1 Divergence Theorem

Let $F(x, y, z)$ be a space vector field, and p a point in the domain of F. Let $\Sigma \subset \mathbb{R}^3$ be a very small sphere around p, positively oriented (outwards). Then the integral

$$\iint_\Sigma F \cdot ds$$

measures how much F is pushing things away from p. It turns out we have another quantity that measures precisely that:

$$\mathrm{div}(F)(p).$$

Overall, if we have a region $E \subset \mathbb{R}^3$ with boundary ∂E oriented positively, the two integrals

$$\iint_{\partial E} F \cdot ds$$
$$\iiint_E \mathrm{div}(F)\, dA$$

measure how much the vector field is pointing outside of E. The Divergence Theorem (aka Gauss Theorem) states that they agree.

Theorem 1 (Divergence Theorem)**.** Let $F(x, y, z)$ be a space vector field, and $E \subset \mathbb{R}^3$ a region inside the domain of F. Then

$$\iint_{\partial E} F \cdot ds = \iiint_E \mathrm{div}(F)\, dV$$

Exercise 1 *Let $\Sigma \subset \mathbb{R}^3$ be the unit sphere, oriented positively. Compute*

$$\iint_\Sigma \langle x^2 + 3, 5xy + 2, 2z - 4\rangle \cdot dS$$

If $F(x, y, z) := \langle x^2 + 3, 5xy + 2, 2z - 4\rangle$, we can compute

$$\mathrm{div}(F) = 2x + 5x + 2 = 7x + 2$$

Then by the Divergence Theorem,

$$\iint_\Sigma \langle x^2 + 3, 5xy + 2, 2z - 4\rangle \cdot dS = \iiint_B (7x + 2)\, dV$$

where $B \subset \mathbb{R}^3$ is the unit ball. By symmetry, the integral of x over B is zero, so we get

$$\iiint_B (7x + 2)\, dV = 2 \cdot \mathrm{vol}(B) = \frac{8}{3}\pi$$

Exercise 2 *Let $\Sigma \subset \mathbb{R}^3$ be the boundary of the box $[-2,1]\times[-1,4]\times[1,6]$, oriented inwards. Compute*

$$\iint_\Sigma \langle 3x+y^2-5z, 2xy+3x, x\cos y+e^z\rangle \cdot dS$$

If $F(x,y,z) := \langle 3x+y^2-5z, 2xy+3x, x\cos y+e^z\rangle$, we can compute

$$\mathrm{div}(F) = 3+2x+e^z$$

Then by the Divergence Theorem,

$$\iint_\Sigma \langle 3x+y^2-5z, 2xy+3x, x\cos y+e^z\rangle \cdot dS = -\int_{-2}^{1}\int_{-1}^{4}\int_{1}^{6}(3+2x+e^z)\,dzdydx$$

(the minus sign comes from the orientation). Then

$$\begin{aligned}\int_{-2}^{1}\int_{-1}^{4}\int_{1}^{6}(3+2x+e^z)\,dzdydx &= 225-25(2^2-1)+15(e^6-e)\\ &= 15(10+e^6-e)\end{aligned}$$

so we conclude

$$\iint_\Sigma \langle 3x+y^2-5z, 2xy+3x, x\cos y+e^z\rangle \cdot dS = -15(10+e^6-e)$$

Exercise 3 *Let $\Sigma \subset \mathbb{R}^3$ be the portion of the paraboloid $z = 4-x^2-y^2$ oriented upwards. Compute*

$$\iint_\Sigma \langle x^2-e^y, 5+2x+\sin z, 3x+12y^2+3\rangle \cdot dS$$

If $F(x,y,z) := \langle x^2-e^y, 5+2x+\sin z, 3x+12y^2+3\rangle$, we can compute

$$\mathrm{div}(F) = 2x$$

Let $E \subset \mathbb{R}^3$ be the region below Σ and above the xy-plane, and ∂E its boundary oriented positively. Then

$$\iint_{\partial E} \langle x^2-e^y, 5+2x+\sin z, 3x+12y^2+3\rangle \cdot dS = \iiint_E 2x\,dV$$

Note that E is symmetric with respect to yz-plane, so the integrals above are zero. Also note that ∂E consists of two parts: Σ, and the disk $x^2+y^2 \leq 4$ in the xy-plane, oriented downwards, which we call Σ_1. Then we compute

$$\begin{aligned}
&\iint_{\Sigma} \langle x^2 - e^y, 5 + 2x + \sin z, 3x + 12y^2 + 3\rangle \cdot dS \\
&\qquad = -\iint_{\Sigma_1} \langle x^2 - e^y, 5 + 2x + \sin z, 3x + 12y^2 + 3\rangle \cdot dS \\
&\qquad = -\iint_{\Sigma_1} \langle x^2 - e^y, 5 + 2x + \sin z, 3x + 12y^2 + 3\rangle \cdot \langle 0, 0, -1\rangle \, dS \\
&\qquad = \iint_{\Sigma_1} (3x + 12y^2 + 3) \, dS \\
&\qquad = 0 + 48\pi + 12\pi \\
&\qquad = 60\pi
\end{aligned}$$