Hint: *The sides of the parallogram have slopes 2 and $-1/3$*

Exercise 41 *Using an appropriate change of variables, compute*

$$\iint_D (x+y)\, dA$$

where $D \subset \mathbb{R}^2$ *is the region bounded by*

- *the lines* $y = -x$ *and* $y = -x + 2$
- *the portion of the parabola* $y = x^2$ *with* $x \geq 0$
- *the portion of the parabola* $y = x(x-2)$ *with* $x \geq 1$

Hint: *If*

$$\begin{aligned} x &= u + v \\ y &= u^2 - v \end{aligned}$$

then the region can be described as

$$\begin{aligned} 0 \leq u \leq 1 \\ 0 \leq v \leq 2 \end{aligned}$$

24 Curves and line integrals of scalar functions

Exercise 42 *Find the curve of intersection of the cylinder* $(x-3)^2+(z+1)^2 = 9$ *and the plane* $2x + 3y - 2z = 5$

Hint: *First find* $x(t)$ *and* $z(t)$ *so that no matter how you define* $y(t)$*, the curve you get lives in the cylinder and wraps around once.*

Exercise 43 *Find parametrizations of the curves that form the sides of the triangle* Σ*, where* Σ *is the intersection of the plane* $-x - 2y + z = 7$ *with the first octant.*

Hint: *For points* p *and* q*, the line segment from* p *to* q *can be parametrized as* $\gamma(t) = p + t(q - p)$ *with* $0 \leq t \leq 1$*.*

Exercise 44 *Find parametrizations of the curves that form the edges of the surface* Σ*, where* Σ *is the portion of the sphere* $(x-3)^2+(y-2)^2+(z-5)^2 = 36$ *with* $x \geq 3$ *and* $z \geq 5$*.*

Hint: Σ *is like the peel of an orange slice. The edges are two half-circles going from* $(3, -4, 5)$ *to* $(3, 8, 5)$

Exercise 45 *Determine whether the curve*

$$\gamma(t) = (t \sin t, t^2 - 1, 3t + \cos t)$$

passes through the point $(\pi/2, \pi^2/4 - 1, 3\pi/2)$*. If it does, determine*

- *the times at which it does*
- *the lines tangent to the curve at those times*
- *the speed of the curve at those times*

Exercise 46 *Determine whether the curve*

$$\gamma(t) = (t^2 - 2, t^3 - 2, 5t + t^2)$$

passes through the point $(7, 25, 24)$*. If it does, determine*

- *the times at which it does*
- *the lines tangent to the curve at those times*
- *the speed of the curve at those times*

Exercise 47 *Determine whether the curve*

$$\gamma(t) = (5 + t, t^2 - 1, 3t^2 - 2t)$$

passes through the point $(7, 2, 8)$. *If it does, determine*

- *the times at which it does*
- *the lines tangent to the curve at those times*
- *the speed of the curve at those times*

Exercise 48 *Determine whether the curve*

$$\gamma(t) = (t^2 + 2t + 1, t^3 + 3t^2 - t + 2, t^3 - 7t + 3)$$

passes through the point $(4, 5, -3)$. *If it does, determine*

- *the times at which it does*
- *the lines tangent to the curve at those times*
- *the speed of the curve at those times*

Exercise 49 *Find the length of the curve*

$$\gamma(t) = (e^t \cos t, e^t \sin t)$$

with $0 \leq t \leq 8\pi$. *Draw a sketch of the curve.*

Exercise 50 *Find the length of the curve*

$$\gamma(t) = (t - \sin t, 1 - \cos t)$$

with $0 \leq t \leq 4\pi$. *Draw a sketch of the curve.*

Exercise 51 *Find the length of the curve*

$$\gamma(t) = (2t, e^t + e^{-t})$$

with $-\log 2 \leq t \leq \log 2$. *Draw a sketch of the curve.*

Exercise 52 *Compute the line integral*

$$\int_C (x + 3y + z^2)\, ds$$

where C *is the curve with parametrization* $\gamma : [0, 2] \to \mathbb{R}^3$ *given by*

$$\gamma(t) = (t - 2, 2t + 3, 5 - t)$$

Exercise 53 *Compute the line integral*

$$\int_C (-8x + 3y - 5z)\, ds$$

where C *is the curve with parametrization* $\gamma : [-1, 3] \to \mathbb{R}^3$ *given by*

$$\gamma(t) = (t^2 - 4, t^2 - 2t - 5, 3 - t^2)$$

Exercise 54 *Compute the line integral*

$$\int_C (x + y + z)\, ds$$

where C *is the curve with parametrization* $\gamma : [0, 3\pi/2] \to \mathbb{R}^3$ *given by*

$$\gamma(t) = (\cos t, \sin t, t)$$

Exercise 55 *Compute the line integral*

$$\int_C x\, ds$$

where C *is the portion of the parabola* $y = x^2 + 1$ *between* $(0, 1)$ *and* $(2, 5)$

Exercise 56 *Compute the line integral*

$$\int_C xe^y\, ds$$

where C *is the line segment from* $(0, 2)$ *to* $(5, 0)$

Exercise 57 *Compute the line integral*

$$\int_C \sqrt{2 + 4x + 8y}\, ds$$

where C *is the portion of the curve of intersection of the cylinder* $y = x^2$ *and the plane* $x + y + z = 0$ *from* $(0, 0, 0)$ *to* $(2, 4, -6)$

25 Surfaces and surface integrals of scalar functions

Exercise 58 *Find a parametrization of the portion of the sphere* $x^2+y^2+z^2 = 9$ *above the plane* $z = 3/2$*. Specify the domain and the expression of the parametrization.*

Exercise 59 *Find a parametrization of the portion of the cylinder* $x^2+y^2 = 16$ *between the planes* $x + y + z = 0$ *and* $2x + y - z = -10$*. Specify the domain and the expression of the parametrization.*

Exercise 60 *Find a parametrization of the portion of the cone* $z = \sqrt{3x^2 + 3y^2}$ *outside the sphere* $x^2+y^2+(z-2)^2 = 4$ *and inside the sphere* $x^2+y^2+(z-5)^2 = 25$*. Specify the domain and the expression of the parametrization.*

Exercise 61 *Find a parametrization of the portion of the graph* $z = \sin x + \sin y$ *between the lines* $x = 0$*,* $x = \pi$*,* $y = x$*, and* $y = \pi$*. Specify the domain and the expression of the parametrization.*

Exercise 62 *For each of the previous problems, take your parametrization*

$$\varphi(u, v) = (x(u, v), y(u, v), z(u, v))$$

and consider the vector $\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}$*. Identify which side of the surface it comes out of. Construct a second parametrization of the same surface where the corresponding vector points in the opposite direction.*

Exercise 63 *Using a parametrization and its Jacobian, find the area of the portion of the plane* $5x+3y+4z = 60$ *in the first octant* $(x, y, z \geq 0)$*. Assumme a metal lamina has this shape and its density is given by* $\rho(x, y, z) = \sin x + 3$*. Find its mass.*

Exercise 64 *Find the area of the portion of the paraboloid* $x = 9 - y^2 - z^2$ *with* $x \geq -16$*. Assume a metal lamina has this shape and its density is given by* $\rho(x, y, z) = x - y + 21$*. Find its mass.*

Exercise 65 *Find*

$$\iint_\Sigma (z^2 + x + 3)\, dS$$

where Σ *is the portion of the sphere* $x^2 + y^2 + z^2 = 16$ *above the plane* $z = -2$*.*

Exercise 66 *Find*

$$\iint_\Sigma (x^2 + 3x + 2y^2 - 2y + z^2 + 3)\, dS$$

where Σ *is the portion of the cone* $z = \sqrt{3x^2 + 3y^2}$ *outside the sphere* $x^2 + y^2 + (z-2)^2 = 4$ *and inside the sphere* $x^2 + y^2 + (z-5)^2 = 25$*.*

Exercise 67 *Consider the surface* $\Sigma \subset \mathbb{R}^3$ *with parametrization*

$$\varphi : [0, 2\pi] \times [0, 2\pi] \to \mathbb{R}^3$$

given by

$$\varphi(u, v) = ((3 + \cos v) \cos u, (3 + \cos v) \sin u, \sin v)$$

Describe the surface Σ *and find its area. Compute*

$$\iint_{\Sigma} (x + 2y + z^2 + 3)\, dS$$

Exercise 68 *Compute*

$$\iint_{\Sigma} (xz + y)\, dS$$

where $\Sigma \subset \mathbb{R}^3$ *is the helicoid with parametrization*

$$\varphi(u, v) = (u \cos v, u \sin v, v)$$

with $1 \leq u \leq 3$, $0 \leq v \leq 4\pi$*.*

26 Vector fields, curl, and divergence

Exercise 69 *Find the curl and divergence of the vector field*

$$F(x, y) = \langle x, y \rangle.$$

Determine if the vector field is conservative or not. If it is, find a potential function.

Exercise 70 *Find the curl and divergence of the vector field*

$$F(x, y) = \langle y, x \rangle.$$

Determine if the vector field is conservative or not. If it is, find a potential function.

Exercise 71 *Find the curl and divergence of the vector field*

$$F(x, y) = \langle -y, x \rangle.$$

Determine if the vector field is conservative or not. If it is, find a potential function.

Exercise 72 *Find the curl and divergence of the vector field*

$$F(x, y) = \langle ye^{xy}, xe^{xy} \rangle.$$

Determine if the vector field is conservative or not. If it is, find a potential function.

Exercise 73 *Find the curl and divergence of the vector field*

$$F(x, y, z) = \langle 2xyz, x^2z, x^2yz \rangle.$$

Determine if the vector field is conservative or not. If it is, find a potential function.

Exercise 74 *Find the curl and divergence of the vector field*

$$F(x, y, z) = \langle yz^2, xz^2, 2xyz \rangle.$$

Determine if the vector field is conservative or not. If it is, find a potential function.

Exercise 75 *Find the curl and divergence of the vector field*

$$F(x, y, z) = \langle e^x + z, \sin z - y, x + y\cos z \rangle.$$

Determine if the vector field is conservative or not. If it is, find a potential function.

Exercise 76 *Find a vector field $F(x, y)$ that has positive curl at the point $(0, 0)$ and negative curl at the point $(10, 0)$.*

Exercise 77 *Find a vector field $F(x, y)$ that has positive divergence at the point $(0, 0)$ and negative divergence at the point $(0, 10)$.*

Exercise 78 *Find a vector field $F(x, y)$ that has positive curl at the point $(1, 1)$ and negative divergence at the point $(-1, -1)$.*

Exercise 79 *Find a vector field $F(x, y)$ with the property that:*

- *The flow-line that starts at the point $(1, 0)$ passes through the point $(2, 0)$ at a later time.*
- *The flow-line that starts at the point $(0, 1)$ passes through the point $(0, 2)$ at a later time.*

Exercise 80 *Find a vector field $F(x, y)$ with the property that:*

- *The flow-line that starts at the point $(1, 0)$ passes through the point $(2, 0)$ at a later time.*
- *The flow-line that starts at the point $(0, 2)$ passes through the point $(0, 1)$ at a later time.*

Exercise 81 *Find a vector field $F(x, y)$ with the property that:*

- *The flow-line that starts at the point $(1, 0)$ passes through the point $(0, 1)$ at a later time.*
- *The flow-line that starts at the point $(2, 0)$ passes through the point $(0, 2)$ at a later time.*

Exercise 82 *Sketch a vector field $F(x, y)$ with the property that:*

- *The flow-line that starts at the point $(2, 0)$ is at the point $(0, 1)$ after one unit of time.*
- *The flow-line that starts at the point $(0, 2)$ is at the point $(1, 0)$ after one unit of time.*

27 Line integrals of vector fields

Exercise 83 *Let $C \subset \mathbb{R}^3$ be the curve with parametrization $\gamma : [0,1] \to \mathbb{R}^3$ given by*

$$\gamma(t) = (t + t^2, t^4, 2t + t^2 - 3).$$

Compute

$$\int_C \langle -y, x+2, z+1 \rangle \cdot ds$$

Exercise 84 *Let $C \subset \mathbb{R}^3$ be the curve with parametrization $\gamma : [0,\pi] \to \mathbb{R}^3$ given by*

$$\gamma(t) = (\sin t, e^t, \cos t).$$

Compute

$$\int_C \langle x+y, y+z, z+x \rangle \cdot ds$$

Exercise 85 *Let $C \subset \mathbb{R}^3$ be the curve with parametrization $\gamma : [0,1] \to \mathbb{R}^3$ given by*

$$\gamma(t) = (t^2 \cos(4\pi t), t^2 + t^3, t^3 + t^4 + t^5).$$

Compute

$$\int_C \left[(yz + 2x)\,dx + (xz + z)\,dy + (xy + y + 3z^2)\,dz\right]$$

Exercise 86 *Let $C \subset \mathbb{R}^3$ be the piece of the parabola $x = y^2$ from $(0,0)$ to $(4,2)$. Compute*

$$\int_C \left[e^{y^2}\,dx + x\,dy\right]$$

Exercise 87 *Let $C \subset \mathbb{R}^3$ be the unit circle in the xy-plane travelled counter-clockwise when viewed from above. Compute*

$$\int_C \langle e^{x^2}, \sin(y^2), \cos(z^3) \rangle \cdot ds$$

Exercise 88 *Let $C \subset \mathbb{R}^3$ be portion from $(-2,0,2)$ to $(2,0,-2)$ of the curve of intersection of the plane $x+y+z=0$ and the cylinder $x^2+z^2=8$. Compute*

$$\int_C \langle x+z, \cos y, e^z + x \rangle \cdot ds$$

Exercise 89 *Let $C \subset \mathbb{R}^3$ be the curve with parametrization $\gamma : [0,1] \to \mathbb{R}^3$ given by*

$$\gamma(t) = (t^2 + 2t - 1, (3+t)\sqrt{t^2+1}, e^t + 2t^2).$$

Compute

$$\int_C \langle e^x(z-1), z\cos y, e^x + \sin y\rangle \cdot ds$$

Exercise 90 *Let $C \subset \mathbb{R}^3$ be the curve with parametrization $\gamma : [0, \pi/2] \to \mathbb{R}^3$ given by*

$$\gamma(t) = (2\sin t, \sin t, 3\cos t).$$

Compute

$$\int_C [yz\,dx + xy\,dy + xz\,dz]$$

Part III

Final Guide

28 Stokes Theorem

Exercise 91 *Let $C \subset \mathbb{R}^3$ be the curve of intersection...*