1 Green's Theorem

Let $F(x, y)$ be a planar vector field, and p a point in the domain of F. Let $C \subset \mathbb{R}^2$ be a very small circle around p, positively oriented (counterclockwise). Then the integral

$$\int_C F \cdot d\gamma$$

measures how much counterclockwise rotation around p is generated by F. It turns out we have another quantity that measures precisely that:

$$\text{curl}(F)(p).$$

Overall, if we have a region D with boundary ∂D oriented positively, the two integrals

$$\int_{\partial D} F \cdot d\gamma$$

$$\iint_D \text{curl}(F)\, dA$$

measure the overall amount of counterclockwise rotation generated by F in the region D. Green's Theorem states that they agree.

Theorem 1 (Green's Theorem)**.** Let $F(x, y)$ be a planar vector field, $D \subset \mathbb{R}^2$ a region inside the domain of F, and ∂D its boundary oriented positively. Then

$$\int_{\partial D} F \cdot d\gamma = \iint_D \text{curl}(F)\, dA$$

Example 1. Let $F = \langle P, Q \rangle$ and $D = [a_1, b_1] \times [a_2, b_2]$ the rectangle with vertices (a_1, b_1), (a_1, b_2), (a_2, b_2), (a_2, b_1). Let C_1, C_2, C_3, C_4 be the bottom, right, top, and left sides of the rectangle, respectively, oriented counterclockwise. Then using the Fundamental Theorem of Calculus,

$$\begin{aligned}
\iint_D \text{curl}(F)\, dA &= \int_{a_1}^{b_1} \int_{a_2}^{b_2} \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] dy dx \\
&= \int_{a_2}^{b_2} \int_{a_1}^{b_1} \frac{\partial Q}{\partial x} dx dy - \int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{\partial P}{\partial y} dy dx \\
&= \int_{a_2}^{b_2} [Q(b_1, y) - Q(a_1, y)] dy - \int_{a_1}^{b_1} [P(x, b_2) - P(x, a_2)] dx \\
&= \int_{C_2} F \cdot d\gamma_2 + \int_{C_4} F \cdot d\gamma_4 + \int_{C_3} F \cdot d\gamma_3 + \int_{C_1} F \cdot d\gamma_1 \\
&= \int_{\partial D} F \cdot d\gamma
\end{aligned}$$

Exercise 1 *Let $C \subset \mathbb{R}^2$ be the curve that travels along straight lines from $(-3,0)$ to $(2,0)$, then from $(2,0)$ to $(0,4)$, and then from $(0,4)$ back to $(-3,0)$. Compute*

$$\int_C \langle 2y + x^2 \cos x - 5, y^2 e^y - 3x \rangle \cdot d\gamma$$

Applying Green's Theorem, the integral above coincides with the integral

$$\iint_D \operatorname{curl}(F)\, dA$$

where $F = \langle 2y + x^2 \cos x - 5, y^2 e^y - 3x \rangle$ and D is the interior of the triangle. So we compute

$$\operatorname{curl} F = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -3 - 2 = -5$$

Then the integral becomes

$$\iint_D \operatorname{curl}(F)\, dA = -5 \cdot \operatorname{Area}(D) = -50$$

Exercise 2 *Let $C \subset \mathbb{R}^2$ be the ellipse $(x-3)^2 + 4(y-2)^2 = 16$ oriented clockwise. Compute*

$$\int_C \langle 3x^2 + 5x^2 - 2y + 1, 3xy - e^y \rangle \cdot d\gamma$$

Let D be the interior of the ellipse, and $F = \langle 3x^2 + 5x^2 - 2y + 1, 3xy - e^y \rangle$. Then

$$\operatorname{curl}(F) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 3y - 2$$

By Green's Theorem, we have

$$\int_C \langle 3x^2 + 5x^2 - 2y + 1, 3xy - e^y \rangle \cdot d\gamma = -\iint_D (3y-2)\, dA$$

(the minus sign appeared because the curve is oriented clockwise). Since the average y-coordinate of D is 2, we have

$$\iint_D y = 2 \cdot \operatorname{Area}(D)$$

Consequently,

$$\iint_D (3y-2)\, dA = (6-2) \cdot \operatorname{Area}(D) = 4 \cdot \operatorname{Area}(D)$$

The area of the ellipse is

$$\operatorname{Area}(D) = \pi \cdot 4 \cdot 2 = 8\pi$$

so we conclude

$$\int_C \langle 3x^2 + 5x^2 - 2y + 1, 3xy - e^y \rangle \cdot d\gamma = -32\pi$$

Example 2 (Area formula)**.** Let $D \subset \mathbb{R}^2$ be a compact region. If ∂D is oriented positively,

$$\text{Area}(D) = \int_{\partial D} \langle 0, x \rangle \cdot d\gamma = -\int_{\partial D} \langle y, 0 \rangle \cdot d\gamma$$

This is immediate from Green's Theorem because

$$\begin{aligned} \text{curl}(\langle 0, x \rangle) &= 1 \\ \text{curl}(\langle y, 0 \rangle) &= -1 \end{aligned}$$

Hence

$$\begin{aligned} \text{Area}(D) &= \iint_D 1 \, dA \\ &= \iint_D \text{curl}(\langle 0, x \rangle) \, dA \\ &= \int_{\partial D} \langle 0, x \rangle \cdot d\gamma \end{aligned}$$

$$\begin{aligned} \text{Area}(D) &= \iint_D 1 \, dA \\ &= -\iint_D \text{curl}(\langle y, 0 \rangle) \, dA \\ &= -\int_{\partial D} \langle y, 0 \rangle \cdot d\gamma \end{aligned}$$

Notation 1. When an oriented curve C is the boundary of a region, and C is oriented in such a way that it has the region on its left, for a planar vector field F we denote

$$\oint_C F \cdot d\gamma = \int_C F \cdot d\gamma$$