1 Exterior derivatives

There is a notion of derivatives of forms that generalizes the notions of classical derivative, gradient, curl, and divergence. The derivative of a smooth k-form is a smooth $(k+1)$-form.

Definition 1. Let

$$\omega := \sum_{i_1<\ldots<i_k} P_{i_1,\ldots,i_k}\, dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$

be a smooth k-form in $\mathbb{R}^n$. Then its exterior derivative $d\omega$ is the smooth $(k+1)$-form defined as

$$d\omega := \sum_{i_1<\ldots<i_k} \sum_{j=1}^{n} \frac{\partial P_{i_1,\ldots,i_k}}{\partial x^j}\, dx^j \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}.$$

Example 1. In $\mathbb{R}$, if $\omega = x^3 + \sin x$, then

$$d\omega = (3x^2 + \cos x)\, dx$$

Example 2. In $\mathbb{R}^2$, if $\omega = x^2 \cos y - ye^x$, then

$$d\omega = (2x \cos y - ye^x)\, dx + (-x^2 \sin y - e^x)\, dy$$

Example 3. In $\mathbb{R}^3$, if $\omega = z^3 e^x + 2z \cos y$, then

$$d\omega = e^x\, dx - 2z \sin y\, dy + (3z^2 e^x + 2 \cos y)\, dz$$

Example 4. In $\mathbb{R}^2$, if $\omega = xy^2\, dx + y \cos x\, dy$, then

$$\begin{aligned} d\omega &= y^2\, dx \wedge dx + 2xy\, dy \wedge dx - y \sin x\, dx \wedge dy + \cos x\, dy \wedge dy \\ &= (-y \sin x - 2xy)\, dx \wedge dy \end{aligned}$$

because the terms $dx \wedge dx$ and $dy \wedge dy$ are zero. Note that if we think of ω as the vector field

$$F = \langle xy^2, y \cos x \rangle,$$

then

$$\mathrm{curl}(F) = -y \sin x - 2xy$$

Example 5. In $\mathbb{R}^3$, if $\omega = yz\, dx + ye^z\, dy + x^2\, dz$, then

$$\begin{aligned} d\omega &= z\, dy \wedge dx + y\, dz \wedge dx + e^z\, dy \wedge dy + ye^z\, dz \wedge dy + 2x\, dx \wedge dz \\ &= -ye^z\, dy \wedge dz + (y - 2x)\, dz \wedge dx - z\, dx \wedge dy \end{aligned}$$

Note that if we think of ω as the vector field

$$F = \langle yz, ye^z, x^2 \rangle,$$

then

$$\mathrm{Curl}(F) = \langle -ye^z, y - 2x, -z \rangle$$

Example 6. In $\mathbb{R}^3$, if $\omega = x^2y\,dy \wedge dz + e^y \cos z\,dz \wedge dx + x^2 \sin z\,dx \wedge dy$, then

$$\begin{aligned} d\omega &= 2xy\,dx \wedge dy \wedge dz + e^y \cos z\,dy \wedge dz \wedge dx + x^2 \cos z\,dz \wedge dx \wedge dy \\ &= (2xy + e^y \cos z + x^2 \cos z)\,dx \wedge dy \wedge dz \end{aligned}$$

Note that if we think of ω as the vector field

$$F = \langle x^2y, e^y \cos z, x^2 \sin z \rangle,$$

then

$$\operatorname{div}(F) = 2xy + e^y \cos z + x^2 \cos z$$

Example 7. In $\mathbb{R}$, if $\omega = f(x)$, then

$$d\omega = f'(x)\,dx$$

Example 8. In $\mathbb{R}^2$, if $\omega = f(x, y)$, then

$$d\omega = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$$

Example 9. In $\mathbb{R}^3$, if $\omega = f(x, y, z)$, then

$$d\omega = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz$$

Example 10. In $\mathbb{R}^2$, if $\omega = P\,dx + Q\,dy$, then

$$d\omega = \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dx \wedge dy$$

Example 11. In $\mathbb{R}^3$, if $\omega = P\,dx + Q\,dy + R\,dz$, then

$$d\omega = \left[\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right] dy \wedge dz + \left[\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right] dz \wedge dx + \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dx \wedge dy$$

Example 12. In $\mathbb{R}^3$, if $\omega = P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy$, then

$$d\omega = \left[\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right] dx \wedge dy \wedge dz$$

Theorem 1 (Fundamental Theorem of Calculus)**.** Let ω be a smooth k-form in $\mathbb{R}^n$, and $M \subset \mathbb{R}^n$ an oriented $(k+1)$-dimensional manifold with boundary. Then

$$\int \cdots \int_{\partial M} \omega = \int \cdots \int_M d\omega$$