1 Oriented curves

An oriented curve is a curve with a choice of direction. Each curve has two orientations: forward and backward. A parametrization of an oriented curve is a parametrization that travels the curve in the correct direction.

Note: the words "forward" and "backward" are subjective.

When a plane curve $\gamma : [a, b] \to \mathbb{R}^2$ is closed and simple ($\gamma(a) = \gamma(b)$, and the curve doesn't self intersect), then we say it is positively oriented if it is oriented with counterclockwise direction and negatively oriented if it is oriented with clockwise direction.

For example, the curve

$$\gamma(t) = (\cos t, \sin t)$$

with $0 \le t \le 2\pi$ is positively oriented, while the curve

$$\sigma(t) = (\sin t, \cos t)$$

with $0 \le t \le 2\pi$ is negatively oriented.

2 Line integrals of vector fields

Definition 1. Let C be an oriented curve, $\gamma : [a, b] \to \mathbb{R}^3$ a parametrization, and

$$F(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$$

a vector field. The integral of F along C is given by

$$\int_C F \cdot d\gamma := \int_C (P\,dx + Q\,dy + R\,dz) := \int_a^b F(\gamma(t)) \cdot \gamma'(t)\,dt.$$

Note: if you use a parametrization in the opposite direction, the integral changes sign.

If we let

$$T(t) := \frac{\gamma'(t)}{|\gamma'(t)|}$$

be the unit tangent vector, then the line integral above becomes

$$\begin{aligned}\int_C F \cdot d\gamma &= \int_a^b F(\gamma(t)) \cdot \gamma'(t)\,dt \\ &= \int_a^b F(\gamma(t)) \cdot \frac{\gamma'(t)}{|\gamma'(t)|}\,|\gamma'(t)|\,dt \\ &= \int_C [F \cdot T]\;ds\end{aligned}$$

The dot product $F \cdot T$ represents "how much is F pointing in the direction in which γ is moving". Therefore the integral can be interpreted as "how much did the force field F help γ perform its trajectory".

Example 1. Let $F = \langle 1, 0 \rangle$ and $\alpha : [0,1] \to \mathbb{R}^2$, $\beta : [0,1] \to \mathbb{R}^2$, $\gamma : [0,1] \to \mathbb{R}^2$ be given by

$$\alpha(t) = (t, 0)$$
$$\beta(t) = (0, t)$$
$$\gamma(t) = (-t, 0)$$

This means:

$$\alpha \text{ is moving right}$$
$$\beta \text{ is moving up}$$
$$\gamma \text{ is moving left}$$

Since F is pointing right, it is helping α perform its trajectory, it is not helping nor preventing β from performing its trajectory, and is pushing γ against its trajectory. Let $A \subset \mathbb{R}^2$ be the image of α, $B \subset \mathbb{R}^2$ be the image of β, $C \subset \mathbb{R}^2$ be the image of γ. From here we intuitively deduce that

$$\int_A F \cdot d\alpha \text{ is positive}$$
$$\int_B F \cdot d\beta \text{ is zero}$$
$$\int_C F \cdot d\gamma \text{ is negative}$$

This can be easily computed from the dot products:

$$F(\alpha(t)) \cdot \alpha'(t) = \langle 1, 0 \rangle \cdot \langle 1, 0 \rangle = 1$$
$$F(\beta(t)) \cdot \beta'(t) = \langle 1, 0 \rangle \cdot \langle 0, 1 \rangle = 0$$
$$F(\gamma(t)) \cdot \gamma'(t) = \langle 1, 0 \rangle \cdot \langle -1, 0 \rangle = -1$$

Basically:

- if α goes with the flow of F, the integral $\displaystyle\int_A F \cdot d\alpha$ is positive.
- if γ is swimming against the current, the integral $\displaystyle\int_C F \cdot d\gamma$ is negative.

The integral $\displaystyle\int_C F \cdot d\gamma$ is also called the work of F along the trajectory γ.

3 Exercises on line integrals

Exercise 1 *Let* $F(x,y,z) = \langle -2y, z^2 + 3x, x-1\rangle$ *and* C *the curve with parametrization* $\gamma : [0,2] \to \mathbb{R}^3$ *given by*

$$\gamma(t) = (t^2 - 3, 2t, t^3).$$

Find

$$\int_C F \cdot d\gamma$$

Using the change of variables

$$\begin{aligned} x(t) &= t^2 - 3 \\ y(t) &= 2t \\ z(t) &= t^3 \end{aligned}$$

we get

$$F(\gamma(t)) = \langle -4t, t^6 + 3t^2 - 9, t^2 - 4\rangle$$

We also need

$$\gamma'(t) = \langle 2t, 2, 3t^2\rangle$$

Then

$$\begin{aligned} \int_C F \cdot d\gamma &= \int_0^2 \langle -4t, t^6 + 3t^2 - 9, t^2 - 4\rangle \cdot \langle 2t, 2, 3t^2\rangle \, dt \\ &= \int_0^2 \left[-8t^2 + 2t^6 + 6t^2 - 18 + 3t^4 - 12t^2\right] \, dt \\ &= -\frac{64}{3} + \frac{256}{7} + \frac{48}{3} - 36 + \frac{96}{5} - 32 \end{aligned}$$

Exercise 2 *Let* $F(x,y,z) = \langle z+1, x, y\rangle$ *and* C *the curve with parametrization* $\gamma : [0,3] \to \mathbb{R}^3$ *given by*

$$\gamma(t) = (e^t, -t^2, t).$$

Find

$$\int_C F \cdot d\gamma$$

Using the change of variables

$$\begin{aligned} x(t) &= e^t \\ y(t) &= -t^2 \\ z(t) &= t \end{aligned}$$

we get

$$F(\gamma(t)) = \langle t+1, e^t, -t^2 \rangle$$

We also need

$$\gamma'(t) = \langle e^t, -2t, 1 \rangle$$

Then

$$\begin{aligned}
\int_C F \cdot d\gamma &= \int_0^3 \langle t+1, e^t, -t^2 \rangle \cdot \langle e^t, -2t, 1 \rangle \, dt \\
&= \int_0^3 \left[te^t + e^t - 2te^t - t^2 \right] \, dt \\
&= \left[-te^t + 2e^t - t^3/3 \right] |_{t=0}^3 \\
&= -3e^3 + 2e^3 - 9 - 2 \\
&= -e^3 - 11
\end{aligned}$$

4 Fundamental Theorem of Calculus II

The classic Fundamental Theorem of Calculus says that the integral of the derivative of a function $F(x)$ is the function F itself:

$$\int_a^b F'(x) \, dx = F(b) - F(a)$$

Something similar happens when we take the line integral of a gradient. Consider a differentiable function $f : D \to \mathbb{R}$ with $D \subset \mathbb{R}^3$ and its gradient vector field ∇f. Also take an oriented curve $C \subset \mathbb{R}^3$ and a parametrization $\gamma : [a, b] \to \mathbb{R}^3$. Recall that by the chain rule, we have

$$\frac{d}{dt}(f(\gamma(t))) = \nabla f(\gamma(t)) \cdot \gamma'(t).$$

Therefore

$$\begin{aligned}
\int_C \nabla f \cdot d\gamma &= \int_a^b \nabla f(\gamma(t)) \cdot \gamma'(t) \, dt \\
&= \int_a^b \frac{d}{dt}(f(\gamma(t))) \, dt \\
&= f(\gamma(b)) - f(\gamma(a))
\end{aligned}$$

Theorem 1. Let $D \subset \mathbb{R}^3$ be a region, $f : D \to \mathbb{R}^3$ a differentiable function, $C \subset \mathbb{R}^3$ an oriented curve with parametrization $\gamma : [a, b] \to \mathbb{R}^3$. Then

$$\int_C \nabla f \cdot d\gamma = f(\gamma(b)) - f(\gamma(a))$$

In particular, the integral $\int_C \nabla f \cdot d\gamma$ does only depend on the endpoints of C and not on the trajectory.

Exercise 3 *Let $F(x, y, z) = \langle x, \cos y, e^z \rangle$ and C the curve with parametrization $\gamma : [0, \pi] \to \mathbb{R}^3$ given by*

$$\gamma(t) = (t^2\sqrt{t^2 + 1}, e^{t^2}, t^2 \cos t).$$

Find

$$\int_C F \cdot d\gamma$$

Note that

$$\mathrm{curl}(F) = \langle 0, 0, 0 \rangle,$$

and the domain of F is $\mathbb{R}^3$, so F is conservative. To find the potential function, we do partial integration,

$$\begin{aligned} f(x, y, z) &= x^2/2 + g_1(y, z) \\ f(x, y, z) &= \sin y + g_2(x, z) \\ f(x, y, z) &= e^z + g_3(x, y) \end{aligned}$$

Matching terms, we get

$$f(x, y, z) = x^2/2 + \sin y + e^z + C$$

On the other hand, the endpoints of C are

$$\begin{aligned} \gamma(0) &= (0, 1, 0) \\ \gamma(\pi) &= (\pi^2\sqrt{\pi^2 + 1}, e^{\pi^2}, -\pi^2) \end{aligned}$$

Therefore,

$$\begin{aligned} \int_C F \cdot d\gamma &= f(\pi^2\sqrt{\pi^2 + 1}, e^{\pi^2}, -\pi^2) - f(0, 1, 0) \\ &= \pi^4(\pi^2 + 1)/2 + \sin(e^{\pi^2}) + e^{-\pi^2} - \sin(1) - 1 \end{aligned}$$

Exercise 4 *Show that the vector field*

$$F(x, y) = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle$$

is not conservative, even though it has zero curl.

Let $C \subset \mathbb{R}^2$ be the unit circle oriented counterclockwise. Take the parametrization $\gamma(t) = (\cos t, \sin t)$ with $0 \le t \le 2\pi$. Then

$$\begin{aligned} F(\gamma(t)) &= \left\langle \frac{-\sin t}{\cos^2 t + \sin^2 t}, \frac{\cos t}{\cos^2 t + \sin^2 t} \right\rangle \\ &= \langle -\sin t, \cos t \rangle \end{aligned}$$

Also,

$$\gamma'(t) = \langle -\sin t, \cos t \rangle$$

Therefore,

$$\begin{aligned}
\int_C F \cdot d\gamma &= \int_0^{2\pi} F(\gamma(t)) \cdot \gamma'(t)\, dt \\
&= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt \\
&= \int_0^{2\pi} 1\, dt \\
&= 2\pi \neq 0
\end{aligned}$$

If F was conservative, we would have $F = \nabla f$ for some scalar function $f(x, y)$. By the Fundamental Theorem of Calculus, we would have

$$\begin{aligned}
\int_C F \cdot d\gamma &= f(\gamma(2\pi)) - f(\gamma(0)) \\
&= f(1, 0) - f(1, 0) \\
&= 0
\end{aligned}$$