1 Stokes Theorem

Let $F(x, y, z)$ be a space vector field, p a point in the domain of F, and v a unit vector based at p. Let $C \subset \mathbb{R}^2$ be a very small circle centered at p, around v, which travels counterclockwise when seen from the tip of v. Then the integral

$$\int_C F \cdot d\gamma$$

measures how much rotation is generated by F around the axis v. It turns out we have another quantity that measures precisely that:

$$\mathrm{Curl}(F)(p) \cdot \mathrm{v}.$$

Overall, if we have an oriented surface $\Sigma \subset \mathbb{R}^3$ with boundary $\partial\Sigma$ oriented counterclockwise, the two integrals

$$\int_{\partial\Sigma} F \cdot d\gamma$$

$$\iint_{\Sigma} (\mathrm{Curl}(F) \cdot N)\, dS$$

measure how much rotation the vector field F generates over Σ (with axis perpendicular to the surface). Stokes Theorem states that they agree.

Theorem 1 (Stokes Theorem)**.** Let $F(x, y, z)$ be a space vector field, and $\Sigma \subset \mathbb{R}^3$ an oriented surface with boundary $\partial\Sigma$, oriented counterclockwise, when seen from the tip of N. Then

$$\int_{\partial\Sigma} F \cdot d\gamma = \iint_{\Sigma} \mathrm{Curl}(F) \cdot dS$$

Exercise 1 *Let $C \subset \mathbb{R}^3$ be the circle $x^2 + y^2 = 25$ in the xy-plane, oriented counterclockwise, when seen from above. Compute*

$$\int_C \langle x^2 - 2z - 5, 3x - e^y, z^3 - xy + 1\rangle \cdot d\gamma$$

Set $F = \langle x^2 - 2z - 5, 3x - e^y, z^3 - xy + 1\rangle$ and compute

$$\mathrm{Curl}(F) = \langle -x, -2 + y, 3\rangle$$

If we define Σ to be disk $x^2 + y^2 \leq 25$ in the xy-plane, oriented upward, we have

$\partial\Sigma = C$, and by Stokes Theorem,

$$\begin{aligned}
&\int_C \langle x^2 - 2z - 5, 3x - e^y, z^3 - xy + 1\rangle \cdot d\gamma \\
&\quad = \iint_\Sigma \langle -x, -2 + y, 3\rangle \cdot dS \\
&\quad = \iint_\Sigma \langle -x, -2 + y, 3\rangle \cdot \langle 0, 0, 1\rangle \, dS \\
&\quad = \iint_\Sigma 3 \, dS \\
&\quad = 75\pi
\end{aligned}$$

Exercise 2 *Let $C \subset \mathbb{R}^3$ be the curve that travels along straight lines first from $(1, 0, 0)$ to $(0, 0, 1)$, then from $(0, 0, 1)$ to $(0, 1, 0)$, and then from $(0, 1, 0)$ to $(1, 0, 0)$. Compute*

$$\int_C \langle 3x + y, e^y - 2x, 5z - 4x - 1\rangle \cdot d\gamma$$

Set $F = \langle 3x + y, e^y - 2x, 5z - 4x - 1\rangle$ and compute

$$\text{Curl}(F) = \langle 0, 4, -3\rangle$$

If we define Σ to be the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, oriented upward, by Stokes Theorem we have

$$\int_C \langle 3x + y, e^y - 2x, 5z - 4x - 1\rangle \cdot d\gamma = -\iint_\Sigma \langle 0, 4, -3\rangle \cdot dS$$

where the minus sign appears because when we look at C from the tip of $N = \langle 1, 1, 1\rangle$, it goes clockwise. Note that $\langle 0, 4, -3\rangle \cdot N = 4 - 3 = 1$, so

$$\iint_\Sigma \langle 0, 4, -3\rangle \cdot dS = \iint_\Sigma 1 \, dS = \text{Area}(\Sigma)$$

The triangle is equilateral with side $\sqrt{2}$, so its area is $\frac{1}{2}$, and

$$\int_C \langle 3x + y, e^y - 2x, 5z - 4x - 1\rangle \cdot d\gamma = -\frac{1}{2}$$

Exercise 3 *Let $C \subset \mathbb{R}^3$ be the curve that goes from $(1, 0)$ to $(-1, 0)$ along the arc $x^2 + y^2 = 1$, $y \geq 0$, in the xy-plane, followed by the curve that goes back to $(1, 0)$ along the arc $x^2 + z^2 = 1$, $z \geq 0$, in the xz-plane. Compute*

$$\int_C \langle 3x + 2 + z^2, 4\cos y - z^2, 2y - 5\rangle \cdot d\gamma$$

Set $F = \langle 3x + 2 + z^2, 4\cos y, 2y - 5\rangle$. We compute

$$\mathrm{Curl}(F) = \langle 2, 2z, 0\rangle$$

If we define Σ to be the portion of the sphere $x^2+y^2+z^2 = 1$ with $y \geq 0$, $z \geq 0$, oriented upward, we get $\partial\Sigma = C$. Then by Stokes Theorem,

$$\int_C \langle 3x + 2 + z^2, 4\cos y, 2y - 5\rangle \cdot d\gamma = \iint_\Sigma \langle 2, 2z, 0\rangle \cdot dS$$

Parametrizing Σ, we use $\varphi : [0, \pi] \times [0, \pi/2] \to \Sigma$ with

$$\varphi(u, v) = (\cos u \sin v, \sin u \sin v, \cos v),$$

then

$$\begin{aligned}
\frac{\partial\varphi}{\partial u} &= \langle -\sin u \sin v, \cos u \sin v, 0\rangle \\
\frac{\partial\varphi}{\partial v} &= \langle \cos u \cos v, \sin u \cos v, -\sin v\rangle \\
\frac{\partial\varphi}{\partial u} \times \frac{\partial\varphi}{\partial v} &= \langle -\cos u \sin^2 v, -\sin u \sin^2 v, -\sin v \cos v\rangle
\end{aligned}$$

Note that $\dfrac{\partial\varphi}{\partial u} \times \dfrac{\partial\varphi}{\partial v}$ points inwards, so φ is NOT compatible with the orientation. The integrand becomes

$$\begin{aligned}
&\langle 2, 2\cos v, 0\rangle \cdot \langle -\cos u \sin^2 v, -\sin u \sin^2 v, -\sin v \cos v\rangle \\
&\quad = -2\cos u \sin^2 v - 2\sin u \cos v \sin^2 v
\end{aligned}$$

Then (recalling that φ was not compatible with the orientation), we conclude

$$\begin{aligned}
\iint_\Sigma \langle 2, 2z, 0\rangle \cdot dS &= -\int_0^\pi \int_0^{\pi/2} [-2\cos u \sin^2 v - 2\sin u \cos v \sin^2 v]\, dv du \\
&= 0 + 4\int_0^{\pi/2} \cos v \sin^2 v\, dv \\
&= 4\left[\frac{\sin^3 v}{3}\right]_{v=0}^{\pi/2} \\
&= \frac{4}{3}
\end{aligned}$$